Georgian Mathematical Journal
Volume 11 (2004), Number 2, 283–299

KAROUBI–VILLAMAYOR K-THEORY, WEAKLY STABLE C^*-CATEGOROIDS, AND KK-THEORY

T. KANDELAKI

Abstract. The aim of the paper is to give, according to Karoubi–Villamayor K-groups, an interpretation of Kasparov KK-groups. It continues the study of KK-theory by the methods of K-theory, focusing attention on the problem posed and discussed in the author's papers published in 2000 and 2001. But the methods used in those papers are based on the excision property and Morita invariance of algebraic and topological K-theories on the category of C^*-algebras, which are not applicable to Karoubi–Villamayor K-theory, since excision holds only for some sub class of short exact sequences of C^*-algebras. In this paper we introduce and study a weak stability property of the C^*-category $\mathrm{Rep}(A, B)$, which is the key to our problem.

2000 Mathematics Subject Classification: 19K35, 19D25, 46L80,46L99.
Key words and phrases: Karoubi–Villamayor K-theory of C^*-categories, Kasparov KK-theory.

Introduction

An interpretation of Kasparov KK-theory is an interesting and useful problem. There are various points of view and directions in the study of the theory. Such interpretations open new application areas.

This article continues the study of KK-theory by the methods of K-theory, focusing attention on the problem posed and studied in the author's article [5]. The study of the additive C^*-category $\mathrm{Rep}(A, B)$ by the methods of K-theory and homological algebra seems to be a interesting problem. In particular, the following result was announced in [5] (see also [8]):

(1) Quillen's algebraic K-groups of $\mathrm{Rep}(A, B)$ are isomorphic, up to a shift of dimension, to Kasparov $KK_n(A, B)$-groups;

The proof of this result is based on the excision property and Morita invariance of the algebraic K-theory. It is not applicable to Karoubi–Villamayor K-theory, since excision holds only for some sub-class of short exact sequences.

In this article we prove the analogue of the main result of [5] for Karoubi–Villamayor K-theory. (Note that the method of the present paper is applicable to the case pointed out above. However, the proof of this interesting fact is omitted here.)

In more detail the contents of this paper are the following.

In Section 1 we recall the definition and properties of C^*-categoroids (see [7]).

In Section 2 we study the weak continuity property of Karoubi–Villamayor K-groups of additive C^*-categoroids. Let A be an additive C^*-categoroid. There

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

is an inductive system of abelian groups $\{KV_n(\mathcal{L}(a))\}$ where $a \in \mathrm{Ob}A$ and $\mathcal{L}(a) = \hom(a,a)$ and a natural isomorphism

$$\varinjlim_a KV_n(\mathcal{L}(a)) \simeq KV_n(A).$$

In Section 3 we define Karoubi–Villamayor K-functors on the category of C^*-categoroids and $*$-functoroids. By the weak continuity property we prove that the functors KV_n and $KV_n(-\otimes\mathcal{K})$ are naturally isomorphic on the category of weakly stable additive C^*-categoroids and additive $*$-functoroids. In the concluding section, we recall the definition of the additive C^*-category $\mathrm{Rep}(A,B)$ and show that Karoubi–Villamayor K-groups are naturally isomorphic to topological K-groups on the category of weakly stable additive C^*-categoroids. Since the additive C^*-category $\mathrm{Rep}(A,B)$ is a weakly stable C^*-category, one has a natural isomorphism

$$KV_n(\mathrm{Rep}(A,B)) \simeq K^t(\mathrm{Rep}(A,B)).$$

Now, according to the main result of [5], one has a natural isomorphism

$$KV_n(\mathrm{Rep}(A,B)) \simeq KK_{n-1}(A,B).$$

1. C^*-Categories and C^*-Categoroids

In this section, we discuss some elementary properties of C^*-categories and C^* -categoroids, and also a natural categorical generalization of unital C^* -algebras and C^*-algebras. We give the basic definitions, constructions and properties without proofs, but for details we refer the reader to [7] (cf. [1], [14]).

Recall that *a diagram scheme* D consists of a class of objects $\mathrm{Ob}D$ and a set $\hom(a,b)$ for any $a,b \in \mathrm{Ob}D$. By a $\mathbb{C}$-*scheme* we mean a diagram scheme D such that $\hom(a,b)$ has the structure of a $\mathbb{C}$-linear space, where $\mathbb{C}$ is the field of complex numbers. D is called an *involutive* $\mathbb{C}$-scheme if:

(a) an anti-linear map $* : \hom(a,b) \to \hom(b,a)$ is given for each $a,b \in \mathrm{Ob}D$.

(b) the bilinear composition law

$$\hom(a,b) \times \hom(b,a) \to \hom(a,a),$$
$$\hom(a,b) \times \hom(b,b) \to \hom(a,b),$$
$$\hom(a,a) \times \hom(a,b) \to \hom(a,b)$$

is defined and it is associative for any $a,b \in \mathrm{Ob}D$.

(c) $(f^*)^* = f$, and $(fg)^* = g^*f^*$ if the composition fg exists.

By a C^*-*scheme* is meant an involutive $\mathbb{C}$-scheme D such that:

(1) $\hom(a,b)$ is a Banach space;

(2) involution is an isometry;

(3) $\|f\|^2 = \|f^*f\|$ for any $f \in \hom(a,b)$;

(4) the morphism f^*f is a positive element in the C^*-algebra $\hom(a,a)$ for any $f \in \hom(a,b)$ and $a,b \in ObD$.

A diagram scheme $\mathcal{D}$ is called *a categoroid* if it satisfies all the axioms of a category except the existence of the identities of objects. Let a and b be objects from $\mathcal{D}$. Then $\hom(a,b)$ denotes a set of morphisms from a to b. The definition of morphisms between categoroids is analogous to that of a functor, and is called *a functoroid.* If $\mathcal{F} : D \to D'$ is a functoroid of categoroids and there exists the composition of morphisms f, g in $\mathcal{D}$, then $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$.

A categoroid A is called a C^**-categoroid* if it has the structure of a C^*-scheme such that

(1) the composition of morphisms is bilinear and $\|fg\| \leq \|f\| \cdot \|g\|$;

(2) if A is both a category and a C^*-categoroid, then it is called a C^*-category.

Remark. The underlying categoroid of a C^*-categoroid" is assumed to be small unless otherwise specified.

Examples.

(1) The category with Hilbert spaces as objects and all bounded linear maps as morphisms is a large C^*-category.

(2) Let A be a C^*-algebra. The category $\mathcal{H}(A)$ with countably generated right A-modules as objects and all bounded A-linear maps which have adjoints as morphisms is a C^*-category.

(3) A C^*-algebra is a C^*-categoroid with one object $\diamond$ and elements of the C^*-algebra as morphisms.

(4) The category of countably generated Hilbert right B-modules as objects and compact B-linear maps as morphisms has the structure of a C^*-categoroid.

Let A and B be C^*-categoroids. A $*$-functoroid $\mathcal{F} : A \to B$ is given if $\mathcal{F}$ maps the objects and morphisms of A into the objects and morphisms of B, respectively, so that:

a) $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$;

b) $\mathcal{F}(f + g) = \mathcal{F}(f) + \mathcal{F}(g)$;

c) $\mathcal{F}(\lambda f) = \lambda\mathcal{F}(f)$;

d) $\mathcal{F}(f^*) = \mathcal{F}(f)^*$ when the left side is defined.

