Georgian Mathematical Journal
Volume 11 (2004), Number 2, 337–348

ON THE BEHAVIOR OF SOLUTIONS OF LINEAR NEUTRAL INTEGRODIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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Abstract. A useful inequality for solutions of linear neutral integrodifferential equations with unbounded delay is established, using a real root of the corresponding characteristic equation. This inequality is used to obtain an estimate for solutions, which leads to a stability criterion.

2000 Mathematics Subject Classification: 34K20, 34K25, 34K40.
Key words and phrases: Integrodifferential equation, neutral equation, unbounded delay, asymptotic behavior, stability.

1. Introduction And Statement Of The Main Results

During the past four decades, the theories of Volterra integral equations and Volterra integrodifferential equations have undergone rapid developments. Various classical problems in the theory of differential equations (ordinary or partial) lead to integral or integrodifferential equations and, in many cases, can be dealt with in a more satisfactory manner using these (integral or integrodifferential equations) than directly with differential equations. Also, various problems in applied science are conducive to integral and integrodifferential equations in a natural way, these equations thus emerging as competent mathematical tools in modelling phenomena and processes encountered in those fields of investigation. In particular, Volterra integral and integrodifferential equations are widely used in mathematical ecology, especially in population dynamics (cf., e.g., Gopalsamy [8]). For the basic theory of Volterra integral and integrodifferential equations, we choose to refer to the books by Burton [1, 2], Corduneanu [3], and Miller [17]; also, for equations with unbounded delay, the reader is referred to the survey article by Corduneanu and Lakshmikantham [4] and the book by Lakshmikantham, Wen and Zhang [16].

In recent years there has been some research activity concerning the existence and/or the nonexistence of positive solutions of integrodifferential equations. See, for example, Györi and Ladas [10], Kiventidis [11], Kordonis and Philos [14], Ladas, Philos and Sficas [15], Philos [18, 19], and Philos and Sficas [22].

Driver, Sasser and Slater [6] obtained some significant results on the asymptotic behavior and the stability for first order linear delay differential equations with constant coefficients and one constant delay; see Driver [5] for some similar asymptotic and stability results for first order linear autonomous delay differential equations with infinitely many distributed delays. The results given in [6] have been improved and extended in several directions by Graef and Qian

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[9], Kordonis, Niyianni and Philos [12], Philos [20], and Philos and Purnaras [21]. Motivated by the results in [6] (as well as by those in [12], [20] and [21]), Kordonis and Philos [13] established some results on the behavior of solutions of linear integrodifferential equations with unbounded delay.

This paper deals with the behavior of solutions of linear *neutral* integrodifferential equations with unbounded delay. A useful inequality for solutions is established. This inequality is a tool to obtain an estimate for solutions, which leads to a criterion for the stability and the asymptotic stability of the trivial solution. Our results are derived by the use of a real root (with an appropriate property) of the corresponding characteristic equation. The results of the present paper can be applied to the corresponding *non-neutral* integrodifferential equations. An improved version of the results given by Kordonis and Philos [13] for linear (non-neutral) integrodifferential equations with unbounded delay can be obtained (as a special case) from the results of this paper. The techniques applied in obtaining our results originate in the methods used in [13]. Note that nothing but elementary calculus will be used. Neutral integrodifferential equations with unbounded delay have been investigated by many authors (see, for example, the book by Lakshmikantham, Wen and Zhang [16]). These equations belong to a very wide class of the neutral functional differential equations with infinite delay.

Consider the linear neutral integrodifferential equation with unbounded delay

$$\left[x(t) + \int_{-\infty}^{t} G(t-s)x(s)ds\right]' = ax(t) + \int_{-\infty}^{t} K(t-s)x(s)ds, \tag{E}$$

where *a is a real number, and G and K are continuous real-valued functions on the interval $[0, \infty)$. It will be supposed that K is not eventually identically zero.*

As usual, a continuous real-valued function x defined on the real line $\mathbf{R}$ will be called a *solution* of the neutral integrodifferential equation (E) if the function $x(t) + \int\limits_{-\infty}^{t} G(t-s)x(s)ds$ is a continuously differentiable real-valued function for $t \geq 0$ and x satisfies (E) for all $t \geq 0$.

