Georgian Mathematical Journal
Volume 11 (2004), Number 2, 349–359

INVARIANT REGIONS AND GLOBAL EXISTENCE OF SOLUTIONS FOR REACTION-DIFFUSION SYSTEMS WITH A FULL MATRIX OF DIFFUSION COEFFICIENTS AND NONHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. The purpose of this paper is the construction of invariant regions in which we establish the global existence of solutions for reaction-diffusion systems with a general full matrix of diffusion coefficients without balance law'condition ($f + g \equiv 0$) and with nonhomogeneous boundary conditions. Our techniques are based on invariant regions and Lyapunov functional methods. The nonlinear reaction term has been supposed to be of polynomial growth.

2000 Mathematics Subject Classification: 35K57, 35K45.
Key words and phrases: Reaction diffusion systems, invariant regions, Lyapunov functionals, global existence.

1. Introduction

We consider the reaction-diffusion system

$$\frac{\partial u}{\partial t} - a_{11}\Delta u - a_{12}\Delta v = f(u,v) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - a_{21}\Delta u - a_{22}\Delta v = g(u,v) \quad \text{in} \quad \mathbb{R}^+ \times \Omega \tag{1.2}$$

with the boundary conditions

$$\lambda u + (1-\lambda)\frac{\partial u}{\partial \eta} = \beta_1 \quad \text{and} \quad \lambda v + (1-\lambda)\frac{\partial v}{\partial \eta} = \beta_2 \quad \text{on} \quad \mathbb{R}^+ \times \partial\Omega \tag{1.3}$$

and the initial data

$$u(0,x) = u_0(x), \qquad v(0,x) = v_0(x) \quad \text{in} \quad \Omega, \tag{1.4}$$

where

$$\text{(i)} \;\; 0 < \lambda < 1 \quad \text{and} \quad \beta_i \in \mathbb{R}, \quad i = 1,2$$

(Robin nonhomogeneous boundary conditions), or

$$\text{(ii)} \;\; \lambda = \beta_i = 0, \quad i = 1,2$$

(homogeneous Neumann boundary conditions), or

$$\text{(iii)} \;\; 1 - \lambda = \beta_i = 0, \quad i = 1,2$$

(homogeneous Dirichlet boundary conditions).

ISSN 1072-947X / $8.00 /

Ω is an open bounded domain of the class $\mathbb{C}^1$ in $\mathbb{R}^N$, with boundary $\partial\Omega$, $\dfrac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$, the constants a_{ij} $(i,j = 1,2)$ are supposed to be positive and $(a_{12}+a_{21})^2 < 4a_{11}a_{22})$ which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is positive definite. The eingenvalues λ_1 and λ_2 $(\lambda_1 < \lambda_2)$ of A are positive. If we put

$$\underline{a} = \min\{a_{11}, a_{22}\} \quad \text{and} \quad \overline{a} = \max\{a_{11}, a_{22}\},$$

then the positivity of diffusion coefficients implies that

$$\lambda_1 < \underline{a} \le \overline{a} < \lambda_2.$$

The initial data are assumed to be in the region

$$\Sigma = \begin{cases} \left\{(u_0, v_0) \in \mathbb{R}^2 \text{ such that } \mu_2 v_0 \le u_0 \le \mu_1 v_0\right\} & \text{if } \mu_2\beta_2 \le \beta_1 \le \mu_1\beta_2, \\ \left\{(u_0, v_0) \in \mathbb{R}^2 \text{ such that } \frac{1}{\mu_2} u_0 \le v_0 \le \frac{1}{\mu_1} u_0\right\} & \text{if } \frac{1}{\mu_2}\beta_1 \le \beta_2 \le \frac{1}{\mu_1}\beta_1, \end{cases}$$

where

$$\mu_1 \equiv \frac{\underline{a} - \lambda_1}{a_{21}} > 0 > \mu_2 \equiv \frac{\underline{a} - \lambda_2}{a_{21}}.$$

