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INTEGRABILITY AND L^1-CONVERGENCE OF MODIFIED SINE SUMS

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Abstract. New modified sine sums are introduced and a criterion for the L^1-convergence of these modified sine sums under a new class K is obtained. Also a necessary and sufficient condition for the L^1-convergence of the cosine series is deduced as a corollary.

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1. Introduction

Consider the cosine series

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.1}$$

with partial sums defined by $S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$ and let

$$g(x) = \lim_{n\to\infty} S_n(x).$$

Concerning the L^1-convergence of cosine series (1.1) Kolmogorov [2] proved the following theorem:

Theorem A. *If $\{a_n\}$ is a quasi-convex null sequence, then for the L^1-convergence of the cosine series* (1.1) *it is necessary and sufficient that* $\lim\limits_{n\to\infty} a_n \log n = 0$.

The case of this theorem in which the sequence $\{a_n\}$ is convex, was established by Young [7]. That is why this Theorem A is sometimes known as Young–Kolmogorov Theorem.

Rees and Stanojević [5] have introduced modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} (\triangle a_j) \cos kx.$$

Garret and Stanojević [1], Ram [4] and Singh and Sharma [6] studied the L^1-convergence of these cosine sums under different sets of conditions on the coefficients a_n.

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Later on, Kumari and Ram [3], introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle\Big(\frac{a_j}{j}\Big) k \cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle\Big(\frac{a_j}{j}\Big) k \sin kx$$

and have studied their L^1- convergence under the condition that the coefficients a_n belong to different classes of sequences. Also, they deduced some results about the L^1-convergence of cosine and sine series as corollaries.

We introduce here new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\triangle a_{j-1} - \triangle a_{j+1}) \sin kx. \tag{1.2}$$

The aim of this paper is to study the L^1-convergence of these modified sine sums $K_n(x)$ and to obtain an analogue of Theorem A of Kolmogorov for a newly defined class $\mathbf{K}$ of coefficient sequences defined as follows:

Definition. If $a_k = o(1)$, $k \to \infty$, and

$$\sum_{k=1}^{\infty} k \left|\triangle^2 a_{k-1} - \triangle^2 a_{k+1}\right| < \infty \quad (a_0 = 0), \tag{1.3}$$

then we say that $\{a_k\}$ belongs to the class $\mathbf{K}$.

2. Main Result

The main result is the following theorem:

Theorem 1. *Let the sequence $\{a_n\}$ belong to the class $\mathbf{K}$, then $K_n(x)$ converges to $g(x)$ in L^1-norm.*

Proof. We have

$$\begin{aligned} S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx = \frac{1}{2\sin x} \sum_{k=1}^{n} a_k \cos kx 2 \sin x \\ &= \frac{1}{2\sin x} \sum_{k=1}^{n} a_k \left[\sin(k+1)x - \sin(k-1)x\right] \\ &= \frac{1}{2\sin x} \sum_{k=1}^{n} (a_{k-1} - a_{k+1}) \sin kx + a_{n+1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x}. \end{aligned}$$

Applying Abel's transformation, we have

$$S_n(x) = \frac{1}{2\sin x}\left(\sum_{k=1}^{n} (\triangle a_{k-1} - \triangle a_{k+1})\, \tilde{D}_k(x) + (a_n - a_{n+2})\tilde{D}_n(x)\right) + a_{n+1}\frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x},$$

where $\tilde{D}_k(x)$ denotes Dirichlet conjugate kernel. Thus

$$g(x) = \lim_{n\to\infty} S_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{\infty} (\triangle a_{k-1} - \triangle a_{k+1})\, \tilde{D}_k(x),$$

if the series is convergent. Also,

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n} (\triangle a_{j-1} - \triangle a_{j+1}) \sin kx$$
$$= \frac{1}{2\sin x}\left(\sum_{k=1}^{n} (a_{k-1} - a_{k+1}) \sin kx - (a_n - a_{n+2})\tilde{D}_n(x)\right).$$

Applying Abel's transformation, we have

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n} (\triangle a_{k-1} - \triangle a_{k+1})\, \tilde{D}_k(x).$$

Since $\left|\frac{\tilde{D}_k(x)}{2\sin x}\right| = O(k)$ and

$$\sum_{k=1}^{\infty} k \left|\triangle^2 a_{k-1} - \triangle^2 a_{k+1}\right| < \infty,$$

the series

$$\frac{1}{2\sin x}\sum_{k=1}^{\infty} (\triangle a_{k-1} - \triangle a_{k+1})\, \tilde{D}_k(x)$$

converges.

Hence $\lim_{n\to\infty} K_n(x) = g(x)$ exists.

Thus

$$g(x) - K_n(x) = \frac{1}{2\sin x}\sum_{k=n+1}^{\infty} (\triangle a_{k-1} - \triangle a_{k+1})\, \tilde{D}_k(x)$$
$$= \lim_{m\to\infty}\left(\frac{1}{2\sin x}\sum_{k=n+1}^{m} (\triangle a_{k-1} - \triangle a_{k+1})\, \tilde{D}_k(x)\right).$$