If A and B are categories and $\mathcal{F}(1_a) = 1_{\mathcal{F}(1_a)}$ for any $a \in \mathrm{Ob}A$, then $\mathcal{F}$ is called a $*$-functor. We say that a $*$-functoroid between C^*-categoroids is faithful if canonical maps between objects and between morphisms are injections.

Any $*$-functoroid is norm-decreasing. Moreover, a faithful $*$-functoroid is norm preserving.

Let A be a C^*-categoroid and $I \subset \mathrm{Mor}A$. Let $\hom_I(a, b) = \hom(a, b) \cap I$. Then I is called a left ideal if $\hom_I(a, b)$ is a linear subspace of $\hom(a, b)$ and $f \in \hom_I(a, b), g \in \hom(b, c)$ implies $gf \in \hom_I(a, c)$. A right ideal is defined similarly. I is a two-sided ideal if it is both a left and a right ideal. An ideal I is closed if $\hom_I(a, b)$ is closed in $\hom(a, b)$ for each pair of objects. I determines an equivalence relation on the morphisms of $A : f \sim g$, if $f - g \in I$. If $I = I^*$ is an ideal of A, the set of equivalence classes A/I can be made into a $*$-categoroid in a unique way.

Arguing as for C^*-algebras, one can show that if A is a C^*-categoroid and I a closed two-sided ideal of A, then $I = I^*$ and A/I is a C^*-categoroid.

Let A be a C^*-categoroid. A C^*-category B is called a *categorization* of A if A is contained in B as an ideal.

There exists *a smallest categorization* A^+ of A. This C^*-category has the same objects as A, while $\hom_{A^+}(a,a)$ is $\hom_A(a,a)^+$ if the C^*-algebra $\hom(a,a)$ is non-unital and $\hom_A(a,a)$ otherwise.

Any $*$-functoroid $\mathcal{F}$ from A into the large C^*-category of Hilbert spaces and bounded linear maps is said to be *a representation*. If $\mathcal{F}$ is faithful, then it is called *a faithful representation*.

Let A be a C^*-categoroid. Then there exists a faithful representation.

A C^*-category $M(A)$ is said to be *the multiplier C^*-category* of A if A is a closed two-sided ideal in $M(A)$ and has the following universal property: let D be a C^*-categoroid containing A as a closed two-sided ideal; then there exists a unique $*$-functoroid $d: D \to M(A)$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\subset"] \arrow[d, "\|"'] & D \arrow[d, "d"] \\ A \arrow[r, "\subset"] & M(A) \end{tikzcd}$$

is commutative. There exists the multiplier C^*-category for any C^*-categoroid. This is the largest categorization of a C^*-categoroid.

A C^*-categoroid A is said to be an *additive C^*-categoroid* if there exists an additive C^*-category containing A as a closed two-sided ideal. Of course, in this situation the multiplier C^*-category must be an additive C^*-category. A functoroid $f: A \to A'$ is said to be additive if it is the restriction of some additive functor between additive C^*- categories containing the relevant categoroids as ideals.

Let A be a C^*-categoroid. Then there exists a unique additive C^*-categoroid $\mathcal{F}(A)$ satisfying the following conditions:

(1) A is a full sub-categoroid of $\mathcal{F}(A)$;

(2) any object in $\mathcal{F}(A)$ is a finite sequence of objects in A;

(3) any morphism from $a = (a_1, \dots, a_n)$ to $a' = a'_1, \dots, a'_m$ is an $n \times m$-matrix of the form (α_{ij}), where $\alpha_{ij} : a_i \to a'_j$ is a morphism in A;

(4) a composition is the product of matrices.

The structure of a C^*-category is defined as follows. Let $\phi : A \to \mathcal{H}(\mathbb{C})$ be a faithful representation in the large additive C^*-category of Hilbert spaces. Then it has an extension to an additive faithful representation $\mathcal{F}(\phi) : \mathcal{F}(A) \to \mathcal{H}(\mathbb{C})$. Thus one has an induced C^*-norm on $\mathcal{F}(A)$.

Let A be a C^*-category. The pseudo-abelian C^*-category of the additive C^*-category $\mathcal{F}(A)$ is denoted by $\mathcal{P}(A)$.

In the sequel, we will need the notion of the C^*-tensor product of a C^*-categoroid on a C^*-algebra $\mathcal{K}$ of compact operators on a countably generated Hilbert space:

- Let A be a C^*-categoroid. Let $A \odot \mathcal{K}$ be an involutive categoroid with the set of objects that is equal to the set objects of A and

$$\hom(a,a')_{A\odot\mathcal{K}} = \hom(a,a') \odot \mathcal{K},$$

where "$\odot$" is the algebraic tensor product over the given field. Since there exists a faithful embedding of A in the category of Hilbert spaces, there also exists the standard faithful embedding of $A \odot \mathcal{K}$ in the category of Hilbert spaces. Consider the category $A \otimes \mathcal{K}$ with $\mathrm{Ob}(A \otimes \mathcal{K}) = \mathrm{Ob}(A \odot \mathcal{K})$, and $\hom(a, a')_{A \otimes \mathcal{K}}$ is the completion of $\hom(a, a')_{A \odot \mathcal{K}}$ with respect to the induced C^*-norm from the category of Hilbert spaces. The constructed C^*-category is said to be the C^*-tensor product of A and $\mathcal{K}$.

Lemma 1.1. *Let A be an additive C^*-categoroid. Then $A \otimes \mathcal{K}$ is an additive C^*-categoroid too.*

Proof. Let A' be an additive C^*-category containing A as an ideal. Then it is easy to check that $A' \otimes \mathcal{K}^+$ is an additive C^*-category, too. One has an exact sequence of C^*-algebras

$$0 \to \hom(a, a)_{A \otimes \mathcal{K}} \otimes \mathcal{K}^+ \to \hom(a, a)_{A' \otimes \mathcal{K}^+} \otimes \mathcal{K}^+.$$

This implies, according to the definition of a C^*-categoroid, that $A \otimes \mathcal{K}$ is an ideal in the additive C^*-category $A' \otimes \mathcal{K}^+$. This means that $A \otimes \mathcal{K}$ is an additive C^*-categoroid. $\square$

2. Some Properties of Karoubi–Villamayor K-Theory

In this section we want to establish a property of Karoubi–Villamayor K-functors KV_n, $n \geq 1$, which we call *weak continuity*. This property will play one of the major role in the proof of our main result.

Remark. The groups K^{-n}, $n \geq 1$, from [11] are denoted here by KV_n.

Let A be an additive C^*-category. Then one has a sequence of additive categories

(1) $A^{(1)} = A$;
(2) $A^{(n)} = A[x_1, \dots, x_{n-1}]$, if $n > 1$.

The objects of the category $A[x_1, \dots, x_n]$ are the objects of A and a morphism from a into b is a formal polynomial

$$\sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where $a_{i_1, \dots, i_n} \in \hom(a, b)$. A composition is like to the product of polynomials.

Any additive functor $\varphi : A \to B$ may be extended to a functor $\varphi^{(n)} : A^{(n)} \to B^{(n)}$ which is given on morphisms by the map

$$\sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} \mapsto \sum_{i_1, \dots, i_n} \varphi(a_{i_1, \dots, i_n}) x_1^{i_1} \cdots x_n^{i_n}.$$

Let $\mathcal{E}^{(n)}(\varphi)$ be a set of pairs (a, α), where $a \in \mathrm{Ob}A^{(n)} = \mathrm{Ob}A$ and α is an automorphism of a in $A^{(n)}$ such that $\varphi^{(n)}(\alpha) = 1_{\varphi(a)}$; moreover, if $n > 1$ and one of x_i is 0 or 1, then $\alpha = \alpha(x_1, \dots, x_i, \dots, x_{n-1}) = 1$.

