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PROJECTIVE BUNDLES ON INFINITE-DIMENSIONAL COMPLEX SPACES

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Abstract. Let V be a complex localizing Banach space with countable unconditional basis and E a rank r holomorphic vector bundle on $\mathbf{P}(V)$. Here we study the holomorphic embeddings of $\mathbf{P}(E)$ into products of projective spaces and the holomorphic line bundles on $\mathbf{P}(E)$. In particular we prove that if $r \geq 3$, then $H^1(\mathbf{P}(E), L) = 0$ for every holomorphic line bundle L on $\mathbf{P}(E)$.

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1. Introduction

For any complex vector space let $\mathbf{P}(V)$ be the projective space of all one-dimensional linear subspaces of V. Unless otherwise stated all cohomology groups of sheaves are sheaf cohomology groups or, equivalently because $\mathbf{P}(V)$ is metrizable and hence paracompact if V is a Banach space, Čech cohomology groups. In this paper we prove the following geometrical properties of the complex manifold $\mathbf{P}(E)$. For the notion of localizing complex manifold, see [9], p. 509; we just note that V is localizing if and only if $\mathbf{P}(V)$ is localizing.

Theorem 1. *Fix an integer $r \geq 2$. Let V be a localizing Banach space with countable unconditional basis and E a rank r holomorphic vector bundle on $\mathbf{P}(V)$. Set $X := \mathbf{P}(E)$ and let $\pi : X \to \mathbf{P}(V)$ be the projection. Call $\mathbf{O}(1)$ the tautological line bundle on X with degree one on each fiber, i.e., the only line bundle on X such that $\pi_*(\mathbf{O}(1)) \cong E$. We have $\mathrm{Pic}(X) \cong \mathbf{Z}^{\oplus 2}$ and we may take $\pi^*(\mathcal{O}_{\mathbf{P}(V)}(1))$ and $\mathbf{O}(1)$ as a basis of $\mathrm{Pic}(X)$. If $r \geq 3$ we have $H^1(X, L) = 0$ for every $L \in \mathrm{Pic}(X)$. If $r = 2$ we have $H^1(X, L) = 0$ for every $L \in \mathrm{Pic}(X)$ such that the degree of L with respect to the fibration π is at least -1.*

Theorem 2. *Fix integers $r \geq 2$, $s > 0$, $n \geq 0$, infinite dimensional Banach spaces V, $V_1, \dots, V_s$ and a rank r vector bundle E on $\mathbf{P}(V)$. Assume that V is localizing and with countable unconditional basis. Then:*

(a) *If $E \ncong \mathcal{O}_{\mathbf{P}(V)}(t)^{\oplus r}$ for any integer t there is no closed embedding $j : \mathbf{P}(E) \to \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n$ such that $j(\mathbf{P}(E))$ has finite codimension.*
(b) *Let $j : \mathbf{P}(E) \to \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n$ be a closed embedding such that $j(\mathbf{P}(E))$ has finite codimension. Then $E \cong \mathcal{O}_{\mathbf{P}(V)}(t)^{\oplus r}$ for some*

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integer t (i.e., $\mathbf{P}(E) \cong \mathbf{P}(V) \times \mathbf{P}^{r-1}$), $s = 1$, $n \geq r-1$, $V_1 \cong V \oplus \mathbf{C}^x$ for some integer $x \geq 0$ and j embeds linearly each slice $\mathbf{P}(V) \times \{P\}$, $P \in \mathbf{P}^{r-1}$, as a codimension x closed linear subspace of $\mathbf{P}(V_1) \times \{Q\}$ for some $Q \in \mathbf{P}^n$.

Remark 1. Take V and E as in the statement of Theorem 2 with $E \ncong \mathcal{O}_{\mathbf{P}(V)}(t)^{\oplus r}$ for any integer t. By part (a) of Theorem 2 the " algebraic manifold " $\mathbf{P}(E)$ cannot be embedded as a finite codimensional closed analytic subset in any finite product of projective spaces.

