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# SOME FAMILIES OF GENERATING FUNCTIONS FOR THE BESSEL AND RELATED FUNCTIONS

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**Abstract.** The authors apply a certain novel technique based on the combined use of operational methods and of some special multivariable and multi-index polynomials to derive several families of generating functions involving the products of Bessel and related functions. The possibility of extending this technique to the derivation of generating functions of hybrid nature (involving, for example, the product of a Bessel function and Laguerre polynomials) is also investigated.

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## 1. Introduction

Using certain operational rules, Dattoli *et al.* [4] derived generating functions of the form

$$S_{\{p\}}\big(\{x\};t\big) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, J_{n+p_1}(x_1)\cdots J_{n+p_m}(x_m) \tag{1}$$
$$\big(\{x\} = x_1,\ldots,x_m; \quad \{p\} = p_1,\ldots,p_m\big),$$

where $J_n(x)$ is a cylindrical Bessel function (see [1] and [8] for details).

The indices $p_1,\ldots,p_m$, which appear on the right-hand side of equation (1), are not necessarily integers. The technique applied in the derivation of such generating functions as (1) is based on the combined use of operational methods and some families of special functions involving many indices and many variables [5]. It can also be extended to the case of spherical Bessel functions.

The main objective of this paper is to provide an extension of the aforementioned technique and to show how such extended procedure leads to further generalizations including (for example) generating functions of hybrid nature.

We begin by illustrating the derivation of a well-known generating function (cf. [7, p. 427, equation 8.4 (56)]), which we present here as Proposition 1.

**Proposition 1.** *The following generating function relationship holds true:*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, J_{n+\nu}(x) = \left(\frac{x}{x-2t}\right)^{\nu/2} J_\nu\big(\sqrt{x^2-2xt}\big) \quad (\nu\in\mathbb{C})\,. \tag{2}$$

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*Proof.* If we multiply both sides of the familiar derivative formula [8, p. 46, equation 3.2 (6)]

$$\left(\frac{1}{x}\,\frac{d}{dx}\right)^n \left(x^{-\nu} J_\nu\left(x\right)\right) = \frac{\left(-1\right)^n}{x^{n+\nu}}\; J_{n+\nu}\left(x\right) \tag{3}$$
$$\left(\nu \in \mathbb{C};\quad n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}\right)$$

by $\tau^n/n!$ and sum each side from $n = 0$ to $n = \infty$, then we find from (3) that

$$\exp\left(\frac{\tau}{x}\,\frac{d}{dx}\right)\left(x^{-\nu}\; J_\nu\left(x\right)\right) = x^{-\nu}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\tau}{x}\right)^n\; J_{n+\nu}\left(x\right)\quad\left(\nu \in \mathbb{C}\right). \tag{4}$$

The use of the operational identity [6]

$$\exp\left(\frac{\tau}{x}\,\frac{d}{dx}\right) f\left(x\right) = f\left(\sqrt{x^2 + 2\tau}\right) \tag{5}$$

in (4) readily yields (2) after setting $\tau = -xt$.

An alternative procedure is based on the use of the so-called Tricomi–Bessel function defined by (see, e.g., [5])

$$C_n\left(x\right) := \sum_{r=0}^{\infty}\frac{\left(-1\right)^r\;\, x^r}{r!\left(n+r\right)!}\,, \tag{6}$$

which is related to the cylindrical Bessel function $J_n\left(x\right)$ by

$$C_n\left(x\right) = x^{-n/2}\; J_n\left(2\sqrt{x}\right)\quad\text{and}\quad J_n\left(x\right) = \left(\frac{x}{2}\right)^n\; C_n\left(\frac{x^2}{4}\right), \tag{7}$$

with the generating function

$$\sum_{n=-\infty}^{\infty} t^n\; C_n\left(x\right) = \exp\left(t - \frac{x}{t}\right).$$

It is fairly straightforward to observe from definition (6) that

$$\left(\frac{d}{dx}\right)^n\; C_l\left(x\right) = \left(-1\right)^n\;\, C_{n+l}\left(x\right) \tag{8}$$

so that

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\; C_{n+l}\left(x\right) = \exp\left(-t\,\frac{d}{dx}\right) C_l\left(x\right) = C_l\left(x-t\right),$$

which, in the light of (7), yields the generating function (2). This evidently completes our *alternative* (operational) derivation of the generating function (2) *without* using the operational identity (5). □

*Remark* 1. Albeit simple, the above examples illustrate how operational methods and special functions can be combined to end up with the explicit derivation of a generating function.