Let $\mathcal{H}^{(n)}(\varphi)$ be a set of pairs of the form (a,h), where $h = h[x_1,\dots,x_{n-1},t]$ such that

(1) $h[x_1,\dots,x_i,\dots,x_{n-1},t] = 1_a^A$, if $x_i = 1$ or 0.
(2) $\varphi^{(n)}(h[x_1,\dots,x_i,\dots,x_{n-1},t]) = 1_{\varphi(a)}$;

Pairs (a,α_0) and (a,α_1) are homotopic if there exists a pair $(a,h) \in \mathcal{H}^{(n)}(\varphi)$ such that

- $h(x_1,\dots,x_{n-1},0) = \alpha_0$ and $h(x_1,\dots,x_{(n-1)},1) = \alpha_1$.

Pairs (a,α) and (b,β) are said to be equivalent if there exist a pair $(d,1_d)$ such that

$$\alpha \oplus 1_b \oplus 1_d \ \text{ is homotopic to } \ 1_a \oplus \beta \oplus 1_d.$$

By definition $KV_n(\varphi)$ is the set of classes of equivalent pairs in $\mathcal{E}^n(\varphi)$. The latter is an abelian group with respect to the sum

$$(a,\alpha) + (b,\beta) = (a \oplus b, \alpha \oplus \beta)$$

(cf. [11]).

Proposition 2.1. *Let I be an ideal in the additive categories A and B. Let $\Gamma : A \to B$ be an additive functor which is the identity on I. Let $\varphi_I^A : A \to A/I$ and $\varphi_I^B : B \to B/I$ be natural additive functors. Then Γ induces an isomorphism $K_n(\varphi_I^A) \approx K_n(\varphi_I^B)$, $n \geq 1$.*

Proof. Indeed, let $(a, \alpha[x_1,\dots,x_{n-1}]) \in \mathcal{E}^{(n)}(\varphi_I^A)$ and 1_a^A be the identity morphism of an object a in A. Then $1_a^A - \alpha[x_1,\dots,x_{n-1}] \in I[x_1,\dots,x_{n-1}]$. Let 1_a^B be the identity morphism of a in B. It is clear that $\Gamma^{(n)}(\alpha) = 1_a^B + (1_a^A - \alpha[x_1,\dots,x_{n-1}])$. Conversely, if $(a,\beta[x_1,\dots,x_{n-1}]) \in \mathcal{E}^{(n)}(\varphi_I^B)$, then $\alpha = 1_a^A + (1_a^B - \beta[x_1,\dots,x_{n-1}]) \in \mathcal{E}^{(n)}(\varphi_I^A)$ such that $\Gamma^{(n)}(\alpha) = \beta$. Thus $\Gamma^{(n)}$ induces the bijection

$$\mathcal{E}^{(n)}(\varphi_I^A) \approx \mathcal{E}^{(n)}(\varphi_I^B).$$

Consider the induced map $\Gamma_{\mathcal{H}}^n : \mathcal{H}^{(n)}(\varphi_I^A) \to \mathcal{H}^{(n)}(\varphi_I^B)$ defined by the equation

$$\Gamma_{\mathcal{H}}^n(a,h) = (a, \Gamma^{(n+1)}(h)).$$

Let us show that $\Gamma_{\mathcal{H}}^n$ is a bijection. Indeed,

$$h - 1_a^{A[x_1,\dots,x_{n-1},t]} \in I[x_1,\dots,x_{n-1},t] \ \text{ and }$$
$$\Gamma^{(n+1)}(h) = 1_a^{B[x_1,\dots,x_{n-1},t]} + (h[x_1,\dots,x_{n-1},t] - 1_a^{A[x_1,\dots,x_{n-1},t]}).$$

This equation shows that $\Gamma_{\mathcal{H}}^n$ is an injection. $\Gamma_{\mathcal{H}}^n$ is a surjection, too. Indeed, if $(a,l) \in \mathcal{H}^{(n)}(\varphi_I^B)$, then

$$l[x_1,\dots,x_{n-1},t] - 1_a^{B[x_1,\dots,x_{n-1},t]} \in I[x_1,\dots,x_{n-1},t]$$

and

$$h[x_1,\dots,x_{n-1},t] = 1_a^{A[x_1,\dots,x_{n-1},t]} + (l[x_1,\dots,x_{n-1},t] - 1_a^{B[x_1,\dots,x_{n-1},t]}) \in \mathcal{H}^{(n)}(\varphi_I^A).$$

One has $\Gamma_{\mathcal{H}}^n(h[x_1,\dots,x_{n-1},t]) = l[x_1,\dots,x_{n-1},t]$. Now, according to the definition of the group $K_n(\varphi)$, it is clear that $K_n(\varphi_I^A) \approx K_n(\varphi_I^B)$, $n \geq 1$. □

Recall that a morphism $s : a \to b$ in A is said to be an *isometry* if $s^*s = 1_a$. It is clear that $p_s = ss^*$ is a projection.

Let $s : a \to b$ be an isometry and α be an automorphism of a; then $ad_s(\alpha) = s\alpha s^* + (1_b - p_s)$ is an automorphism of b in $A^{(n)}$.

Lemma 2.2. *Let $s : a \to b$ be an isometry. Then (a, α) and $(b, ad_s(\alpha))$ are equivalent pairs.*

Proof. Remark that

(1) $(a, \alpha) \sim (a \oplus b \oplus b, \alpha \oplus 1_b \oplus 1_b)$.
(2) $(a, ad_s(\alpha)) \sim (a \oplus b \oplus b, 1_a \oplus ad_s(\alpha) \oplus 1_b)$.

On the other hand, there exists an isomorphism of the pair $(a \oplus b \oplus b, \alpha \oplus 1_b \oplus 1_b)$ to the pair $(a \oplus b \oplus b, 1_a \oplus ad_s(\alpha) \oplus 1_b)$ induced by a unitary isomorphism $u : a \oplus b \oplus b \to a \oplus b \oplus b$, where

$$u = \begin{pmatrix} 0 & 0 & s^* \\ s & (1_b - p_s) & 0 \\ 0 & p_s & (1_b - p_s) \end{pmatrix}; \quad u^{-1} = \begin{pmatrix} 0 & s^* & 0 \\ 0 & (1_b - p_s) & p_s \\ s & 0 & (1_b - p_s) \end{pmatrix}.$$

But, by Lemma 6.1 in [11], the latter pairs in (1) and (2) are equivalent. Thus (a, α) and $(b, ad_s(\alpha))$ are equivalent pairs, too. □

Definition 2.3. Pairs (a, α) and (b, β) are said to be ι-equivalent if there exist isometries $s : a \to d$ and $s' : b \to d$ such that the pairs $(d, ad_s(\alpha))$ and $(d, ad_{s'}(\beta))$ are homotopic.