One will treat the first case, the second one will be discussed in the last section. We suppose that the reaction terms f and g are continuously differentiable, polynomially bounded on Σ, $(f(r,s), g(r,s))$ is in Σ for all (r,s) in $\partial\Sigma$ (we say that (f,g) points into Σ on $\partial\Sigma$); i.e.,

$$\mu_2 g(\mu_2 s, s) \le f(\mu_2 s, s) \quad \text{and} \quad f(\mu_1 s, s) \le \mu_1 g(\mu_1 s, s), \quad \text{for all } s \ge 0, \tag{1.5}$$

and for positive constants C and $\alpha > -\mu_2$ sufficiently close to $-\mu_2$, we have

$$f(u,v) + Cg(u,v) \le C_1(u + \alpha v + 1) \quad \text{for all } u \text{ and } v \text{ in } \Sigma \tag{1.6}$$

where C_1 is a positive constant.

In the trivial case where $a_{12} = a_{21} = a_{11} - a_{22} = 0$; nonnegative solutions exist globally in time. In diagonal case where $a_{12} = a_{21} = 0$, $a_{11} \ne a_{22}$ and homogeneous Neumann boundary conditions, Alikakos [1] established the global existence and L^∞-bounds of solutions for positive initial data when

$$g(u,v) = -f(u,v) = uv^\beta, \tag{$*$}$$

and $1 < \beta < \frac{(N+2)}{N}$. The reactions given by $(*)$ satisfy in fact a condition analogous to (1.5) and form a special case since (f,g) point into Σ on $\partial\Sigma$ by taking $\Sigma = \mathbb{R}^+ \times \mathbb{R}^+$. Masuda [21] showed that solutions to this system exist globally for every $\beta > 1$ and converge to a constant vector as $t \to +\infty$. Haraux and Youkana [6] generalized the method of Masuda [21] to handle nonlinearities $uF(v)$ that form a particular case since they also assumed that $\Sigma = \mathbb{R}^+ \times \mathbb{R}^+$. Recently Kouachi and Youkana [19] generalized the method of Haraux and Youkana [6] to the triangular case $(a_{12} = 0)$ by taking nonlinearities $f(u,v)$ of a weak exponential growth. Kanel and Kirane [10] proved the global existence, in the case where $g(u,v) = -f(u,v) = -uv^n$ and n is an odd integer, under the embarrassing condition

$$|a_{12} - a_{21}| < C_p,$$

where C_p contains a constant from Solonnikov's estimate. Later they improved their results in [11] to obtain the global existence under the restrictions

H_1. $a_{22} < a_{11} + a_{21}$,

H_2. $a_{12} < \varepsilon_0 \equiv \left(\dfrac{a_{11}a_{22}(a_{11} + a_{21} - a_{22})}{a_{11}a_{22} + (a_{11} + a_{21} - a_{22})} \right)$ if $a_{11} \leq a_{22} < a_{11} + a_{21}$,

H_3. $a_{12} < \min\left\{ \dfrac{1}{2}(a_{11} + a_{21}), \varepsilon_0 \right\}$,

and

$$|F(v)| \leq C_F(1 + |v|^{1+\varepsilon},$$

where ε and C_F are positive constants with $\varepsilon < 1$ sufficiently small and $g(u,v) = -f(u,v) = uF(v)$. All the techniques used by the authors cited above showed their limitations because some are based on the embedding theorem of Sobolev (Alikakos [1], Hollis, Martin, and Pierre [8], Masuda [21]), while others (Kanel and Kirane [11]) use the properties of the Neumann function for a heat equation, for which one of its restriction is that the coefficient of $-\Delta u$ in equation (1.1) must be larger than that of $-\Delta v$ in equation (1.2), whereas it is not the case with problem (1.1)–(1.4).

This article is the continuation of [16], where $a_{11} = a_{22}$ and $\sigma g + \rho f \equiv 0$ with σ and ρ being any positive constants and the function $g(u,v)$ is positive and polynomially bounded. In [16] we considered the homogeneous Neumann boundary conditions and established the global existence of solutions with initial data in an invariant region, which is a special case of the one considered here. In [18] and in the case where $a_{11} = a_{22}$, we eliminate the balance condition and replace it by a condition analogous to (1.6).