In what follows, by S we will denote the (nonempty) set of all continuous real-valued functions ϕ on $(-\infty, 0]$, which are such that

$$\Phi_G(t) = \int_{-\infty}^{t} G(t-s)\phi(s)ds \quad \text{for} \quad t \geq 0$$

is a continuously differentiable real-valued function on $[0, \infty)$, and

$$\Phi_K(t) = \int_{-\infty}^{0} K(t-s)\phi(s)ds \quad \text{for} \quad t \geq 0$$

is a continuous real-valued function on $[0, \infty)$.

It is known (see, for example, [16]) that, for any given *initial function* ϕ in S, there exists a unique solution x of the neutral integrodifferential equation (E) which satisfies the *initial condition*

$$x(t) = \phi(t) \quad \text{for} \ \ t \in (-\infty, 0]; \tag{C}$$

we call this function x the solution of the *initial problem* (E)–(C) or, more briefly, the solution of (E)–(C).

With the neutral integrodifferential equation (E) we associate its *characteristic equation*

$$\lambda \left[1 + \int_0^\infty e^{-\lambda s} G(s) ds \right] = a + \int_0^\infty e^{-\lambda s} K(s) ds. \tag{$*$}$$

To obtain the results of this paper, we will make use of a real root λ_0 of the characteristic equation $(*)$ with the property

$$\int_0^\infty e^{-\lambda_0 s} (1 + |\lambda_0|\, s)\, |G(s)|\, ds + \int_0^\infty e^{-\lambda_0 s} s\, |K(s)|\, ds < 1. \tag{P(λ_0)}$$

In the sequel, if λ_0 is a real root of $(*)$ with the property (P(λ_0)), we will denote by $S(\lambda_0)$ the (nonempty) subset of S which contains all functions ϕ in S such that $e^{-\lambda_0 t}\phi(t)$ is bounded for $t \leq 0$.

The basic result of this paper is the following theorem which establishes a useful inequality for solutions of the neutral integrodifferential equation (E).

Theorem. *Let λ_0 be a real root of the characteristic equation $(*)$ with the property* (P(λ_0)) *and set*

$$\gamma(\lambda_0) = \int_0^\infty e^{-\lambda_0 s} (1 - \lambda_0 s)\, G(s)\, ds + \int_0^\infty e^{-\lambda_0 s} s\, K(s)\, ds$$

and

$$\mu(\lambda_0) = \int_0^\infty e^{-\lambda_0 s} (1 + |\lambda_0|\, s)\, |G(s)|\, ds + \int_0^\infty e^{-\lambda_0 s} s\, |K(s)|\, ds.$$

Then, for any $\phi \in S(\lambda_0)$, the solution x of (E)–(C) *satisfies*

$$\left| e^{-\lambda_0 t} x(t) - \frac{L(\lambda_0; \phi)}{1 + \gamma(\lambda_0)} \right| \leq \mu(\lambda_0) M(\lambda_0; \phi) \quad \text{for all} \ \ t \geq 0,$$

where

$$L(\lambda_0; \phi) = \phi(0) + \int_0^\infty G(s) \left[\phi(-s) - \lambda_0 e^{-\lambda_0 s} \int_{-s}^0 e^{-\lambda_0 r} \phi(r) dr \right] ds + \int_0^\infty e^{-\lambda_0 s} K(s) \left[\int_{-s}^0 e^{-\lambda_0 r} \phi(r)\, dr \right] ds$$

and

$$M(\lambda_0; \phi) = \sup_{t \leq 0} \left| e^{-\lambda_0 t} \phi(t) - \frac{L(\lambda_0; \phi)}{1 + \gamma(\lambda_0)} \right|.$$

Note. The property (P(λ_0)) guarantees that $0 < \mu(\lambda_0) < 1$. Also, since $|\gamma(\lambda_0)| \leq \mu(\lambda_0)$, it holds $-1 < \gamma(\lambda_0) < 1$ and so, in particular, we have $1 + \gamma(\lambda_0) > 0$. Moreover, from (P($\lambda_0$)) and the definition of $S(\lambda_0)$ it follows that (for any $\phi \in S(\lambda_0)$) $L(\lambda_0; \phi)$ is a real number. Furthermore, by the definition of $S(\lambda_0)$, $M(\lambda_0; \phi)$ is a nonnegative constant (for any $\phi \in S(\lambda_0)$).