Proposition 2.4. *Pairs (a, α) and (b, β) are equivalent if and only if they are ι-equivalent.*

Proof. Let (a, α) and (b, β) be equivalent pairs. Consider the isometries $s : a \to a \oplus b \oplus d$ and $s' : b \to a \oplus b \oplus d$ defined by matrices

$$\begin{pmatrix} 1_a \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1_b \\ 0 \end{pmatrix},$$

respectively. Of course, $ad_s(\alpha) = \alpha \oplus 1_b \oplus 1_d$ and $ad_{s'}(\beta) = 1_a \oplus \beta \oplus 1_d$. One has the pairs

$$(a \oplus b \oplus d, \alpha \oplus 1_b \oplus 1_d) \text{ and } (a \oplus b \oplus d, 1_a \oplus \beta \oplus 1_d).$$

By assumption, there exists d such that the latter pairs are homotopic. Thus (a, α) and (b, β) be ι-equivalent pairs. Conversely, let (a, α) and (b, β) are ι-equivalent. By assumption, there exist isometries $s : a \to d$ and $s' : b \to d$ such that the pairs $(d, ad_s(\alpha))$ and $(d, ad_{s'}(\beta))$ are homotopic. But, by Lemma 2.2, $(a, \alpha) \sim (d, ad_s(\alpha))$ and $(b, \beta) \sim (d, ad_{s'}(\beta))$. Therefore (a, α) and (b, β) are equivalent pairs, too. □

Let A' be a full additive subcategory of an additive category A. It is said to be *cofinal* if for any object a in A there exists an isometry $s : a \to b$, where b is an object in A'.

Proposition 2.5. *Let $\varphi : A \to B$ be an additive functor, A' be a cofinal additive subcategory of A and φ' be the restriction of φ to A'. Then $KV_n(\varphi') = KV_n(\varphi)$.*

Proof. Consider the natural homomorphism $KV_n(\varphi') \to KV_n(\varphi)$ induced by the map $(b, \beta) \mapsto (b, \beta)$. Sippose an element in $KV_n(\varphi)$ be represented by the pair (a, α). Consider an isometry $s : a \to b$, where b is an object in A'. Then the pair $(b, ad_s(\alpha))$ defines an element in $KV_n(\varphi')$. But by Lemma 2.2 the pairs (a, α) and $(b, ad_s\alpha)$ are equivalent. Thus, we have shown that the homomorphism is an epimorphism. Now, we have to show that the natural homomorphism is a monomorphism. Let $x = d(b, \beta)$ be an element in $KV_n(\varphi')$ such that the corresponding element in $KV_n(\varphi)$ is zero. Then by Proposition 2.4 there exists an isometry $s : b \to a$ such that $(a, ad_s(\alpha))$ is homotopic to $(a, 1_a)$. By the cofinality there exists an isometry $s' : a \to c$ where c is an object in A'. Note that from the fullness of A' in A it follows that the composition $s's$ is an isometry in A'. It is clear that the pair $(c, ad_{s's}(\beta))$ is homotopic to the pair $(c, 1_c)$. This means that $d(b, \alpha) = 0$ in $KV_n(\varphi')$. □

Let $\varphi : A \to B$ be as above. Consider an object a in A. Let A_a be a full subcategory of A which consists of all finite sums of the form $a^n = a \oplus \cdots \oplus a$ and the zero object. Let $\varphi_a : A_a \to B$ be the restriction of φ to A_a. Let $s : a \to b$ be an isometry. Then one has the isometries $s^n : a^n \to b^n$ for any natural number n, where $s^n = s \oplus \cdots \oplus s$ (n summands). There is a homomorphism

$$\tau^s_{ab} : KV_n(\varphi_a) \to KV_n(\varphi_b)$$

induced by the map $(a^n, \alpha) \mapsto (b_n, ad_{s^n}(\alpha))$. Then Lemma 2.2 implies that the diagram

$$\begin{array}{ccc} KV_n(\varphi_a) & \xrightarrow{\tau_s} & KV_n(\varphi_b) \\ {\scriptstyle \tau_a}\downarrow & & \downarrow{\scriptstyle \tau_b} \\ KV_n(\varphi) & = & KV_n(\varphi) \end{array}$$

commutes, where the vertical arrows are induced by the natural functors $A_a \subset A$ and $A_b \subset A$.

Let $t : a \to b$ be another isometry. According to Lemma 2.2, one concludes that $\tau_s = \tau_t$, since

$$(b_n, ad_s(\alpha)) \sim (b_n, ad_t(\alpha)) \tag{2.1}$$

in the category A_b.

Consider the ordering "$\leq$" on the set of objects in A:

- $a \leq b$ if and only if there exists an isometry $s : a \to b$.

Thus one has a well-defined direct system of abelian groups $\{KV_n(\varphi_a), \tau_{ab}\}$ over the directed system $(\mathrm{Ob}A; \leq)$ and a natural homomorphism

$$\tau : \varinjlim_a KV_n(\varphi_a) \to KV_n(\varphi). \tag{2.2}$$

Proposition 2.6 (Weak Continuity)**.** *Let A and B be additive categories and $\varphi : A \to B$ be an additive functor. Then the natural homomorphism* (2.2) *is an isomorphism.*

Proof. 1. *τ is an epimorphism.* Let (a, α) represent an element f in $KV_n(\varphi)$. This pair represents an element f' in $KV_n(\varphi_a)$ too. It is clear that $\tau_a(f') = f$;

2. *τ is a monomorphism.* Let (a, α) represent an element f in $KV_n(\varphi_a)$ such that the corresponding element in $KV_n(\varphi)$ is the zero element. By Proposition 2.4, there exists an isometry $s : a \to b$ such that $(b, ad_s(\alpha))$ is homotopic to $(b, 1_b)$. This implies that $\tau^s_{ab}(f) = 0$ in $KV_n(\varphi_b)$, and also in $\varinjlim_a KV_n(\varphi_a)$. □

3. Karoubi–Villamayor K-Groups and Weakly Stable C^*-Categoroids

In this section, we define Karoubi–Villamayor K-functors for a pair (A, I), where A is an additive C^*-category and I is a closed ideal. We shall show that this group is independent of the C^*-category A. Thus there exists a natural definition of the Karoubi–Villamayor K-groups on the category of additive C^*-categoroids and additive $*$-functoroids. Using this construction, we will extend this definition to the category of (not necessarily additive) C^*-categoroids. It will be shown that Karoubi–Villamayor K-groups and topological K-groups on the category of weak stable C^*-categoroids are naturally isomorphic (cf. [11], [4]).

Definition 3.1. Let A be an additive C^*-category and let I be an C^*-ideal. Let $\pi : A \to A/I$ be the natural additive functor. We define Karoubi–Villamayor KV-functors, for $n \geq 1$, by

$$KV_n(A, I) = KV_n(\pi), \quad n \geq 1. \tag{3.1}$$

Proposition 3.2. *Let I be an additive C^*-categoroid which is a C^*-ideal in an additive C^*-category A. Then*

$$KV_n(A, I) = KV_n(M(I), I), \;\; n \geq 1.$$

Proof. By the universality property of $M(A)$, there exists a natural additive functor $\omega : A \to M(I)$ which is the identity functor on I. Then Proposition 2.6 guaranties the assertion. □

From Proposition 3.2 it follows that if I is an additive C^*-category, then $KV_n(A, I)$ is naturally isomorphic to $KV_n(A', I)$, where A and A' are additive C^*-categories containing I as an ideal. Thus, if I is an additive C^*-categoroid, we can define

$$KV_n(I) = KV_n(A, I),$$

where A is an additive C^*-category containing I as an ideal. This group is independent of the choice of an additive C^*-category containing I as a closed ideal.