2. The Proofs

Lemma 1. *Let V be a Banach space. Then $H^2(\mathbf{P}(V), \mathbf{Z}) \cong \mathbf{Z}$ and any hyperplane of $\mathbf{P}(V)$ induces a generator of $H^2(\mathbf{P}(V), \mathbf{Z})$.*

Proof. Since the result is well-known when V has finite dimension, we may assume that V is infinite-dimensional. By [13], Proposition 1.2, all cohomology groups of $V \backslash \{0\}$ vanishes. We point out that for several infinite-dimensional Banach spaces (e.g., if V is a Hilbert space) $V \backslash \{0\}$ is homeomorphic to V ([7]) and even diffeomorphic to it ([4] or see [8], p. 21); every separable Banach space is homeomorphic to the separable Hilbert space; it easily follows that any separable Banach space V is homeomorphic to $V \backslash \{0\}$ and hence $V \backslash \{0\}$ is contractible. There is a locally trivial fiber bundle $m : V \backslash \{0\} \to \mathbf{P}(V)$ with $\mathbf{C}^*$ as fiber. Since $H^2(V \backslash \{0\}, \mathbf{Z}) = 0$, the Leray spectral sequence of m gives the existence of an injective homomorphism $H^2(\mathbf{P}(V), \mathbf{Z}) \to H^1(\mathbf{C}^*, \mathbf{Z}) \cong \mathbf{Z}$. Hence to check the lemma it is sufficient to show that the class $a \in H^2(\mathbf{P}(V), \mathbf{Z})$ represented by a hyperplane H is not torsion. Take a line $D \subset \mathbf{P}(V)$ not contained in H. Thus $H \cap D$ is a point, P. By the controvariance of cohomology groups the class a induces a non-zero multiple of the class b represented by P in $H^2(D, \mathbf{Z}$. Since $H^2(D, \mathbf{Z}) \cong \mathbf{Z}$ and b is a generator of $H^2(D, \mathbf{Z})$, b is not torsion. □

Lemma 2. *Let U be an open subset of a Banach space and m, k non-negative integers with $m + k > 0$. Let f be a holomorphic function on $U \times (\mathbf{C}^*)^k \times \mathbf{C}^m$. Then there is a Laurent expansion*

$$f = \sum_{i_1 \in \mathbf{Z}, \ldots, i_k \in \mathbf{Z}, j_1 \geq 0, \ldots j_m \geq 0} a_{i_1, \ldots, i_k, j_1, \ldots, j_m} z_1^{i_1} \ldots z_k^{i_k} z_{k+1}^{j_1} \ldots z_{k+m}^{j_m} \tag{1}$$

in which each $a_{i_1, \ldots, i_k, j_1, \ldots, j_m}$ is a holomorphic function on U. For all subsets A, B of $\{1, \ldots, k\}$ let $\sum_{A(\geq 0), B(<0)}[f]$ be the formal expansion given by (1) in which every index $i \in A$ varies only on the set of all non-negative integers and in which every index $i \in B$ varies only on the set of all negative integers. Then the expansion $\sum_{A(\geq 0), B(<0)}[f]$ defines a holomorphic function on the product of U, $\mathbf{C}^*$ for every $i \in \{1, \ldots, k\} \backslash (A \cup B)$, $\mathbf{C}$ for every $i \in A$ and $\mathbf{P}^1 \backslash \{0\}$ for every $i \in B$.

Proof. It is sufficient to use the classical proof by integration of the Laurent expansion in finitely many variables, just as done in [12], 45–52, for the Cauchy formula. Just to simplify the notation we will only write down the case $k = 1$ and $m = 0$. Fix $a \in \mathbf{C}^*$ and integrate $f(w, z)$ with respect to the variable z in the union of a circle centered at 0 and with large radius going counterclockwise and a circle with center at 0 and small radius going clockwise. $\square$

Remark 2. Let V be a localizing Banach space. By [1], Remark 5, for every integer t we have $H^1(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(t)) = 0$ (sheaf cohomology).

Remark 3. A Banach space V has countable unconditional basis if and only if $V \oplus \mathbf{C}$ has countable unconditional basis.

Unfortunately, since in Remark 2 we are able to handle only the first cohomology group, in this paper we will use only the case $q = 1$ of the following result.