In Section 2 of this paper, we will provide a generalization of the above results by suitably combining various known results and identities with different special functions including, for example, some non-standard forms of Bessel and related functions.

## 2. A Class of Multivariable Bessel Functions

An important rôle in pure and applied mathematics is also played by some families of Bessel functions with more than one variable. For example, we have a two-variable one-parameter Bessel function defined by the generating function [5]

$$\sum_{n=-\infty}^{\infty} t^n \, J_n(x,y;\tau) = \exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right]$$

and given explicitly by the series:

$$J_n(x,y;\tau) = \sum_{l=-\infty}^{\infty} \tau^l \, J_{n-2l}(x)\, J_l(y)\,.$$

It is convenient, for our purposes, to introduce the two-variable one-parameter counterpart of the Tricomi–Bessel function, which satisfies each of the following identities:

$$\sum_{n=-\infty}^{\infty} t^n \, C_n(x,y;\tau) = \exp\left(t-\frac{x}{t}+t^2\tau-\frac{y}{t^2\tau}\right), \tag{9}$$

$$C_n(x,y;\tau) = \sum_{l=-\infty}^{\infty} \tau^l \, C_{n-2l}(x)\, C_l(y)\,,$$

$$J_n(x,y;\tau) = \left(\frac{x}{2}\right)^n C_n\left(\frac{x^2}{4},\frac{y^2}{4};\frac{2y}{x^2}\,\tau\right),$$

and

$$C_n(x,y;\tau) = x^{-n/2}\, J_n\left(2\sqrt{x},2\sqrt{y};\frac{x}{\sqrt{y}}\,\tau\right).$$

*Remark* 2. It can also be proved in a fairly direct way that

$$(-1)^s\,\frac{\partial^s}{\partial x^s}\, C_n(x,y;\tau) = C_{n+s}(x,y;\tau) \tag{10}$$

and

$$(-\tau)^s\,\frac{\partial^s}{\partial y^s}\, C_n(x,y;\tau) = C_{n+2s}(x,y;\tau)\,, \tag{11}$$

which can be applied to derive the generating functions [4]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \, C_{n+l}(x,y;\tau) = C_l(x-t,y;\tau),$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \, C_{2n+l}(x,y;\tau) = C_l(x,y-\tau t;\tau), \tag{12}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \, J_{n+l}(x,y;\tau) = \left(\frac{x}{x-2t}\right)^{l/2} \, J_l\left(\sqrt{x^2-2xt}, y; \frac{\tau(x-2t)}{x}\right),$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \, J_{2n+l}(x,y;\tau) = J_l\left(x, \sqrt{y^2-2y\tau t}; \tau\sqrt{\frac{y}{y-2\tau t}}\right).$$

With a view to further generalizing the above results, we recall that Bessel functions can be extended to the case with more than two variables and one parameter. Indeed, note that the three-variable two-parameter Tricomi–Bessel function given by

$$C_n(x,y,z;\tau_1,\tau_2) = \sum_{l=-\infty}^{\infty} \tau_2^l \, C_{n-3l}(x,y;\tau_1) \, C_l(z),$$

satisfies the generating function

$$\sum_{n=-\infty}^{\infty} t^n \, C_n(x,y,z;\tau_1,\tau_2) = \exp\left(t - \frac{x}{t} + t^2 \, \tau_1 - \frac{y}{t^2 \, \tau_1} + t^3 \, \tau_2 - \frac{z}{t^3 \, \tau_2}\right)$$

and the derivative formula

$$(-\tau_2)^s \, \frac{\partial^s}{\partial z^s} \, C_n(x,y,z;\tau_1,\tau_2) = C_{n+3s}(x,y,z;\tau_1,\tau_2)$$

along with

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \, C_{3n+l}(x,y,z;\tau_1,\tau_2) = C_l(x,y,z-\tau_2 t;\tau_1,\tau_2).$$