It is easy to check that if $\phi : I \to I'$ is an additive functoroid then there is the induced homomorphism $\phi_n : KV_n(I) \to KV_n(I')$.

Now, we are ready to extend the definition of Karoubi–Villamayor functors to the category of C^*-categories and $*$-functoroids.

Let A be a C^*-categoroid. We need two C^*-categories A^+ (see section one) and $\mathbb{C}_A$, satisfying the following properties: $Ob(A^+) = Ob(A)$ and $Ob(\mathbb{C}_A) = Ob(A)$; besides that

$$\hom_{A^+}(a, a') = \begin{cases} \hom_A(a, a') & \text{if } a \neq a', \\ \hom_A(a, a) & \text{if } a = a' \text{ and } \hom_A(a, a) \text{ is unital}, \\ \hom_A(a, a)^+ & \text{otherwise} \end{cases}$$

and

$$\hom_{\mathbb{C}_A}(a, a') = \begin{cases} 0 & \text{if } a \neq a', \\ 0 & \text{if } a = a' \text{ and } \hom_A(a, a) \text{ is unital}, \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

Let $\chi_A : A^+ \to \mathbb{C}_A$ be the natural $*$-functor which is the identity map on the set of objects and the induced $*$-homomorphism $\chi_a : \hom_{A^+}(a, a) \to \mathbb{C}$ is the natural projection, defined by the map $(a, \lambda) \mapsto \lambda$, $\lambda \in \mathbb{C}$. The functor χ_A induces the additive functor $\mathcal{P}(\chi_A)$:$\mathcal{P}(A^+) \to \mathcal{P}(\mathbb{C}_A)$.

Definition 3.3. Let A be a C^*-categoroid. By definition

$$\mathbb{KV}_n(A) = KV_n(\mathcal{P}(\chi_A)).$$

So extended Karoubi–Villamayor groups define functors from the category of C^*-categoroids and $*$-functoroids into the category abelian groups. Indeed, if $f : A \to B$ is a $*$-functoroid, then one has the naturally extended $*$-functoroid $f^+ : A^+ \to B^+$ and the additive $*$-functor $\mathcal{P}(f^+) : \mathcal{P}(A^+) \to \mathcal{P}(B^+)$. Thus one has the induced homomorphism

$$f_n : \mathbb{KV}_n(A) \to \mathbb{KV}_n(B).$$

The following lemma shows that above functors are a natural extension of Karoubi–Villamayor groups from the category of additive C^*-categories and additive $*$-functors to the category of C^*-categoroids and $*$-functoroids.

Lemma 3.4. *The functors KV_n and $\mathbb{KV}_n$ are isomorphic on the category of additive C^*-categories and additive $*$-functors.*

Proof. Let A be an additive C^*-category and ϑ be the trivial additive category containing only zero object. Let $\tau : A \to \vartheta$ be the trivial additive functor. By definition $KV_n(A) = KV_n(\vartheta)$. Since A is a C^*-category, $A^+ = A$ and $\mathbb{C}_A \simeq \vartheta$. Thus $\mathbb{KV}_n(A) = KV_n(\mathcal{P}(A))$. According to Proposition 2.5, it is easy to check that $KV_n(A) = KV_n(\mathrm{P}(A))$, where $\mathrm{P}(A)$ is the pseudo-abelian C^*-category of A. Since A is an additive C^*-category, the natural imbedding $A \hookrightarrow \mathcal{F}(A)$ is an equivalence of additive categories. Thus it induces equivalence of categories $\mathrm{P}(A)$ and $\mathcal{P}(A)$, and $KV_n(\mathrm{P}(A)) = KV_n(\mathcal{P}(A))$. Therefore, $KV_n(A) = \mathbb{KV}_n(A)$. □

Definition 3.5. A C^*-categoroid A is said to be directed if for any two objects a and a' in A there exists an object b and isometries $s : a \to b$ and $s' : a' \to b$ in an $M(A)$.

For example, any additive C^*-categoroid is a directed C^*-categoroid.

Let A be a directed C^*-categoroid. Then one can form, as in the case of additive C^*-categoroids, a direct system of abelian groups $\{\mathbb{KV}_n(\mathcal{L}(a)), \tau_{ab}\}$ over directed set $(\mathrm{Ob}A; \leq)$. Proposition 2.6 implies that the natural homomorphism

$$\tau : \varinjlim_{a} \mathbb{KV}_n(\mathcal{L}(a)) \to \mathbb{KV}_n(A) \tag{3.2}$$

is an isomorphism. The following property is said to be the *weak continuity* of functors $\mathbb{KV}_n$:

(1) Let A be a directed C^*-categoroid. Let $s_a : a \to a'$ be an isometry in a C^*-category A' containing A as a closed ideal. Let $ad(s) : \mathcal{L}(a) \to \mathcal{L}(a')$ be the $*$-homomorphism defined by the map $f \mapsto s_a f s_a^*$ and let $i_a : \mathcal{L}(a) \to A$ be the natural $*$-functoroid defined by the maps $a \mapsto a$ and $f \mapsto f$, $f \in \mathcal{L}(a)$. Then the diagram

$$\begin{tikzcd} \mathbb{KV}_n(\mathcal{L}(a)) \arrow[r, "(ad(s))_n"] \arrow[d, "(i_a)_n"'] & \mathbb{KV}_n(\mathcal{L}(a')) \arrow[d, "(i_{a'})_n"] \\ \mathbb{KV}_n(A) \arrow[r, equal] & \mathbb{KV}_n(A) \end{tikzcd}$$

commutes.

(2) Let A be a directed C^*-category. For any element $\alpha \in \mathbb{KV}_n(A)$, there exist an object $a \in A$ and an element $\alpha_a \in \mathbb{KV}_n(\mathcal{L}(a))$ such that $(i_a)_n(\alpha_a) = \alpha$.

Definition 3.6. Let A be a C^*-categoroid. A $*$-functoroid $G : A \to A$ is said to be inner if for any object $a \in A$ there exists an isometry $s_a : a \to G(a)$ in a C^*-category A', such that $G(f) = s_a f s_a^*$, $\forall f \in \mathcal{L}(a)$.

Proposition 3.7. *Let A be a directed C^*-categoroid and let G be an inner additive $*$-functoroid. Then*

$$G_n : \mathbb{KV}_n(A) \to \mathbb{KV}_n(A)$$

is the identity homomorphism.