The components $u(t,x)$ and $v(t,x)$ represent either chemical concentrations or biological population densities and system (1.1)–(1.2) is a mathematical model describing various chemical and biological phenomena (Cussler [2], Garcia-Ybarra and Clavin [4], De Groot and Mazur [5], Jorne [9], Kirkaldy [14], Lee and Hill [20] and Savchik, Changs, and Rabitz [23].

2. Local Existence and Invariant Regions

In this section, we prove that if (f,g) points into Σ on $\partial\Sigma$, then Σ is an invariant region for problem (1.1)–(1.4), i.e., the solution remains in Σ for any initial data in Σ. Once the invariant regions are constructed, both problems of the local and global existence become easier to be established: for the first problem we demonstrate that system (1.1)–(1.2) with the boundary conditions (1.3) and the initial data in Σ is equivalent to a problem for which the local existence throughout the time interval $[0, T_{\max}[$ can be obtained by the known procedure and for the second one, since we use usual techniques based on Lyapunov functionals which are not directly applicable to problem (1.1)–(1.4) and need invariant regions (Kirane and Kouachi [12], [13], Kouachi [15], [16] and Kouachi and Youkana [19]).

The main result of this subsection is

Proposition 2.1. *Suppose that* (f, g) *points into* Σ *on* $\partial\Sigma$, *then for any* (u_0, v_0) *in* Σ *the solution* $(u(t, .), v(t, .))$ *of problem* (1.1)–(1.4) *remains in* Σ *for all* t *in* $[0, T^*[$.

Proof. Let $\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$ and $\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$ be the eigenvectors of the matrix A^t associated with its eigenvalues λ_1 and λ_2 ($\lambda_1 < \lambda_2$). For fixed $i = 1, 2$, multiplying equation (1.1) by x_{i1} and equation (1.2) by x_{i2} and adding the resulting equations, we get

$$\frac{\partial w_1}{\partial t} - \lambda_1 \Delta w_1 = x_{11} f + x_{12} g = F_1(w_1, w_2) \quad \text{in} \quad]0, T^*[\times \Omega, \tag{2.1}$$

$$\frac{\partial w_2}{\partial t} - \lambda_2 \Delta w_2 = x_{21} f + x_{22} g = F_2(w_1, w_2) \quad \text{in} \quad]0, T^*[\times \Omega, \tag{2.2}$$

with the boundary conditions

$$\lambda w_i + (1 - \lambda) \frac{\partial w_i}{\partial \eta} = \rho_i, \quad i = 1, 2, \quad \text{on} \quad]0, T^*[\times \partial\Omega, \tag{2.3}$$

and the initial data

$$w_i(0, x) = w_i^0(x), \quad i = 1, 2, \quad \text{in} \quad \Omega, \tag{2.4}$$

where

$$\begin{aligned} w_i &= (x_{i1} u + x_{i2} v)(t, x), \quad i = 1, 2, \quad \text{in} \quad]0, T^*[\times \Omega, \\ \rho_i &= x_{i1} \beta_1 + x_{i2} \beta_2, \quad i = 1, 2, \end{aligned} \tag{2.5}$$

and

$$F_i(w_1, w_2) = x_{i1} f + x_{i2} g, \quad i = 1, 2, \quad \text{for all } u \text{ and } v \text{ in } \Sigma. \tag{2.6}$$