Furthermore, the three-variable two-parameter Bessel function given by

$$J_n(x,y,z;\tau_1,\tau_2) = \sum_{l=-\infty}^{\infty} \tau_2^l \, J_{n-3l}(x,y;\tau_1) \, J_l(z)$$

with the generating function

$$\sum_{n=-\infty}^{\infty} t^n \, J_n(x,y,z;\tau_1,\tau_2) = \exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right) + \frac{y}{2}\left(t^2 \, \tau_1 - \frac{1}{t^2 \, \tau_1}\right) + \frac{z}{2}\left(t^3 \, \tau_2 - \frac{1}{t^3 \, \tau_2}\right)\right]$$

can be expressed in terms of the corresponding Tricomi–Bessel functions as follows:

$$J_n(x,y,z;\tau_1,\tau_2) = \left(\frac{x}{2}\right)^n \; C_n\left(\frac{x^2}{4}, \frac{y^2}{4}, \frac{z^2}{4}; \frac{2y}{x^2}\;\tau_1, \frac{4z}{x^3}\;\tau_2\right) \tag{13}$$

and (conversely)

$$C_n(x,y,z;\tau_1,\tau_2) = x^{-n/2}\; J_n\left(2\sqrt{x}, 2\sqrt{y}, 2\sqrt{z}; \frac{x}{\sqrt{y}}\;\tau_1, \tau_2\sqrt{\frac{x^3}{z}}\right)$$

so that we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\; J_{3n+l}(x,y,z;\tau_1,\tau_2) = J_l\left(x, y, \sqrt{z^2 - 2\tau_2\, tz}; \tau_1, \tau_2\sqrt{\frac{z}{z-2\tau_2\, t}}\right).$$

The extension of relation (13) to the case with $m$ $(> 3)$ variables is straightforward and we merely observe here that

$$J_n\left(\{x\}_1^m; \{\tau\}_1^{m-1}\right) = \left(\frac{x_1}{2}\right)^n \; C_n\left(\frac{1}{4}\{x^2\}_1^m; \{\sigma\}_2^m\right)$$

$$\Bigg(\{a\}_1^m = a_1,\dots,a_m; \quad \{a^2\}_1^m = a_1^2,\dots,a_m^2;$$

$$\{\sigma\}_2^m = \frac{2^{r-1}\,x_r}{x_1^r}\;\tau_{r-1} \quad (r = 2,\dots,m)\Bigg).$$

## 3. Generating Functions for Hermite–Bessel Functions

For the two-variable one-parameter Tricomi–Bessel function $C_n(x,y;\tau)$ defined by (9), it readily follows from (10) and (11) that

$$-\tau\;\frac{\partial}{\partial y}\;C_n(x,y;\tau) = \frac{\partial^2}{\partial x^2}\;C_n(x,y;\tau).$$

We also find from (10) and the generating function (12) that

$$C_n(x, y-\tau; \tau) = \exp\left(\frac{\partial^2}{\partial x^2}\right) C_n(x,y;\tau).$$

The above observations are particularly interesting because they allow a conceptual step forward. With this point in view, we first recall that (*cf.* [2])

$$\exp\left(a\;\frac{\partial^2}{\partial x^2}\right) x^n = H_n(x,a) := n! \sum_{r=0}^{[n/2]} \frac{a^r\; x^{n-2r}}{r!\,(n-2r)!}, \tag{14}$$

where $H_n(x,y)$ denotes the Hermite–Kampé de Fériet polynomials in two variables, given also by

$$H_n(x,y) = \left(i\sqrt{x}\right)^n \; H_n\left(\frac{y}{2i\sqrt{x}}\right) = g_n^2(x,y) \qquad \left(i := \sqrt{-1}\right) \tag{15}$$