Proof. According to the weak continuity properties (1) and (2), for any element $\alpha \in \mathbb{KV}_n(A)$ there exist an object $a \in A$ and an element $\alpha_a \in \mathbb{KV}_n(\mathcal{L}(a))$ such that $(i_a)_n(\alpha_a) = \alpha$, and for any isometry $s_a : a \to a'$ one has a commutative diagram

$$\begin{tikzcd} \mathbb{KV}_n(\mathcal{L}(a)) \arrow[r, "(ad(s_a))_n"] \arrow[d, "(i_a)_n"] & \mathbb{KV}_n(\mathcal{L}(G(a))) \arrow[d, "(i_{G(a)})_n"] \\ \mathbb{KV}_n(A) \arrow[r, equal] & \mathbb{KV}_n(A). \end{tikzcd}$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{L}(a) & \xrightarrow{G_a} & \mathcal{L}(G(a)) \\ {\scriptstyle i_a}\downarrow & & \downarrow{\scriptstyle i_{G(a)}} \\ A & \xrightarrow{G} & A \end{array}$$

where G_a is a $*$- homomorphism which is the restriction of G to the object a. Thus the following diagram

$$\begin{array}{ccc} \mathbb{KV}_n(\mathcal{L}(a)) & \xrightarrow{(G_a)_n} & \mathbb{KV}_n(\mathcal{L}(G(a))) \\ {\scriptstyle (i_a)_n}\downarrow & & \downarrow{\scriptstyle (i_{G(a)})_n} \\ \mathbb{KV}_n(A) & \xrightarrow{G_n} & \mathbb{KV}_n(A) \end{array}$$

commutes. Since G is inner, there exists an isometry $s_a : a \to G(a)$ such that $G_a(f) = ad(s_a)(f)$, $f \in \mathcal{L}(a)$ (definition 3.6) and the diagram

$$\begin{array}{ccc} \mathbb{KV}_n(\mathcal{L}(a)) & \xrightarrow{(ad(s_a))_n} & \mathbb{KV}_n(\mathcal{L}(G(a))) \\ {\scriptstyle (i_a)_n}\downarrow & & {\scriptstyle (i_{G(a)})_n}\downarrow \\ \mathbb{KV}_n(A) & \xrightarrow{G_n} & \mathbb{KV}_n(A) \end{array}$$

commutes. Comparing the first and the last commutative diagrams one concludes that G_n is the identity homomorphism. $\square$

Let $\mathcal{K}$ be the C^*-algebra of compact linear maps in $\mathcal{H}$. Consider $\mathcal{K}$ as a categoroid with one object. Let A be a directed C^* -categoroid. Then $A \otimes \mathcal{K}$ is a directed C^*-categoroid. Indeed, it is enough to remark that if s is an isometry in the C^*-category A', containing A as a closed ideal, then $s \otimes 1_{\mathcal{H}}$ is an isometry in $A \otimes \mathcal{L}(\mathcal{H})$ containing $A \otimes \mathcal{K}$ as a closed ideal. Define a $*$ -functoroid

$$e_A : A \to A \otimes \mathcal{K}$$

to be the identity on objects and defined on the morphisms by a map $f \mapsto f \otimes p$, where f is a morphism in A and p is a rank one projection in $\mathcal{K}$.

Definition 3.8. A directed C^*-categoroid A is said to be weak stable if there exists a $*$-functoroid $G_A : A \otimes \mathcal{K} \to A$ such that the composition $G_A \circ e_A$ is an inner functoroid.

Consider a functor

$$\mathbb{KV}_n^{\mathcal{K}} = \mathbb{KV}_n(- \otimes \mathcal{K})$$

on the category of C^*-categoroids. A simple check shows that Proposition 2.6 holds for $\mathbb{KV}_n^{\mathcal{K}}$, too. The functor $\mathbb{KV}_n^{\mathcal{K}}$ is said to be the *stabilization* of $\mathbb{KV}_n$. The following theorem is one of the main result in this paper.

Theorem 3.9. *The functor $\mathbb{KV}_n$ is isomorphic to $\mathbb{KV}_n^{\mathcal{K}}$, for any $n \geq 1$, on the full subcategory of weakly stable C^*-categoroids.*

Proof. 1. Note that if A is a weakly stable C^*-categoroid, then $A \otimes \mathcal{K}$ is a weakly stable C^*-categoroid, too. Indeed, consider the homomorphism

$$e_{A\otimes\mathcal{K}} : A \otimes \mathcal{K} \to A \otimes \mathcal{K} \otimes \mathcal{K}$$

given by the map $a \otimes k \mapsto a \otimes k \otimes p$, where p is a fixed rank one projection. Then the homomorphism

$$G_{A\otimes\mathcal{K}} : A \otimes \mathcal{K} \otimes \mathcal{K} \to A \otimes \mathcal{K}$$

is defined by the map $a \otimes k \otimes l \mapsto G_A(a \otimes l) \otimes k$. It is clear that

$$G_{A\otimes\mathcal{K}} \circ e_{A\otimes\mathcal{K}} = (G_A \circ e_A) \otimes id_{\mathcal{K}}$$

and the isometry $s_a \otimes id_K$ satisfies

$$ad(s_a \otimes id_K)(f \otimes k) = (G_{A\otimes\mathcal{K}} \circ e_{A\otimes\mathcal{K}})(f \otimes k).$$

2. Let $\rho : A\otimes\mathcal{K}\otimes\mathcal{K} \to A\otimes\mathcal{K}\otimes\mathcal{K}$ be an isomorphism of C^*-categoroids which is the identity on the objects and defined on the morphisms by the twisting map $a \otimes k \otimes l \mapsto a \otimes l \otimes k$. Then the diagram

$$\begin{tikzcd} A \otimes \mathcal{K} \otimes \mathcal{K} \arrow[r, "\rho"] \arrow[d, "G_A\otimes id_{\mathcal{K}}"'] & A \otimes \mathcal{K} \otimes \mathcal{K} \arrow[d, "G_{A\otimes\mathcal{K}}"] \\ A \otimes \mathcal{K} \arrow[r, equal] & A \otimes \mathcal{K} \end{tikzcd}$$

commutes.

3. The functoroid ρ is an inner $*$-functoroid. Let $\rho' : \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H})$ be a $*$-isomorphism defined by a map $f \otimes l \mapsto l \otimes f$. Consider a linear map $\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ defined by $x \otimes y \mapsto y \otimes x$. The map μ is a self-adjoint (bounded) unitary map. Indeed,

$$\langle \mu(x \otimes y); x' \otimes y' \rangle = \langle y; x' \rangle \cdot \langle x; y' \rangle = \langle x \otimes y; \mu(x' \otimes y') \rangle.$$

Let $f \otimes l : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ be a bounded linear map and $x, y \in \mathcal{H}$; then

$$\begin{aligned} Ad_\mu(f \otimes l)(x \otimes y) &= \mu(f \otimes l)\mu(x \otimes y) = \mu(f \otimes l)(y \otimes x) = \mu(f(y) \otimes l(x)) \\ &= l(x) \otimes f(y) = (l \otimes f)(x \otimes y) = \rho'(f \otimes l)(x \otimes y). \end{aligned}$$

Now, we are ready to show that the functoroid ρ is inner. Indeed, consider a unitary morphism $id_E \otimes \mu : E \to E$ in the C^*-category $M(A) \otimes \mathcal{L}(\mathcal{H})) \otimes \mathcal{L}(\mathcal{H}))$. Let $f \otimes (k \otimes l) \in \mathcal{L}(E)$. Then

$$(id_E \otimes \mu)(f \otimes (k \otimes l))(id_E \otimes \mu) = f \otimes (l \otimes k).$$