First, as it has been mentioned above, note that the condition of the parabolicity of system (1.1)–(1.2) implies of the one of system (2.1)–(2.2); since $(a_{12}+a_{21})^2 < 4a_{11}a_{22} \Rightarrow \det A = a_{11}a_{22} - a_{12}a_{21} > 0$. Since λ_1 and λ_2 ($\lambda_1 < \lambda_2$) are the eingenvalues of the matrix A^t, problem (1.1)–(1.4) is equivalent to problem (2.1)–(2.4) and to prove that Σ is an invariant region for system (1.1)–(1.2) it suffices to prove that the region

$$\left\{(w_1^0, w_2^0) \in \mathbb{R}^2 \text{ such that } w_i^0 \geq 0,\, i = 1, 2\right\} = \mathbb{R}^+ \times \mathbb{R}^+, \tag{2.7}$$

is invariant for system (2.1)–(2.2) and that

$$\Sigma = \left\{(u_0, v_0) \in \mathbb{R}^2 \text{ such that } w_i^0 = (x_{i1} u_0 + x_{i2} v_0) \geq 0, \;\; i = 1, 2\right\}. \tag{2.8}$$

Since $\begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix}$ is an eingenvector of A^t associated to the eingenvalue λ_i, $i = 1, 2$, then if we assume without loss of generality that $a_{11} \leq a_{22}$ we have $(a_{11}-\lambda_i)x_{i1}+$

$a_{21}x_{i2} = 0$, $i = 1, 2$. If we choose $x_{12} = \dfrac{1}{\mu_1}$ and $x_{22} = -\dfrac{1}{\mu_2}$, then $x_{i1}u_0 + x_{i2}v_0 \geq 0$, $i = 1, 2, \Leftrightarrow \dfrac{1}{\mu_1}\left(-\dfrac{1}{\mu_1}u_0 + v_0\right) \geq 0$ and $-\dfrac{1}{\mu_2}\left(-\dfrac{1}{\mu_2}u_0 + v_0\right) \geq 0$. Then the first inequality is equivalent to $u_0 \leq \mu_1 v_0$ and since $\mu_2 < 0$, the second one is equivalent to $u_0 \geq \mu_2 v_0$. Then (2.8) is proved and (2.5) can be written

$$w_1 = -u + \mu_1 v \quad \text{and} \quad w_2 = u - \mu_2 v. \tag{2.5$'$}$$

Now, to prove that the region $\mathbb{R}^+ \times \mathbb{R}^+$ is invariant for system (2.1)–(2.2), it suffices to show that $F_1(w_1, w_2) \geq 0$ for all (w_1, w_2) such that $w_1 = 0$ and $w_2 \geq 0$ and $F_2(w_1, w_2) \geq 0$ for all (w_1, w_2) such that $w_1 \geq 0$ and $w_2 = 0$ thanks to the invariant region's method (Smoller [24]). But using expressions (2.7), we get

$$F_1(w_1, w_2) = -f + \mu_1 g \quad \text{and} \quad F_2(w_1, w_2) = f - \mu_2 g. \tag{2.6$'$}$$

Following the same reasoning as above and taking into account that $v_0 \geq 0$ in Σ, we come to condition (1.5). Then Σ is an invariant region for the system (1.1)–(1.3). □

Then system (1.1)–(1.2) with the boundary conditions (1.3) and initial data in Σ is equivalent to system (2.1)–(2.2) with the boundary conditions (2.3) and positive initial data (2.4). As it has been mentioned at the beginning of this section and since ρ_1 and ρ_2 given by

$$\rho_1 = -\beta_1 + \mu_1\beta_2 \text{ and } \rho_2 = \beta_1 - \mu_2\beta_2$$

are positive, we have for any initial data in $\mathbb{C}\left(\overline{\Omega}\right)$ or $\mathbb{L}^p(\Omega)$, $p \in (1, +\infty)$, local existence and uniqueness of solutions to the initial value problem (2.1)–(2.4) and consequently those of problem (1.1)–(1.4) follow from the basic existence theory for abstract semilinear differential equations (Friedman [3], Henry [7] and Pazy [22]). The solutions are classical on $]0, T^*[$, where T^* denotes the eventual blow up time in $\mathbb{L}^\infty(\Omega)$. The local solution is continued globally by a priori estimates.

Once invariant regions are constructed, one can apply the Lyapunov technique and establish the global existence of unique solutions for (1.1)–(1.4).