in terms of the familiar Hermite polynomials $H_n(x)$ and the Gould–Hopper polynomials $g_n^m(x,y)$ with $m = 2$ (cf. [7, p. 76, equation 1.9 (6)]). Next, in the light of the operational representation in (14), we introduce the notion of *H-based* functions as follows:

$$ {}_{H}f(x,a) := \exp\left(a\,\frac{\partial^2}{\partial x^2}\right) f(x) = \sum_{n=0}^{\infty} c_n\; H_n(x,a) $$

$$ \left( f(x) := \sum_{n=0}^{\infty} c_n\; x^n \right), $$

where we have simply replaced the ordinary monomial $x^n$ in the Taylor–Maclaurin expansion of $f(x)$ by the polynomials $H_n(x,a)$ defined by (14) and, just as we have shown above in (15), related closely to the relatively more familiar Hermite polynomials $H_n(x)$ *and* the Gould–Hopper polynomials $g_n^m(x,y)$ (with $m=2$).

Within the above *H-based* framework, we note that the function ${}_{H}C_n(x,y)$ defined by

$$ \exp\left(a\,\frac{\partial^2}{\partial x^2}\right) C_n(x) = {}_{H}C_n(x,a) := \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!}\; H_r(x,a) $$

also satisfies the following generating function:

$$ \sum_{n=-\infty}^{\infty} t^n \cdot {}_{H}C_n(x,a) = \exp\left(t - \frac{x}{t} + \frac{a}{t^2}\right). $$

Clearly, therefore we have

$$ C_n(x, y-\tau; \tau) = {}_{H}C_n(x,y;\tau|1,0), $$

where

$$ {}_{H}C_n(x,y;\tau|a,b) := \sum_{l=-\infty}^{\infty} \tau^l \cdot {}_{H}C_{n-2l}(x,a) \cdot {}_{H}C_l(y,b). $$

The results, which we have just obtained, open new possibilities. Indeed, by combining equations (10) and (11), we find that

$$ \sum_{n=0}^{\infty} \frac{t^n}{n!}\; C_{3n+l}(x,y;\tau) = \exp\left(t\tau\,\frac{\partial^2}{\partial x\,\partial y}\right) C_l(x,y;\tau). $$

The action of the exponential operator containing a mixed derivative on a function of the variables $x$ and $y$ can again be obtained by using the concept of *H-based* functions. We note that if

$$ f(x,y) = \sum_{m,n=0}^{\infty} c_{m,n}\; x^m\; y^n, $$

then

$$ \exp\left(\alpha\,\frac{\partial^2}{\partial x \partial y}\right) f(x,y) = {}_{h}f(x,y;\alpha) := \sum_{m,n=0}^{\infty} c_{m,n}\; h_{m,n}(x,y;\alpha), $$

where

$$h_{m,n}(x,y;\alpha) := m!\ n! \sum_{r=0}^{\min(m,n)} \frac{\alpha^r\ x^{m-r}\ y^{n-r}}{r!\,(m-r)!\,(n-r)!}$$

denotes the *incomplete* two-variable Hermite polynomials considered in (for example) [3] and [9]. We thus find that

$$\exp\left(t\tau\ \frac{\partial^2}{\partial x \partial y}\right) C_l(x,y;\tau) = {}_hC_l(x,y;\tau|t\tau),$$

$${}_hC_l(x,y;\tau|t\tau) = \sum_{r=-\infty}^{\infty} \tau^r \cdot {}_hC_{l-2r,r}(x,y;t\tau),$$

and

$${}_hC_{m,n}(x,y;\tau) := \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s}}{r!\ s!\,(r+m)!\,(s+n)!}\ h_{r,s}(x,y;\tau).$$

*Remark* 3. Various interesting properties of the above families of functions and polynomials were investigated by (among others) Dattoli [2]. Furthermore, it is fairly obvious that the above-detailed considerations, valid for Tricomi–Bessel functions, can be extended without any significant problem to cylindrical Bessel functions as well.