4. Let us return to the main aim. A simple check shows that the family of homomorphisms $\{(e_A)_n\}_A$ is a natural transformation from functor $\mathbb{KV}_n$ to the functor $\mathbb{KV}_n^{\mathcal{K}}$. Thus the diagram

$$\begin{tikzcd} \mathbb{KV}_n(A \otimes \mathcal{K}) \arrow[r, "(e_{A\otimes\mathcal{K}})_n"] \arrow[d, "(G_A)_n"'] & \mathbb{KV}_n^{\mathcal{K}}(A \otimes \mathcal{K}) \arrow[d, "(G_A)_n^{\mathcal{K}}"] \\ \mathbb{KV}_n(A) \arrow[r, "(e_A)_n"] & \mathbb{KV}_n^{\mathcal{K}}(A) \end{tikzcd}$$

commutes. Since ρ is an inner isomorphism, it follows that

$$(G_A)_n^{\mathcal{K}} \cdot (e_{A\otimes\mathcal{K}})_n = (G_A \otimes id_{\mathcal{K}})_n \cdot (e_{A\otimes\mathcal{K}})_n$$
$$= (G_{A\otimes\mathcal{K}})_n \cdot (\rho)_n \cdot (e_{A\otimes\mathcal{K}})_n = (G_{A\otimes\mathcal{K}})_n \cdot (e_{A\otimes\mathcal{K}})_n$$

is the identity. Thus $(e_A)_n$ is an epimorphism. According to the weakly stable property, one has $G_n \circ (e_A)_n = id_{\mathbb{KV}_n(A)}$. This implies that $(e_A)_n$ is a monomorphism. Thus $(e_A)_n$ is an isomorphism. □

It is a well-known fact that the stabilizations of Karoubi–Villamayor K-functors are naturally isomorphic to the topological K-functors K_n^t on the category of C^*-algebras [3]. Similar to Karoubi–Villamayor K-groups, one can establish the weak continuity for topological K-theory. Therefire topological K-functors and stabilizations of Karoubi–Villamayor K-functors are naturally isomorphic. So, in Theorem 3.9, the functors $\mathbb{KV}_n^{\mathcal{K}}$ may be replaced by the functors of topological K-theory. Thus Theorem 3.9 may be formulated in the following form.

Corollary 3.10. *The functor $\mathbb{KV}_n$ is isomorphic to $\mathbb{K}_n^t$, for any $n \geq 1$, on the full subcategory of weakly stable C^*-categoroids.*

4. Karoubi–Villamayor K-Theory and KK-Theory

In this section, we recall the definition of the C^*-category $\mathrm{Rep}(A, B)$, where A is and involutive algebra and B is a σ-unital C^*-algebra (see also [15], where it is denoted by $D(A, B)$). We will show that it is a weakly stable C^*-category and according to the results of Section 3 and the main result of [5], calculate Kasparov KK-groups as the Karoubi–Villamayor K-groups of $\mathrm{Rep}(A, B)$ (up to a shift of dimension).

Let $\mathcal{H}(B)$ be the C^*-category of countably generated right Hilbert B-modules and adjointable B-homomorphisms. The norm of a morphism is defined as the norm of the bounded linear map (see [12]). $\mathcal{H}(B)$ is an additive C^*-category with respect to the sum of Hilbert modules.

The additive C^*-category $Rep(A, B)$ is defined as follows. Objects are all pairs of the form $(E; \phi)$, where E is an object in $\mathcal{H}(B)$ and $\phi : A \to \mathcal{L}(E)$ is a $*$-homomorphism. A morphism $f : (E, \phi) \to (E', \phi')$ is a morphism $f : E \to E'$ in $\mathcal{H}(B)$ such that

$$f\phi(a) - \phi'(a)f \in \mathcal{K}(E, E')$$

for all $a \in A$. The structure of a C^*-category follows from $\mathcal{H}(B)$. It is easy to show that $Rep(A, B)$ is an additive C^*-category (but it is not a pseudo-abelian C^*-category). The universal pseudo-abelian C^*category associated to $Rep(A, B)$ is denoted by $\mathrm{Rep}(A, B)$.

Using the definition of a pseudo-abelian C^*-category, we have the following description of $\mathrm{Rep}(A, B)$. Its objects are triples (E, ϕ, p), where (E, ϕ) is an object of $\mathrm{Rep}(A, B)$ and $p : (E, \phi) \to (E, \phi)$ is a morphism in $Rep(A, B)$ such that $p^* = p$ and $p^2 = p$. A morphism $f : (E, \phi, p) \to (E', \phi', p')$ is a morphism

$f : (E, \phi) \to (E, \phi)$ of $Rep(A, B)$ such that $fp = p'f = f$. More exactly, f has the properties

$$f\phi(a) - \phi'(a)f \in \mathcal{K}(E, F), \quad fp = p'f = f. \tag{4.1}$$

The structure of the C^*-category for $\mathrm{Rep}(A, B)$ is obtained from the corresponding structure of $Rep(A, B)$ [5].

Proposition 4.1. *Let A be an involutive algebra and B be a C^*-algebra. Then the category $Rep(A; B)$ is a weakly stable C^*-category.*

Proof. Let E be a Hilbert B-module, and $\mathcal{H}$ be a separable countably generated Hilbert space. According to [12], one can construct the Hilbert B-module $E \otimes \mathcal{H}$ in the following way. Consider the B-scalar product

$$\langle e_0 \odot h_0; e_1 \odot h_1 \rangle = \langle e_0; e_1 \rangle_E \cdot \langle h_0; h_1 \rangle_{\mathcal{H}}$$

on the algebraic tensor product $E \odot \mathcal{H}$, where $\langle -; - \rangle_E$ is the B-scalar product on E and $\langle -; - \rangle_{\mathcal{H}}$ is a usual scalar product on $\mathcal{H}$. The completion of $E \odot \mathcal{H}$ with respect to the so defined B-scalar product is denoted by $E \otimes \mathcal{H}$. An object of $Rep(A, B) \otimes \mathcal{K}$ is, by definition, an object $\xi = (E, \varphi)$ in $Rep(A, B)$, where $\varphi : A \to \mathcal{L}(E)$ is a $*$-homomorphism. By definition,

$$\hom_{Rep(A,B)\otimes\mathcal{K}}(\xi, \xi') = \hom_{Rep(A,B)}(\xi, \xi') \otimes \mathcal{K}.$$

Let $f \otimes \kappa : (E, \varphi) \to (E', \varphi')$ be a morphism in $Rep(A, B) \otimes \mathcal{K}$. Then

$$f \otimes \kappa : (E \otimes \mathcal{H}, \hat{\varphi}) \to (E \otimes \mathcal{H}, \hat{\varphi}')$$

is an morphism in $Rep(A, B)$, where

$$\hat{\varphi} : A \to \mathcal{L}(E \otimes \mathcal{H}), \quad a \mapsto \varphi(a) \otimes 1_{\mathcal{H}}, \quad a \in A.$$

Indeed,

$$\begin{aligned}(f \otimes \kappa) \cdot (\varphi \otimes 1_{\mathcal{H}})(a) - (\varphi' \otimes 1_{\mathcal{H}})(a) \cdot (f \otimes \kappa) &= f\varphi(a) \otimes \kappa - \varphi'(a)f \otimes \kappa \\ &= (f\varphi(a) - \varphi'(a)f) \otimes \kappa \in \mathcal{K}(E, E') \otimes \mathcal{K} \subset \mathcal{K}(E \otimes \mathcal{H}, E \otimes \mathcal{H}).\end{aligned}$$

Define a $*$-functoroid

$$G : Rep(A, B) \otimes \mathcal{K} \to Rep(A, B)$$

in the following way. Let $\xi = (E, \varphi)$ be an object in $Rep(A, B) \otimes \mathcal{K}$. Then

$$G(\xi) = (E \otimes \mathcal{H}, \hat{\varphi}),$$

and $G(f \otimes \kappa) = f \otimes \kappa$ where $f \otimes \kappa : \xi \to \xi'$ is a morphism in $Rep(A, B) \otimes \mathcal{K}$. It is easy to check that the $*$-functor

$$\mathcal{I} = G \circ e_{Rep(A,B)} : Rep(A, B) \to Rep(A, B)$$

sends an object (E, φ) to $(E \otimes \mathcal{H}, \hat{\varphi})$ and a morphism f to $f \otimes p$. We will show below that $\mathcal{I} = G \circ e_{Rep(A,B)}$ is an inner functoroid.