3. Global Existence

As the determinant of the linear algebraic system (2.5), with respect to variables u and v, is different from zero, to prove global existence of solutions of problem (1.1)–(1.4) one needs to prove it for problem (2.1)–(2.4). To this end, it is well known that (Henry [7]) it suffices to derive a uniform estimate of $\|F_1(w_1, w_2)\|_p$ and $\|F_2(w_1, w_2)\|_p$ on $[0, T^*[$ for some $p > N/2$, where $\|.\|_p$denotes the usual norms in spaces $\mathbb{L}^p(\Omega)$ defined by

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p \, dx, \quad 1 \leq p < \infty, \quad \text{and} \quad \|u\|_\infty = \underset{x \in \Omega}{\text{ess sup}}\, |u(x)|$$

Let us define, for any positive integer n, the finite sequence

$$\theta_i = \theta^{(n-i)^2}, \quad i = 0, 1, \ldots, n, \tag{3.1}$$

where θ is a positive constant sufficiently large such that

$$\theta > \frac{TrA}{2\sqrt{\det A}} \equiv \frac{(a_{11}+a_{22})}{2\sqrt{a_{11}a_{22}-a_{12}a_{21}}}. \tag{3.2}$$

The main result of this subsection is

Theorem 3.1. *Let* $(w_1(t,.), w_2(t,.))$ *be any positive solution of problem* (2.1)–(2.4). *Introduce the functional*

$$t \longrightarrow L(t) = \int_{\Omega} H_n\left(w_1(t,x), w_2(t,x)\right) dx, \tag{3.3}$$

where

$$H_n(w_1, w_2) = \sum_{i=0}^{n} C_n^i \theta_i w_1^i w_2^{n-i}. \tag{3.4}$$

Then the functional L *is uniformly bounded on the interval* $[0, T^*], T^* < T_{\max}$.

Proof. The proof is similar to that in [15]. Differentiating L with respect to t yields

$$\begin{aligned} L'(t) &= \int_{\Omega} \left[\frac{\partial H_n}{\partial w_1}\frac{\partial w_1}{\partial t} + \frac{\partial H_n}{\partial w_2}\frac{\partial w_2}{\partial t}\right] dx \\ &= \int_{\Omega} \left(\lambda_1 \frac{\partial H_n}{\partial w_1}\Delta w_1 + \lambda_2 \frac{\partial H_n}{\partial w_2}\Delta w_2\right) dx + \int_{\Omega} \left(\frac{\partial H_n}{\partial w_1}F_1 + \frac{\partial H_n}{\partial w_2}F_2\right) dx \\ &= I + J. \end{aligned}$$

By simple use of Green's formula we have

$$I = I_1 + I_2,$$

where

$$I_1 = \int_{\partial\Omega} \left(\lambda_1 \frac{\partial H_n}{\partial w_1}\frac{\partial w_1}{\partial \eta} + \lambda_2 \frac{\partial H_n}{\partial w_2}\frac{\partial w_2}{\partial \eta}\right) ds \tag{3.5}$$

(with ds as area element) and

$$I_2 = -\int_{\Omega} \left(\lambda_1 \frac{\partial^2 H_n}{\partial w_1^2}\left|\nabla w_1\right|^2 + (\lambda_1+\lambda_2)\frac{\partial^2 H_n}{\partial w_1 \partial w_2}\nabla w_1 \nabla w_2 + \lambda_2 \frac{\partial^2 H_n}{\partial w_2^2}\left|\nabla w_2\right|^2\right) dx. \tag{3.6}$$

First, let us calculate the first and second partial derivatives of H_n with respect to w_1 and w_2. We have

$$\frac{\partial H_n}{\partial w_1} = \sum_{i=1}^{n} iC_n^i \theta_i w_1^{i-1} w_2^{n-i} \quad \text{and} \quad \frac{\partial H_n}{\partial w_2} = \sum_{i=0}^{n-1} (n-i) C_n^i \theta_i w_1^i w_2^{n-i-1}.$$