Lemma 3. *Let M be a complex manifold locally modelled over open subsets of a Banach space with countable unconditional basis and E a rank r holomorphic vector bundle on M. Let $f : \mathbf{P}(E) \to M$ be the projection. Then $R^q f_*(\mathbf{O}(t)) = 0$ for every pair (q, t) of integers such that either $1 \leq q \leq r-2$ or $q \geq r$ or $q = r-1$ and $t \geq -r+1$.*

Proof. It is sufficient to prove that for every pseudoconvex open subset U of a Banach space with countable unconditional basis and every such pair (q, t) we have $H^q(U \times \mathbf{P}^{r-1}, \pi_2^*(\mathcal{O}_{\mathbf{P}^{r-1}}(t))) = 0$, where $\pi_2 : U \times \mathbf{P}^{r-1} \to \mathbf{P}^{r-1}$ is the projection. We may find a finite open covering $\mathfrak{U}$ of $U \times \mathbf{P}^{r-1}$ such that any finite intersection of open subsets of this covering is isomorphic to $U \times (\mathbf{C}^*)^k \times \mathbf{C}^m$ for some non-negative integers k and m with $k + m = r - 1$. By Remark 3 for all non-negative integers k, m the manifold $U \times (\mathbf{C}^*)^k \times \mathbf{C}^m$ is a pseudoconvex open subset of a Banach space with countable unconditional basis. Thus $H^i(U \times (\mathbf{C}^*)^k \times \mathbf{C}^m, \mathcal{O}_{U \times (\mathbf{C}^*)^k \times \mathbf{C}^m}) = 0$ for every $i > 0$ ([10], Theorem 0.1). Hence $\mathfrak{U}$ is a Leray covering which computes $H^q(U \times \mathbf{P}^{r-1}, \pi_2^*(\mathcal{O}_{\mathbf{P}^{r-1}}(t)))$. By Lemma 2 we have the Laurent expansions which allow us to copy word for word the proof for the case U a point given in [5], pp. 51–55. $\square$

Remark 4. In the set-up of Lemma 3 we only used that M has a basis $\mathfrak{U}$ of open subsets such that $H^i(U, \mathcal{O}_U) = 0$ for every $i > 0$ and every $U \in \mathfrak{U}$. Hence by [11] we may apply Lemma 3 to complex manifolds modelled over certain Fréchet spaces.

Remark 5. Let M be a complex Banach manifold and E a rank r holomorphic vector bundle on M. Let $f : \mathbf{P}(E) \to M$ be the projection and $\mathbf{O}(1)$ the tautological line bundle on $\mathbf{P}(E)$ whose restriction to any fiber of f has degree one. For every holomorphic vector bundle A on M and all integers i, t with $i \geq 0$ we have $R^i f_*(\mathbf{O}(t) \otimes f^*(A)) \cong A \otimes R^i f_*(\mathbf{O}(t))$. If $t < 0$ we have $f_*(\mathbf{O}(t)) = 0$. We have $f_*(\mathcal{O}_{\mathbf{P}(E)}) \cong \mathcal{O}_M$. If $t > 0$ we have $f_*(\mathbf{O}(t)) \cong S^t(E)$, where S^t denotes the symmetric product. Now assume $t \geq 0$, $M = \mathbf{P}(V)$ and E isomorphic to a direct sum of r line bundles. Then $S^t(E)$ is a direct sum of line bundles.

Proof of Theorem 1. By Lemma 3 and Remark 5 we have $\pi_*(\mathcal{O}_X) \cong \mathcal{O}_{\mathbf{P}(V)}$ and $R^1\pi_*(\mathcal{O}_X) = 0$. Hence from the Leray spectral sequence of π we obtain

$H^1(X, \mathcal{O}_X) = H^1(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)})$. The latter sheaf cohomology group vanishes by [9] and [1], Remark 5. From the exponential sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0 \tag{2}$$

we obtain an inclusion $j : \mathrm{Pic}(X) \to H^2(X, \mathbf{Z})$. The Leray spectral sequence of π gives $H^2(X, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 2}$ and that $H^2(X, \mathbf{Z})$ has as generators the first Chern classes of $\pi^*(\mathcal{O}_{\mathbf{P}(V)}(1))$ and of $\mathbf{O}(1)$. This implies the surjectivity of j, proving the first assertion of Theorem 1. All the vanishing results in the statement of Theorem 1 follow from Lemma 3, Remark 5 and the vanishing theorems proved in [9] (see [1], Remark 5). □