An interesting consequence of our theorem is the corollary below, which gives an estimate of solutions of the neutral integrodifferential equation (E) that leads to a stability criterion for the *trivial solution* of (E).

Before stating this corollary, we will give two well-known definitions (see, e.g., [16]).The trivial solution of (E) is said to be *stable* (*at* 0) if, for every $\epsilon > 0$, there exists $\delta \equiv \delta(\epsilon) > 0$ such that, for any ϕ in S with $\|\phi\| \equiv \sup_{t \le 0} |\phi(t)| < \delta$, the solution x of (E)–(C) satisfies $|x(t)| < \epsilon$ for all $t \in \mathbf{R}$. Moreover, the trivial solution of (E) is called *asymptotically stable* (*at* 0) if it is stable (at 0) in the above sense and, in addition, there exists $\delta_0 > 0$ such that, for any ϕ in S with $\|\phi\| < \delta_0$, the solution x of (E)–(C) satisfies $\lim_{t \to \infty} x(t) = 0$.

Corollary. *Let λ_0 be a real root of the characteristic equation* $(*)$ *with the property* $(\mathrm{P}(\lambda_0))$.

Define $\gamma(\lambda_0)$ and $\mu(\lambda_0)$ as in Theorem and set

$$\Theta(\lambda_0) = \frac{[1 + \mu(\lambda_0)]^2}{1 + \gamma(\lambda_0)} + \mu(\lambda_0).$$

Then, for any $\phi \in S(\lambda_0)$, the solution x of (E)–(C) *satisfies*

$$|x(t)| \le \Theta(\lambda_0) N(\lambda_0; \phi) e^{\lambda_0 t} \quad \textit{for all } \ t \ge 0,$$

where

$$N(\lambda_0; \phi) = \sup_{t \le 0} \left[e^{-\lambda_0 t} |\phi(t)| \right].$$

Moreover, the trivial solution of (E) *is stable* (*at* 0) *if $\lambda_0 = 0$ and it is asymptotically stable* (*at* 0) *if $\lambda_0 < 0$.*

Note. Clearly, $\Theta(\lambda_0)$ is a real number with $\Theta(\lambda_0) > 1$. Moreover, by the definition of $S(\lambda_0)$, $N(\lambda_0; \phi)$ is a nonnegative constant (for any $\phi \in S(\lambda_0)$).

The proofs of our theorem and corollary stated above will be given in Section 2.

Now, let us consider the special case where the kernel G is identically zero on $[0, \infty)$, i.e., the case of the linear (non-neutral) integrodifferential equation with unbounded delay

$$x'(t) = ax(t) + \int_{-\infty}^{t} K(t-s) x(s)\, ds. \tag{$\widetilde{\mathrm{E}}$}$$

By a *solution* of the integrodifferential equation $(\widetilde{\mathrm{E}})$ we mean a continuous real-valued function x defined on $\mathbf{R}$, which is continuously differentiable on $[0, \infty)$ and satisfies $(\widetilde{\mathrm{E}})$ for $t \ge 0$. As it concerns the integrodifferential equation $(\widetilde{\mathrm{E}})$, the set S is the (nonempty) set of all continuous real-valued functions x on $(-\infty, 0]$ such that

$$\Phi_K(t) = \int_{-\infty}^{t} K(t-s) \phi(s) ds \quad \text{for} \ \ t \ge 0$$

is a continuous real-valued function on $[0,\infty)$. The *characteristic equation* of $(\widetilde{\mathrm{E}})$ is

$$\lambda = a + \int_0^\infty e^{-\lambda s} K(s)\,ds. \qquad (\widetilde{*})$$

In the special case of the (non-neutral) integrodifferential equation $(\widetilde{\mathrm{E}})$, the property $(\mathrm{P}(\lambda_0))$ (of a real root λ_0 of the characteristic equation $(\widetilde{*})$) takes the form

$$\int_0^\infty e^{-\lambda_0 s} s\,|K(s)|\,ds < 1. \qquad (\widetilde{\mathrm{P}}(\lambda_0))$$