Using the formula

$$iC_n^i = nC_{n-1}^{i-1} \quad \text{for all} \quad i = 1, \ldots, n \tag{3.7}$$

and replacing the index i by $i-1$, we get

$$\frac{\partial H_n}{\partial w_1} = n\sum_{i=0}^{n-1} C_{n-1}^{i}\theta_{i+1}w_1^{i}w_2^{n-1-i}. \tag{3.8}$$

For $\dfrac{\partial H_n}{\partial w_2}$, using (3.7) and the fact that

$$C_n^i = C_n^{n-i} \quad \text{for all} \quad i=0,\dots,n, \tag{3.9}$$

we get

$$\frac{\partial H_n}{\partial w_2} = n\sum_{i=0}^{n-1} C_{n-1}^{i}\theta_{i}w_1^{i}w_2^{n-1-i}. \tag{3.10}$$

Using formulas (3.8) and (3.10), we deduce by analogy

$$\frac{\partial^2 H_n}{\partial w_1^2} = n(n-1)\sum_{i=0}^{n-2} C_{n-2}^{i}\theta_{i+2}w_1^{i}w_2^{n-2-i}, \tag{3.11}$$

$$\frac{\partial^2 H_n}{\partial w_1\partial w_2} = n(n-1)\sum_{i=0}^{n-2} C_{n-2}^{i}\theta_{i+1}w_1^{i}w_2^{n-2-i} \tag{3.12}$$

and

$$\frac{\partial^2 H_n}{\partial w_2^2} = n(n-1)\sum_{i=0}^{n-2} C_{n-2}^{i}\theta_{i}w_1^{i}w_2^{n-2-i}. \tag{3.13}$$

Now we claim that there exists a positive constant C_2 independent of $t \in [0,T_{\max}[$ such that

$$I_1 \le C_2 \quad \text{for all} \quad t\in[0,T_{\max}[\,. \tag{3.14}$$

To see this, we follow the same reasoning as in [18]:

(i) If $0<\lambda<1$, using the boundary conditions (2.3) we get

$$I_1 = \int_{\partial\Omega}\left(\lambda_1\frac{\partial H_n}{\partial w_1}(\gamma_1-\sigma w_1)+\lambda_2\frac{\partial H_n}{\partial w_2}(\gamma_2-\sigma w_2)\right)ds,$$

where $\sigma=\dfrac{\lambda}{1-\lambda}$ and $\gamma_i=\dfrac{\rho_i}{1-\lambda}$, $i=1,\ 2$.

Since

$$\begin{aligned}H(w_1,w_2) &= \lambda_1\frac{\partial H_n}{\partial w_1}(\gamma_1-\sigma w_1)+\lambda_2\frac{\partial H_n}{\partial w_2}(\gamma_2-\sigma w_2)\\ &= P_{n-1}(w_1,w_2)-Q_n(w_1,w_2),\end{aligned}$$

where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $n-1$ and n and since the solution is positive, we obtain

$$\limsup_{(|w_1|+|w_2|)\to+\infty} H(w_1,w_2) = -\infty, \tag{**}$$

which proves that H is uniformly bounded on $\mathbb{R}_+^2$ and consequently (3.14).

(ii) If $\lambda=0$, then $I_1=0$ on $[0,T_{\max}[\,$.

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on $[0,T_{\max}[\times\Omega$ implies $\dfrac{\partial w_1}{\partial\eta}\leq 0$ and $\dfrac{\partial w_2}{\partial\eta}\leq 0$ on $[0,T_{\max}[\times\partial\Omega$. Consequently one again gets (3.14) with $C_2=0$

$$I_2=-n(n-1)\sum_{i=0}^{n-2}C_{n-2}^{i}$$
$$\times\int_{\Omega}w_1^iw_2^{n-2-i}\left(\lambda_1\theta_{i+2}\left|\nabla w_1\right|^2+\left(\lambda_1+\lambda_2\right)\theta_{i+1}\nabla w_1\nabla w_2+\lambda_2\theta_i\left|\nabla w_2\right|^2\right)dx.$$