Remark 6. Let T be a reduced and irreducible finite-dimensional complex space. For every integer $m > 2(\dim(T))$ every holomorphic map $g : \mathbf{P}^m \to T$ is constant because any two fibers of f are disjoint and for any two closed subvarieties A, B of $\mathbf{P}^m$ with $\mathrm{codim}(A) \le \dim(T)$ and $\mathrm{codim}(B) \le \dim(T)$ we have $A \cap B \neq \varnothing$. Thus every holomorphic map $\mathbf{P}(V) \to T$ is constant if V is infinite-dimensional.

Proof of Theorem 2. We have $E \cong \mathcal{O}_{\mathbf{P}(V)}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}(V)}(a_r)$ for some integers $a_1 \ge \cdots \ge a_r$ ([9], Theorems 7.1 and 8.5). Set $X := \mathbf{P}(E)$ and let $f : X \to \mathbf{P}(V)$ be the projection. Assume the existence of a closed embedding $j : X \to \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n$ with $j(X)$ of finite codimension. A section of f is given by a pair (L, u), where $L \in \mathrm{Pic}(\mathbf{P}(V))$ and $u : L \to E$ is a nowhere vanishing inclusion, i.e., an injection as a map of holomorphic bundles, i.e., a map whose dual map $E^* \to L^*$ is surjective. The image of a section of f is biholomorphic to $\mathbf{P}(V)$. Hence by Remark 6 we see that either $n = 0$ or the image of any such section is mapped to a point by the projection $\mu_{s+1} : \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n \to \mathbf{P}^n$. Now we use that $E \cong \mathcal{O}_{\mathbf{P}(V)}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}(V)}(a_r)$. Hence taking $L = \mathcal{O}_{\mathbf{P}(V)}(a_r)$ we see that for every $P \in X$ there is a section of f whose image contains P. Fix any section u of f. We have $\mathrm{codim}((j \circ u)(\mathbf{P}(V))) = \mathrm{codim}(j(X)) + r - 1$ and hence $j \circ u$ is a finite-codimensional embedding of $\mathbf{P}(V)$ into $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n$. Let $\mu_i : \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n \to \mathbf{P}(V_i)$, $1 \le i \le s$, be the projections. As in Lemma 1 and in the proof of Theorem 1 we obtain $H^2(\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n, \mathbf{Z}) \cong \mathbf{Z}^{(s+1)}$ and $\mathrm{Pic}(\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n) \cong \mathbf{Z}^{(s+1)}$ and that both Abelian groups are freely generated by the pull-backs of the classes of the hyperplanes of each of the factors of $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n$. Since $\mathrm{Pic}(\mathbf{P}(V))$ is freely generated by $\mathcal{O}_{\mathbf{P}(V)}(1)$, there are integers x_i, $1 \le i \le s+1$, such that $(\mu_i \circ j \circ u)^*(\mathcal{O}_{\mathbf{P}(V_i)}(1)) \cong \mathcal{O}_{\mathbf{P}(V)}(x_i)$ for $1 \le i \le s$ and $(\mu_{s+1} \circ j \circ u)^*(\mathcal{O}_{\mathbf{P}^n}(1)) \cong \mathcal{O}_{\mathbf{P}(V)}(x_{s+1})$. We have $x_i \ge 0$ for every i and $x_j = 0$ if and only if $\mu_j \circ j \circ u$ is a point.