If λ_0 is a real root of $(\widetilde{*})$ with the property $(\widetilde{\mathrm{P}}(\lambda_0))$, the set $S(\lambda_0)$ is defined as in the general case of the equation (E).

By applying our theorem and its corollary to the special case of the (non-neutral) integrodifferential equation $(\widetilde{\mathrm{E}})$, we obtain the following results respectively:

Let λ_0 be a real root of the characteristic equation $(\widetilde{*})$ with the property $(\widetilde{\mathrm{P}}(\lambda_0))$ and set

$$\widetilde{\gamma}(\lambda_0) = \int_0^\infty e^{-\lambda_0 s} sK(s)ds \quad \text{and} \quad \widetilde{\mu}(\lambda_0) = \int_0^\infty e^{-\lambda_0 s} s\,|K(s)|\,ds.$$

Then, for any $\phi \in S(\lambda_0)$, the solution x of $(\widetilde{\mathrm{E}})$–(C) satisfies

$$\left| e^{-\lambda_0 t} x(t) - \frac{\widetilde{L}(\lambda_0;\phi)}{1+\widetilde{\gamma}(\lambda_0)} \right| \le \widetilde{\mu}(\lambda_0)\widetilde{M}(\lambda_0;\phi) \quad \text{for all } t \ge 0,$$

where

$$\widetilde{L}(\lambda_0;\phi) = \phi(0) + \int_0^\infty e^{-\lambda_0 s} K(s) \left[\int_{-s}^0 e^{-\lambda_0 r} \phi(r) dr \right] ds$$

and

$$\widetilde{M}(\lambda_0;\phi) = \sup_{t\le 0} \left| e^{-\lambda_0 t} \phi(t) - \frac{\widetilde{L}(\lambda_0;\phi)}{1+\widetilde{\gamma}(\lambda_0)} \right|.$$

Let λ_0 be a real root of the characteristic equation $(\widetilde{*})$ with the property $(\widetilde{\mathrm{P}}(\lambda_0))$.

Define $\widetilde{\gamma}(\lambda_0)$ and $\widetilde{\mu}(\lambda_0)$ as above and set

$$\widetilde{\Theta}(\lambda_0) = \frac{[1+\widetilde{\mu}(\lambda_0)]^2}{1+\widetilde{\gamma}(\lambda_0)} + \widetilde{\mu}(\lambda_0).$$

Then, for any $\phi \in S(\lambda_0)$, the solution x of $(\widetilde{E})$–(C) satisfies

$$|x(t)| \le \widetilde{\Theta}(\lambda_0) N(\lambda_0;\phi) e^{\lambda_0 t} \quad \text{for all } t \ge 0,$$

where $N(\lambda_0;\phi)$ is defined as in Corollary.

Moreover, the trivial solution of $(\widetilde{E})$ is stable (at 0) if $\lambda_0 = 0$ and it is asymptotically stable (at 0) if $\lambda_0 < 0$.

The above results for the (non-neutral) integrodifferential equation $(\widetilde{\mathrm{E}})$ can be considered as improved versions of the main results of the paper by Kordonis and Philos [13]. In [13], the following simple result has been established:

Assume that there exists a real number γ *such that*

$$\int_0^\infty e^{-\gamma s} |K(s)| \, ds < \infty$$

and let the following hypothesis be satisfied:

$$\int_0^\infty e^{-\gamma s} K(s) ds > \gamma - a \quad \text{ and } \quad \int_0^\infty e^{-\gamma s} s \, |K(s)| \, ds \leq 1.$$

Then, in the interval (γ, ∞), *the characteristic equation* $(\widetilde{*})$ *has a unique root* λ_0; *this root satisfies* $(\widetilde{\mathrm{P}}(\lambda_0))$.