Let p be a fixed rank one projection on $\mathcal{H}$, and y be a fixed element in $p\mathcal{H}$ such that $\|y\| = 1$. On the one hand side, there is a B-linear map

$$s_E : E \to E \otimes \mathcal{H}, \;\; x \mapsto x \otimes y, \;\; x \in E,$$

and on the other hand, one has a B-linear map

$$s_E^* : E \otimes \mathcal{H} \to E, \quad x \otimes z \mapsto \lambda x, \quad x \in E, \quad z \in \mathcal{H},$$

where λ is a unique number such that $pz = \lambda y$, since $p\mathcal{H}$ is a one-dimensional subspace of $\mathcal{H}$.

Let us show that s^* is an adjoint to s. Indeed, let $x, x' \in E$ and $z \in \mathcal{H}$. Since $p^* = p$ and

$$\langle y; z\rangle = \langle py; z\rangle = \langle y; pz\rangle,$$

we have

$$\begin{aligned}\langle s_E x; x' \otimes z\rangle &= \langle x \otimes y; x' \otimes z\rangle = \langle x; x'\rangle \cdot \langle y; z\rangle = \langle x; x'\rangle \cdot \langle y; pz\rangle \\ &= \lambda\langle x; x'\rangle \cdot \langle y; y\rangle = \lambda\langle x; x'\rangle = \langle x; \lambda x'\rangle = \langle x; s_E^*(x' \otimes z)\rangle.\end{aligned}$$

Let $\varphi : A \to \mathcal{L}(E)$ be a $*$-homomorphism, where A is a separable C^*-algebra. Then one has the induced $*$-homomorphism $\hat{\varphi} : A \to \mathcal{L}(E \otimes \mathcal{H})$. Let us show that

$$s_E\varphi(a) = \hat{\varphi}(a)s_E, \qquad \forall a \in A.$$

Indeed, let $x \in E$ and $a \in A$, then $s_E\varphi(a)(x) = \varphi(a)(x) \otimes y$ and

$$\hat{\varphi}(a)s_E(x) = \hat{\varphi}(a)(x \otimes y) = (\varphi(a) \otimes 1_{\mathcal{H}})(x \otimes y) = \varphi(a)(x) \otimes y.$$

Thus s_E is a morphism from φ into $\hat{\varphi}$ in the category $Rep(A; B)$. Moreover, s_E is an isometry. Indeed, $s_E^* s_E(x) = s_E^*(x \otimes y) = x$ since $py = y$. The $*$-functoroid $\mathcal{I}$ is an inner functoroid. Indeed, for any object φ there is an isometry

$$s_\varphi : \varphi \to \mathcal{I}(\varphi) = \hat{\varphi},$$

and if $f : \varphi \to \varphi$ is a morphism, then $s_\varphi f s_\varphi^* = \mathcal{I}(f)$ since

$$\begin{aligned}s_\varphi f s_\varphi^*(x \otimes z) &= \lambda s_\varphi f(x) = \lambda(f(x) \otimes y) = f(x) \otimes \lambda y \\ & \qquad f(x) \otimes pz = (f \otimes p)(x \otimes z) = \mathcal{I}(f)(x \otimes z),\end{aligned}$$

where $x \in E$, $z \in \mathcal{H}$ and λ is a scalar. □

Consider the natural transformation $\kappa_A : KV_n(A) \to K_n^t(A)$ defined by the map $(E, \alpha) \to (E, \overline{\alpha})$, where $\overline{\alpha}$ is a continuous map from I^{n-1} into $GL(A)$, where $I = [0, 1]$ and $\overline{\alpha}$ is the continuous map associated to the polynomial α. Then, according to Corollary 3.10, Lemma 3.4, Proposition 2.5 and the main result of [5], one immediately gets the following result.

Theorem 4.2. *Let A be a separable C^*-algebra, and let B be a σ-unital C^*-algebra. Then there exist natural isomorphisms*

$$KV_n(\mathrm{Rep}(A, B)) \simeq KV_n(Rep(A, B)) \simeq KK_{n-1}(A, B), \quad n \geq 1,$$

where KV_n stand for Karoubi–Villamayor K-groups from [11].

Remark 1. The main results of this paper also hold for the category of C^*-algebras, over either real or complex numbers, with a fixed action of a compact group. One can also replace the Karoubi–Villamayor K-groups by Quillen's K-groups. In the latter case, see [6], [8].

Acknowledgement

The work was partially supported by the FNRS grant No. 7GEPJ065513.01 and the INTAS grant No. 00566.

The author expresses his gratitude to the referees for their meticulous reading and the advice given by one of them to pay attention to refs. [14] and [15].

References

1. P. Ghez, R. Lima R, and J. E. Roberts, W^*-categories. *Pacific J. Math.* **120**(1985), no. 1, 79–109.
2. N. Higson, C^*-algebra extension theory and duality. *J. Funct. Anal.* **129**(1995), No. 2, 349–363.
3. N. Higson, Algebraic K-theory of stable C^*-algebras. *Adv. in Math.* **67**(1988), No. 1, 140 pp.
4. H. Inassaridze, Algebraic K-theory. *Mathematics and its Applications,* 311. *Kluwer Academic Publishers Group, Dordrecht,* 1995.
5. T. Kandelaki, KK-theory as the K-theory of C^*-categories. *Homology Homotopy Appl.* **2**(2000), 127–145 (electronic).
6. T. Kandelaki, Algebraic and topological bivariant KK-theories of C^*-algebras and their isomorphism. *Proc. A. Razmadze Math. Inst.* **126**(2001), 110–112.
7. T. Kandelaki, Multiplier and Hilbert C^*-categories. *Proc. A. Razmadze Math. Inst.* **127**(2001), 89–111.
8. T. Kandelaki, Algebraic K-theory view on KK-theory (to appear) (see www.arXiv.org. (April, (2003)).
9. M. Karoubi, Algébres de Clifford et K-théorie. Ann. *Sci. École Norm. Sup. (4)* **1**(1968), 161–270.
10. M. Karoubi, K-theory. An introduction. *Grundlehren der Mathematischen Wissenschaften,* Band 226. *Springer-Verlag, Berlin-New York,* 1978.
11. M. Karoubi and O. Villamayor, K-théorie algébrique et K-théorie topologique. I. *Math. Scand.* **28**(1971), 265–307 (1972).
12. G. Kasparov, Hilbert C^*-modules: theorems of Stinespring and Voiculescu. *J. Operator Theory* **4**(1980), No. 1, 133–150.
13. G. Kasparov, The operator K-functor and extensions of C^*-algebras. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **44** (1980), No. 3, 571–636.
14. P. D. Mitchener, C^*-categories. *Proc. London Math. Soc. (3)* **84**(2002), No. 2, 375–404.
15. P. D. Mitchener, KK-theory of C^*-categories and the analytic assembly map. *K-Theory* **26**(2002), No. 4, 307–344.

(Received 4.08.2003; revised 2.04.2004)

Author's address:

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: kandel@rmi.acnet.ge