Using (3.1) and (3.2) we deduce that the quadratic forms (with respect to ∇w_1 and ∇w_2) are positive since

$$\left(\left(\lambda_1+\lambda_2\right)\theta_{i+1}\right)^2-4\lambda_1\lambda_2\theta_i\theta_{i+2}=\theta_{i+1}^2\left(\left(\lambda_1+\lambda_2\right)^2-4\lambda_1\lambda_2\theta^2\right)<0, \qquad (3.15)$$
$$i=0,1,\ldots,n-2.$$

Then

$$I_2\leq 0. \qquad (3.16)$$

(3.8) and (3.10) together imply

$$J=n\sum_{i=0}^{n-1}C_{n-1}^{i}\int_{\Omega}\left[\left(\theta_{i+1}F_1(w_1,w_2)+\theta_iF_2(w_1,w_2)\right)w_1^iw_2^{n-1-i}\right]dx.$$

Using expressions (2.6)′, we get

$$\theta_{i+1}F_1(w_1,w_2)+\theta_iF_2(w_1,w_2)=\left(-\theta_{i+1}+\theta_i\right)f+\left(\mu_1\theta_{i+1}-\mu_2\theta_i\right)g$$
$$=\left(\mu_1\theta_{i+1}-\mu_2\theta_i\right)\left[\frac{\frac{\theta_i}{\theta_{i+1}}-1}{-\mu_2\frac{\theta_i}{\theta_{i+1}}+\mu_1}f+g\right].$$

Since the function $x\rightarrow\dfrac{x-1}{-\mu_2x+\mu_1}$ is increasing with $\lim\limits_{x\rightarrow+\infty}\dfrac{x-1}{-\mu_2x+\mu_1}=-\dfrac{1}{\mu_2}$ and since $\frac{\theta_i}{\theta_{i+1}}$ is sufficiently large when θ is chosen sufficiently large, by using condition (1.6) and relation (2.5)′ successively we get, for an appropriate constant C_3,

$$J\leq C_3\int_{\Omega}\left[\sum_{i=0}^{n-1}\left(w_1+w_2+1\right)C_{n-1}^{i}w_1^iw_2^{n-1-i}\right]dx.$$

Following the same reasoning as in [17], a straightforward calculation shows that

$$J\leq C_4L(t)\quad\text{on}\quad[0,T^*].$$

Then we have

$$\overset{\bullet}{L}(t)\leq C_5L(t)+C_6L^{(p-1)/p}(t)\quad\text{on}\quad[0,T^*].$$

Putting

$$Z=L^{1/p},$$

one gets

$$p\overset{\bullet}{Z} \leq C_5 Z + C_6.$$

The solution of this linear differential inequality gives the uniform boundedeness of the functional L on the interval $[0, T^*]$, which completes the proof of the theorem. □

Corollary 3.1. *Suppose that the functions $f(r,s)$ and $g(r,s)$ are continuously differentiable on Σ, point into Σ on $\partial\Sigma$ and satisfy condition* (1.6). *Then all solutions of* (1.1)–(1.4) *with initial data in Σ and uniformly bounded on Ω are in $\mathbb{L}^\infty(0, T^*; \mathbb{L}^p(\Omega))$ for all $p \geq 1$.*

Proof. The proof is an immediate consequence of Theorem 3.1, the trivial inequality

$$\int_\Omega (w_1(t,x) + w_2(t,x))^p \, dx \leq L(t) \quad \text{on} \quad [0, T^*[$$

and $(2.5)'$. □

Proposition 3.1. *Under the hypothesis of Corollary* 3.1*, if the reactions $f(r,s)$ and $g(r,s)$ are polynomially bounded, then all solutions of* (1.1)–(1.3) *with the initial data in Σ and uniformly bounded on Ω are global in time.*