The results given above for the (non-neutral) integrodifferential equation $(\widetilde{\mathrm{E}})$ have been previously obtained in [13] with the use of the unique root λ_0 of $(\widetilde{*})$ in the interval (γ, ∞) and under the conditions mentioned above (which guarantee the existence and uniqueness of this root of $(\widetilde{*})$). It must be noted that the assumption that the kernel K keeps its sign posed in [13] is not necessary and so this assumption can be removed. No restriction on the sign of K or G is assumed in the present paper.

It is an open problem to find conditions on the coefficient a and the kernels G and K of the neutral integrodifferential equation (E), which are sufficient for the characteristic equation $(*)$ to have a real root λ_0 with the property $(\mathrm{P}(\lambda_0))$.

Before closing this section, we remark that our main results can be extended to a more general case of the linear neutral integro-delay-differential equation with unbounded delay

$$\left[x(t) + \sum_{n=1}^{\infty} c_n x(t-\sigma_n) + \int_{-\infty}^{t} G(t-s)x(s)ds\right]' = ax(t) + \sum_{n=1}^{\infty} b_n x(t-\tau_n) + \int_{-\infty}^{t} K(t-s)x(s)ds, \qquad (\widehat{\mathrm{E}})$$

where c_n and b_n $(n = 1, 2, \dots)$ are real numbers, and σ_n and τ_n $(n = 1, 2, \dots)$ are positive constants with $\sigma_i \neq \sigma_j$ and $\tau_i \neq \tau_j$ $(i, j = 1, 2, \dots;\ i \neq j)$. Equations with unbounded delay of the form $(\widehat{\mathrm{E}})$ have appeared in several investigations; we choose to refer to the paper by Gopalsamy [7], the survey article by Corduneanu and Lakshmikantham [4], and the book by Lakshmikantham, Wen and Zhang [16].

2. Proofs of the Main Results

Proof of Theorem. Consider an arbitrary function ϕ in $\mathrm{S}(\lambda_0)$ and let x be the solution of (E)–(C).

Set

$$y(t) = e^{-\lambda_0 t} x(t) \quad \text{for} \quad t \in \mathbf{R}.$$

Then, for every $t \geq 0$, we obtain

$$\begin{aligned}
& \left[x(t) + \int_{-\infty}^{t} G(t-s)x(s)ds\right]' - ax(t) - \int_{-\infty}^{t} K(t-s)x(s)ds \\
= & \left[x(t) + \int_{0}^{\infty} G(s)x(t-s)ds\right]' - ax(t) - \int_{0}^{\infty} K(s)x(t-s)ds \\
= & e^{\lambda_0 t}\left\{\left[y(t) + \int_{0}^{\infty} e^{-\lambda_0 s}G(s)y(t-s)ds\right]' + \right. \\
& + \lambda_0 \left[y(t) + \int_{0}^{\infty} e^{-\lambda_0 s}G(s)y(t-s)ds\right] - \\
& \left. -ay(t) - \int_{0}^{\infty} e^{-\lambda_0 s}K(s)y(t-s)ds\right\} \\
= & e^{\lambda_0 t}\left\{\left[y(t) + \int_{0}^{\infty} e^{-\lambda_0 s}G(s)y(t-s)ds\right]' + (\lambda_0 - a)y(t) + \right. \\
& \left. + \lambda_0 \int_{0}^{\infty} e^{-\lambda_0 s}G(s)y(t-s)ds - \int_{0}^{\infty} e^{-\lambda_0 s}K(s)y(t-s)ds\right\} \\
= & e^{\lambda_0 t}\left\{\left[y(t) + \int_{0}^{\infty} e^{-\lambda_0 s}G(s)y(t-s)ds\right]' - \right. \\
& - \lambda_0 \left[\int_{0}^{\infty} e^{-\lambda_0 s}G(s)ds\right] y(t) + \left[\int_{0}^{\infty} e^{-\lambda_0 s}K(s)ds\right] y(t) + \\
& \left. + \lambda_0 \int_{0}^{\infty} e^{-\lambda_0 s}G(s)y(t-s)ds - \int_{0}^{\infty} e^{-\lambda_0 s}K(s)y(t-s)ds\right\} \\
= & e^{\lambda_0 t}\left\{\left[y(t) + \int_{0}^{\infty} e^{-\lambda_0 s}G(s)y(t-s)ds\right]' - \right. \\
& - \lambda_0 \int_{0}^{\infty} e^{-\lambda_0 s}G(s)\left[y(t) - y(t-s)\right]ds \; + \\
& \left. + \int_{0}^{\infty} e^{-\lambda_0 s}K(s)\left[y(t) - y(t-s)\right]ds\right\}.
\end{aligned}$$