The use of Abel's transformation gives

$$g(x) - K_n(x) = \frac{1}{2\sin x}\lim_{m\to\infty}\left[\sum_{k=n+1}^{m-1} (k+1)\left(\triangle^2 a_{k-1} - \triangle^2 a_{k+1}\right) \tilde{F}_k(x)\right.$$

$$+(m+1)\left(\triangle a_{m-1}-\triangle a_{m+1}\right)\tilde{F}_m(x)-(n+1)\left(\triangle a_n-\triangle a_{n+2}\right)\tilde{F}_n(x)\Bigg]$$

$$=\frac{1}{2\sin x}\Bigg[\sum_{k=n+1}^{\infty}(k+1)\left(\triangle^2 a_{k-1}-\triangle^2 a_{k+1}\right)\tilde{F}_k(x)$$

$$-(n+1)\left(\triangle a_n-\triangle a_{n+2}\right)\tilde{F}_n(x)\Bigg],$$

where

$$\tilde{F}_k(x)=\frac{1}{k+1}\sum_{j=0}^{k}\tilde{D}_j(x)$$

denotes the conjugate Fejer kernel. Now

$$\int_{-\pi}^{\pi}|g(x)-K_n(x)|dx$$

$$=\int_{-\pi}^{\pi}\Bigg|\frac{1}{2\sin x}\Bigg[\sum_{k=n+1}^{\infty}(k+1)\left(\triangle^2 a_{k-1}-\triangle^2 a_{k+1}\right)\tilde{F}_k(x)$$

$$-(n+1)\left(\triangle a_n-\triangle a_{n+2}\right)\tilde{F}_n(x)dx\Bigg|$$

$$\leq C\Bigg[\sum_{k=n+1}^{\infty}(k+1)\left|\left(\triangle^2 a_{k-1}-\triangle^2 a_{k+1}\right)\right|\int_{-\pi}^{\pi}\left|\tilde{F}_k(x)\right|dx$$

$$+(n+1)\left|\left(\triangle a_n-\triangle a_{n+2}\right)\right|\int_{-\pi}^{\pi}\left|\tilde{F}_n(x)\right|dx\Bigg].$$

But $\frac{1}{\pi}\int_{-\pi}^{\pi}|\tilde{F}_k(x)|dx=1$, and

$$|(\triangle a_n-\triangle a_{n+2})|=\left|\sum_{k=n}^{\infty}\left(\triangle^2 a_k-\triangle^2 a_{k+2}\right)\right|$$

$$=\left|\sum_{k=n+1}^{\infty}\frac{k}{k}\left(\triangle^2 a_{k-1}-\triangle^2 a_{k+1}\right)\right|$$

$$\leq\frac{1}{n+1}\left|\sum_{k=n+1}^{\infty}k\left(\triangle^2 a_{k-1}-\triangle^2 a_{k+1}\right)\right|=o\Big(\frac{1}{n+1}\Big).$$

Thus we have

$$\int_{-\pi}^{\pi}|g(x)-K_n(x)|\,dx=O\Bigg(\sum_{k=n+1}^{\infty}(k+1)\left|\left(\triangle^2 a_{k-1}-\triangle^2 a_{k+1}\right)\right|\Bigg)$$

$$+\,o(1) = o(1)$$

by (1.3). □

Corollary. *If $\{a_n\}$ belongs to the class $\boldsymbol{K}$, then the necessary and sufficient condition for the L^1-convergence of the cosine series* (1.1) *is* $\lim\limits_{n\to\infty} a_n \log n = 0$.

Proof. We have

$$\big\|S_n(x)-g(x)\big\| \le \big\|S_n(x)-K_n(x)\big\| + \big\|K_n(x)-g(x)\big\| = \big\|K_n(x)-g(x)\big\| + \left\|(a_n-a_{n+2})\frac{\tilde{D}_n(x)}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}\right\|.$$

Also,

$$\left\|(a_n-a_{n+2})\frac{\tilde{D}_n(x)}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}\right\| = \big\|K_n(x)-S_n(x)\big\| \le \big\|K_n(x)-g(x)\big\| + \big\|S_n(x)-g(x)\big\|,$$

and

$$\begin{aligned}\big|(a_n-a_{n+2})\big| &= \left|\sum_{k=n}^{\infty}(\triangle a_k - \triangle a_{k+2})\right| \\ &= \left|\sum_{k=n+1}^{\infty}\frac{k}{k}\big(\triangle a_{k-1}-\triangle a_{k+2}-\triangle a_{k+1}+\triangle a_{k+2}\big)\right| \\ &\le \frac{1}{n+1}\left|\sum_{k=n+1}^{\infty} k\left(\triangle^2 a_{k-1}-\triangle^2 a_{k+1}\right)\right| = o\Big(\frac{1}{n}\Big).\end{aligned}$$

Since $\int\limits_{-\pi}^{\pi}\frac{\tilde{D}_n(x)}{2\sin x}dx = O(n)$, we obtain

$$(a_n-a_{n+2})\int\limits_{-\pi}^{\pi}\frac{\tilde{D}_n(x)}{2\sin x}dx = O\big((a_n-a_{n+2})n\big) = o(1).$$

Moreover,

$$\begin{aligned}&\int\limits_{-\pi}^{\pi}\left|a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}\right|dx \\ &\le \int\limits_{-\pi}^{\pi} a_n\left|\frac{\sin nx}{2\sin x} + \frac{\sin(n+1)x}{2\sin x}\right|dx = a_n\int\limits_{-\pi}^{\pi}|D_n(x)|\,dx \sim (a_n\log n).\end{aligned}$$

Since $\|K_n(x)-g(x)\| = o(1)$ $(n\to\infty)$, by Theorem 2.1.

Therefore

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}|g(x)-S_n(x)|\,dx=o(1)$$

if and only if $\lim_{n\to\infty} a_n \log n = 0$. □

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