## 4. A Family of Mixed Generating Functions

In this section, we discuss the possibility of obtaining generating functions of the form

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\ \mathcal{L}_n(x,y)\, C_{n+l}(z),$$

which incidentally is a *hybrid* generating function involving the product of Bessel-like functions and Laguerre-like polynomials $\mathcal{L}_n(x,y)$ defined by (cf. [2])

$$\mathcal{L}_n(x,y) := n! \sum_{r=0}^{n} \frac{(-1)^r\ x^r\ y^{n-r}}{(r!)^2\,(n-r)!} = y^n\ L_n\left(\frac{x}{y}\right), \tag{16}$$

where $L_n(x)$ denotes the *ordinary* Laguerre polynomial of degree $n$ in $x$ (see [1] and [7] for details).

The polynomials $\mathcal{L}_n(x,y)$ defined by (16) are also given by means of the following operational rule (*cf.* [2]):

$$\mathcal{L}_n(x,y) = \left(y - \widehat{\mathcal{D}}_x^{-1}\right)^n (1),$$

where, for convenience,

$$\widehat{\mathcal{D}}_x^{-n}(1) = \frac{x^n}{n!} \qquad (n \in \mathbb{N}_0).$$

It is easily observed that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\ \mathcal{L}_n(x,y)\, C_{n+l}(x) = C_l\left(z - t\left(y - \widehat{\mathcal{D}}_x^{-1}\right)\right)(1). \tag{17}$$

The families of Hermite–Bessel and Laguerre–Bessel functions have recently been investigated rather systematically (see [2] and the references cited therein) as a useful tool for dealing with the solution of partial differential equations associated with some electromagnetic transport problems. In the present case, the Laguerre–Tricomi functions are defined by

$$ {}_{\mathcal{L}}C_n(x,y) := \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!}\ \mathcal{L}_r(x,y), \tag{18} $$

which, in conjunction with (6) and (17), yields

$$ \sum_{n=0}^{\infty} \frac{t^n}{n!}\ \mathcal{L}_n(x,y)\, C_{n+l}(z) = {}_{\mathcal{L}}C_l(-xt, z-yt) $$

or, equivalently,

$$ \sum_{n=0}^{\infty} \frac{t^n}{n!}\ \mathcal{L}_n(x,y)\, J_{n+l}(z) = {}_{\mathcal{L}}C_l\left(-\frac{xzt}{2}, \frac{z^2-2yzt}{4}\right) \tag{19} $$

in terms of the Laguerre–Tricomi function ${}_{\mathcal{L}}C_n(x,y)$ defined by (18). We thus complete the proof of Proposition 2 below.

**Proposition 2.** *The bilateral generating function* (19) *holds true for the Laguerre-like polynomials* $\mathcal{L}_n(x,y)$ *defined by* (16).

Finally, we consider the possibility of deriving generating functions of the form

$$ \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2}\ C_{n+l}(z), $$

which, in view of (6) and (8), assumes the following operational form:

$$ \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2}\ C_{n+l}(z) = C_0\left(t\ \frac{\partial}{\partial z}\right) C_l(z). \tag{20} $$

Since (cf. [2])

$$ C_0\left(t\ \frac{\partial}{\partial x}\right) x^n = \mathcal{L}_n(t,x), $$

we find from (20) that

$$ \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2}\ C_{n+l}(z) = {}_{\mathcal{L}}C_l(t,z), \tag{21} $$

where the Laguerre–Tricomi function ${}_{\mathcal{L}}C_n(x,y)$ is given by (18). Thus we have proved Proposition 3 below.

**Proposition 3.** *In terms of the Laguerre–Tricomi function* ${}_{\mathcal{L}}C_n(x,y)$ *defined by* (18), *the generating function* (21) *holds true for the Tricomi–Bessel function* $C_n(x)$ *defined by* (6).

In this paper we have shown how operational methods may play a significant rôle in the derivation of generating functions involving Bessel-type functions. In a forthcoming investigation, we will apply the results presented here to problems of physical nature.

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