Thus, the fact that x satisfies (E) for $t \geq 0$ is equivalent to the fact that y satisfies

$$\begin{aligned}
\left[y(t) + \int_{0}^{\infty} e^{-\lambda_0 s}G(s)y(t-s)ds\right]' &= \lambda_0 \int_{0}^{\infty} e^{-\lambda_0 s}G(s)\left[y(t) - y(t-s)\right]ds \\
&- \int_{0}^{\infty} e^{-\lambda_0 s}K(s)\left[y(t) - y(t-s)\right]ds \quad \text{for } t \geq 0. \quad (2.1)
\end{aligned}$$

On the other hand, the initial condition (C) takes the equivalent form

$$y(t) = e^{-\lambda_0 t}\phi(t) \quad \text{for } \; t \in (-\infty, 0]. \quad (2.2)$$

Furthermore, by using (2.2) and taking into account the definition of $L(\lambda_0;\phi)$, we can verify that (2.1) is equivalent to

$$y(t)+\int_0^\infty e^{-\lambda_0 s}G(s)y(t-s)ds=\lambda_0\int_0^\infty e^{-\lambda_0 s}G(s)\left[\int_{t-s}^t y(r)dr\right]ds$$
$$-\int_0^\infty e^{-\lambda_0 s}K(s)\left[\int_{t-s}^t y(r)dr\right]ds+L(\lambda_0;\phi)\quad \text{for } t\geq 0. \tag{2.3}$$

Next, we define

$$z(t)=y(t)-\frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\quad \text{for } t\in \mathbf{R}.$$

Then, by the definition of $\gamma(\lambda_0)$, it is not difficult to see that (2.3) can equivalently be written as

$$z(t)+\int_0^\infty e^{-\lambda_0 s}G(s)z(t-s)ds=\lambda_0\int_0^\infty e^{-\lambda_0 s}G(s)\left[\int_{t-s}^t z(r)dr\right]ds$$
$$-\int_0^\infty e^{-\lambda_0 s}K(s)\left[\int_{t-s}^t z(r)dr\right]ds\quad \text{for } t\geq 0. \tag{2.4}$$

Moreover, the initial condition (2.2) is equivalent to

$$z(t)=e^{-\lambda_0 t}\phi(t)-\frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\quad \text{for } t\in(-\infty,0]. \tag{2.5}$$

Because of the definitions of y and z, the proof will be complete by showing that z satisfies

$$|z(t)|\leq \mu(\lambda_0)M(\lambda_0;\phi)\ \text{ for all } t\geq 0. \tag{2.6}$$

In the rest of the proof we will establish (2.6). By the definition of $M(\lambda_0;\phi)$ and in view of (2.5), we have

$$|z(t)|\leq M(\lambda_0;\phi)\ \text{ for } t\in(-\infty,0]. \tag{2.7}$$

We will prove that $M(\lambda_0;\phi)$ is also a bound of z on the whole real line $\mathbf{R}$, i.e.,

$$|z(t)|\leq M(\lambda_0;\phi)\ \text{ for all } t\in\mathbf{R}. \tag{2.8}$$

To this end, let us consider an arbitrary number $\epsilon>0$. Then

$$|z(t)|<M(\lambda_0;\phi)+\epsilon\quad \text{for every } t\in\mathbf{R}. \tag{2.9}$$