(i) First assume $s \ge 2$ and fix $Q \in \mathbf{P}(V_2) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n$. The analytic set $(j \circ u)(\mathbf{P}(V)) \cap \mathbf{P}(V_1) \times \{Q\}$ is closed and of finite codimension in the infinite-dimensional projective space $\mathbf{P}(V_1) \times \{Q\}$. Hence it contains many lines, D (see e.g., [2] or [3]). Since D maps to $\{Q\}$ by the projection $\rho : \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n \to \mathbf{P}(V_2) \times \cdots \times \mathbf{P}(V_s) \times \mathbf{P}^n$ we obtain $\deg(D) = x_1 \deg((j \circ u)^{-1}(D)$. Since the compact Riemann Surface $(j \circ u)^{-1}(D)$ has positive degree, while D is a line, we obtain $x_1 = 1$ and that $(j \circ u)^{-1}(D)$ has degree one, i.e., it is a line. Fix $P \in (j \circ u)^{-1}(D)$ and move the line $(j \circ u)^{-1}(D)$ among the lines of $\mathbf{P}(V)$ through P. For any such nearby line R the set $\rho \circ j \circ u(R)$ is a point. Since $P \in R$, we get $\rho \circ j \circ u(R) = \{Q\}$ for any such R. Since any point of $\mathbf{P}(V)$ is contained in a line containing P, we obtain $\rho \circ j \circ u(\mathbf{P}(V)) = \{Q\}$, i.e., $j \circ u(\mathbf{P}(V)) \subseteq \mathbf{P}(V_1) \times \{Q\}$. Since V_2 has infinite dimension, $j \circ u(\mathbf{P}(V))$ has not finite codimension, contradiction.

(ii) By part (i) we may assume $s = 1$. First assume $s = 1$ and $n = 0$. By [3], Theorem 3, we obtain that for any section u of f the closed analytic subset $j(u(X))$ is a closed finite codimensional linear subspace of $\mathbf{P}(V_1)$. Let α and β be the integers such that $j^*(\mathcal{O}_{\mathbf{P}(V_1)}(1)) \cong \mathbf{O}(\alpha) \otimes f^*(\mathcal{O}_{\mathbf{P}(V)}(\beta))$. Since the restriction of $\mathcal{O}_{\mathbf{P}(V_1)}(1)$ to any compact Riemann Surface contained in $\mathbf{P}(V_1)$ has positive degree, we have $\alpha > 0$ and $\beta > 0$. First assume $a_1 = a_r$. Hence there are several sections u, v of f such that $u(\mathbf{P}(V)) \cap v(\mathbf{P}(V)) = \varnothing$. Since $j(u(\mathbf{P}(V)))$ and $j(v(\mathbf{P}(V)))$ are finite codimensional linear subspaces of $\mathbf{P}(V_1)$ and V_1 has infinite dimension, then $j(u(\mathbf{P}(V))) \cap j(v(\mathbf{P}(V))) \neq \varnothing$, contradicting the injectivity of j. Now assume $a_1 > a_r$. There are nowhere vanishing morphisms $\mathcal{O}_{\mathbf{P}(V)}(a_1) \to E$ and $\mathcal{O}_{\mathbf{P}(V)}(a_r) \to E$ such that the images of the corresponding sections are disjoint. Again, we just obtained a contradiction. Now assume $s = 1$ and $n > 0$. First assume $a_1 > a_r$. In this case we obtain a family $\{u_\alpha\}$ of sections of f whose images $\{u_\alpha(\mathbf{P}(V))\}$ covers X and whose images are mutually intersecting. Since each $\mu(u_\alpha(\mathbf{P}(V)))$ is just a point (Remark 5), we obtain the existence of $P \in \mathbf{P}^n$ such that $j(X) \subseteq \mathbf{P}(V) \times \{P\}$. The case $s = 1$ and $n = 0$ gives a contradiction. Now assume $a_1 = a_r$, i.e., assume $E \cong \mathcal{O}_{\mathbf{P}(V)}(a_1)^{\oplus r}$. Let α', β' be the integers such that $j^*(\mu_1^*(\mathcal{O}_{\mathbf{P}(V_1)}(1)) \otimes \mu_2^*(\mathcal{O}_{\mathbf{P}^n}(1))) \cong \mathbf{O}(\alpha') \otimes f^*(\mathcal{O}_{\mathbf{P}(V)}(\beta'))$. The proof just given for the case $a_1 > a_r$ shows that $\alpha' = 1$. As in the proofs in [2] or [3] (i.e., just a use of Remark 6) we see that j is obtained as a Segre product of a linear embedding of $\mathbf{P}(V)$ as a finite codimensional linear subspace of $\mathbf{P}(V_1)$ with an embedding of $\mathbf{P}^{r-1}$ into $\mathbf{P}^n$, proving the theorem with a more precise form of part (b). □

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