Proof. As it has been mentioned above, it suffices to derive a uniform estimate of $\|F_1(w_1,w_2)\|_p$ and $\|F_2(w_1,w_2)\|_p$ on $[0, T^*[$ for some $p > n/2$. Since the functions $f(u,v)$ and $g(u,v)$ are polynomially bounded on Σ, by using relations (2.5) and (2.6) we get that so are $F_1(w_1,w_2)$ and $F_2(w_1,w_2)$ and the proof becomes an immediate consequence of Corollary 3.1. □

4. Final Remarks

If $\dfrac{1}{\mu_2}\beta_1 \leq \beta_2 \leq \dfrac{1}{\mu_1}\beta_1$, then system (1.1)–(1.2) can be rewritten as

$$\frac{\partial v}{\partial t} - a_{22}\Delta v - a_{21}\Delta u = \widetilde{f}(v,u) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.1$'$}$$

$$\frac{\partial u}{\partial t} - a_{12}\Delta v - a_{11}\Delta u = \widetilde{g}(v,u) \quad \text{in } \mathbb{R}^+ \times \Omega \tag{1.2$'$}$$

with the same boundary conditions (1.3) and initial data (1.4) and where

$$\widetilde{f}(v,u) = g(u,v) \quad \text{and} \quad \widetilde{g}(v,u) = f(u,v) \quad \text{for all} \quad (u,v) \text{ in } \mathbb{R}^2.$$

In this case, the diffusion matrix of the system becomes

$$A = \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}$$

and the new constants μ_1 and μ_2 become $\overline{\mu_1} = \dfrac{\underline{a} - \lambda_1}{a_{12}}$ and $\overline{\mu_2} = \dfrac{\underline{a} - \lambda_2}{a_{12}}$ which are equal respectively to $-\dfrac{1}{\mu_2}$ and $-\dfrac{1}{\mu_1}$. Then all the previous results remain valid in the region

$$\Sigma = \left\{ (u_0, v_0) \in \mathbb{R}^2 \ \text{ such that } \ \frac{1}{\mu_2} u_0 \leq v_0 \leq \frac{1}{\mu_1} u_0 \right\}.$$

$(\widetilde{f}, \widetilde{g})$ points into Σ on $\partial\Sigma$ if

$$\frac{1}{\mu_2}\widetilde{g}(\frac{1}{\mu_2}s, s) \leq \widetilde{f}(\frac{1}{\mu_2}s, s) \quad \text{and} \quad \widetilde{f}(\frac{1}{\mu_1}s, s) \leq \frac{1}{\mu_1}\widetilde{g}(\frac{1}{\mu_1}s, s) \ \text{ for all } \ s \geq 0,$$

which is equivalent to

$$\frac{1}{\mu_2} f(s, \frac{1}{\mu_2}s) \leq g(s, \frac{1}{\mu_2}s) \quad \text{and} \quad g(s, \frac{1}{\mu_1}s) \leq \frac{1}{\mu_1} f(s, \frac{1}{\mu_1}s) \ \text{ for all } \ s \geq 0, \quad (1.5)'$$

and condition (1.6) becomes, for an appropriate constant C_1,

$$\widetilde{f}(v, u) + C\widetilde{g}(v, u) \leq C_1 (v + \alpha u + 1) \ \text{ for all } u \text{ and } v \text{ in } \Sigma$$

for positive constants C and $\alpha > \overline{\mu_1}$ sufficiently close to $\overline{\mu_1}$, which can be interpreted as

$$g(u, v) + Cf(u, v) \leq C_1 (\alpha u + v + 1) \ \text{ for all } u \text{ and } v \text{ in } \Sigma \qquad (1.6)'$$

for positive constants C and $\alpha > -\frac{1}{\mu_2}$ sufficiently close to $-\frac{1}{\mu_2}$.

Acknowledgment

The author would like to thank the anonymous referee for his/her valuable suggestions.

The author thanks the Superior Education Ministry and the National Agency of Development and Scientific Research (ANDRU) for their financial support.

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(Received 20.02.2002; revised 8.12.2003)

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