Indeed, in the case where (2.9) fails, by taking into account (2.7) we can conclude that there exists a point $t^*>0$ such that

$$|z(t)|<M(\lambda_0;\phi)+\epsilon\ \text{ for } t\in(-\infty,t^*),\quad \text{and}\quad |z(t^*)|=M(\lambda_0;\phi)+\epsilon.$$

Then, since $0<\mu(\lambda_0)<1$, from (2.4) we obtain

$$M(\lambda_0;\phi)+\epsilon=|z(t^*)|\leq\int_0^\infty e^{-\lambda_0 s}\left|G(s)\right|\left[|z(t^*-s)|+|\lambda_0|\int_{t^*-s}^{t^*}|z(r)|\,dr\right]ds$$
$$+\int_0^\infty e^{-\lambda_0 s}\left|K(s)\right|\left[\int_{t^*-s}^{t^*}|z(r)|\,dr\right]ds$$

$$\leq \left\{ \int_0^\infty e^{-\lambda_0 s} \left(1 + |\lambda_0| \, s\right) |G(s)| \, ds \right.$$
$$\left. + \int_0^\infty e^{-\lambda_0 s} s \, |K(s)| \, ds \right\} [M(\lambda_0; \phi) + \epsilon]$$
$$\equiv \mu(\lambda_0) \, [M(\lambda_0; \phi) + \epsilon] < M(\lambda_0; \phi) + \epsilon.$$

We have thus arrived at a contradiction and so (2.9) holds true. Furthermore, since $\epsilon > 0$ is arbitrary, (2.9) yields (2.8), i.e., (2.8) is always valid. Finally, by (2.8), from (2.4) we derive for every $t \geq 0$

$$|z(t)| \leq \int_0^\infty e^{-\lambda_0 s} \, |G(s)| \left[|z(t-s)| + |\lambda_0| \int_{t-s}^t |z(r)| \, dr \right] ds$$
$$+ \int_0^\infty e^{-\lambda_0 s} \, |K(s)| \left[\int_{t-s}^t |z(r)| \, dr \right] ds$$
$$\leq \left\{ \int_0^\infty e^{-\lambda_0 s} \left(1 + |\lambda_0| \, s\right) |G(s)| \, ds + \int_0^\infty e^{-\lambda_0 s} s \, |K(s)| \, ds \right\} M(\lambda_0; \phi)$$
$$\equiv \mu(\lambda_0) M(\lambda_0; \phi),$$

which means that (2.6) is satisfied. So, our proof is complete. □

Proof of Corollary. Let ϕ be an arbitrary function in $S(\lambda_0)$ and let x be the solution of (E)–(C). Then our theorem guarantees that the solution x satisfies

$$\left| e^{-\lambda_0 t} x(t) - \frac{L(\lambda_0; \phi)}{1 + \gamma(\lambda_0)} \right| \leq \mu(\lambda_0) M(\lambda_0; \phi) \quad \text{for all} \quad t \geq 0,$$

where $L(\lambda_0; \phi)$ and $M(\lambda_0; \phi)$ are defined as in Theorem. This gives

$$e^{-\lambda_0 t} \, |x(t)| \leq \frac{|L(\lambda_0; \phi)|}{1 + \gamma(\lambda_0)} + \mu(\lambda_0) M(\lambda_0; \phi) \quad \text{for every} \quad t \geq 0.$$

But, from the definitions of $M(\lambda_0; \phi)$ and $N(\lambda_0; \phi)$ it follows that

$$M(\lambda_0; \phi) \ \leq N(\lambda_0; \phi) + \frac{|L(\lambda_0; \phi)|}{1 + \gamma(\lambda_0)}$$

and so we have

$$e^{-\lambda_0 t} \, |x(t)| \leq \frac{1 + \mu(\lambda_0)}{1 + \gamma(\lambda_0)} \, |L(\lambda_0; \phi)| + \mu(\lambda_0) N(\lambda_0; \phi) \quad \text{for} \quad t \geq 0. \tag{2.10}$$

Furthermore, by taking into account the definition of $L(\lambda_0; \phi)$, we obtain

$$|L(\lambda_0; \phi)| \leq |\phi(0)| + \int_0^\infty |G(s)| \left[|\phi(-s)| + |\lambda_0| \, e^{-\lambda_0 s} \int_{-s}^0 e^{-\lambda_0 r} \, |\phi(r)| \, dr \right] ds$$
$$+ \int_0^\infty e^{-\lambda_0 s} \, |K(s)| \left[\int_{-s}^0 e^{-\lambda_0 r} \, |\phi(r)| \, dr \right] ds$$
$$= |\phi(0)| + \int_0^\infty e^{-\lambda_0 s} \left[e^{-\lambda_0 (-s)} \, |\phi(-s)| + |\lambda_0| \int_{-s}^0 e^{-\lambda_0 r} \, |\phi(r)| \, dr \right] |G(s)| \, ds$$

$$+\int_0^\infty e^{-\lambda_0 s}\left[\int_{-s}^0 e^{-\lambda_0 r}\left|\phi(r)\right| dr\right]\left|K(s)\right| ds.$$

Thus, by the definition of $N(\lambda_0;\phi)$ and $\mu(\lambda_0)$, we get

$$\begin{aligned}\left|L(\lambda_0;\phi)\right| \leq \Bigg[1+\int_0^\infty e^{-\lambda_0 s}\left(1+\left|\lambda_0\right| s\right)\left|G(s)\right| ds \\ +\int_0^\infty e^{-\lambda_0 s}\left|K(s)\right| ds\Bigg] N(\lambda_0;\phi) = \left[1+\mu(\lambda_0)\right] N(\lambda_0;\phi).\end{aligned}$$

Hence, (2.10) yields

$$e^{-\lambda_0 t}\left|x(t)\right| \leq \left\{\frac{\left[1+\mu(\lambda_0)\right]^2}{1+\gamma(\lambda_0)}+\mu(\lambda_0)\right\} N(\lambda_0;\phi), \quad t \geq 0,$$

which, by the definition of $\Theta(\lambda_0)$, can be written as

$$\left|x(t)\right| \leq \Theta(\lambda_0) N(\lambda_0;\phi) e^{\lambda_0 t} \quad \text{for all } \ t \geq 0. \tag{2.11}$$

Now, let us suppose that $\lambda_0 \leq 0$. Let ϕ be an arbitrary bounded function in S and define

$$\|\phi\| = \sup_{t\leq 0}\left|\phi(t)\right|.$$

As λ_0 is nonpositive, it follows immediately that ϕ belongs to the set $\mathrm{S}(\lambda_0)$ and, moreover, that

$$N(\lambda_0;\phi) \leq \|\phi\|. \tag{2.12}$$

The solution x of (E)-(C) satisfies (2.11). By (2.12), from (2.11) we obtain

$$\left|x(t)\right| \leq \Theta(\lambda_0)\left\|\phi\right\| e^{\lambda_0 t} \quad \text{for every } \ t \geq 0. \tag{2.13}$$

Since $\lambda_0 \leq 0$, the last inequality gives

$$\left|x(t)\right| \leq \Theta(\lambda_0)\left\|\phi\right\| \quad \text{for any } \quad t \geq 0.$$

So, as $\Theta(\lambda_0) > 1$, it follows that

$$\left|x(t)\right| \leq \Theta(\lambda_0)\left\|\phi\right\| \quad \text{for all } \ t \in \mathbf{R}. \tag{2.14}$$

We have thus proved that, for any bounded function $\phi \in$S, the solution x of (E)-(C) satisfies (2.13) and (2.14). From (2.14) it follows that the trivial solution of (E) is stable (at 0) (provided that $\lambda_0 \leq 0$). Moreover, if $\lambda_0 < 0$, then (2.13) guarantees that

$$\lim_{t\to\infty} x(t) = 0,$$

which means that the trivial solution of (E) is asymptotically stable (at 0).

The proof of our corollary is now complete.

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(Received 10.01.2003)

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