A RECURSION FORMULA FOR THE COEFFICIENTS OF
ENTIRE FUNCTIONS SATISFYING AN ODE WITH
POLYNOMIAL COEFFICIENTS

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Abstract. A recursion formula for the coefficients of entire functions which
are solutions of linear differential equations with polynomial coefficients is
derived. Some explicit examples are developed.

The Newton sum rules for the powers of zeros of a class of entire functions
are constructed in terms of Bell polynomials.

Key words and phrases: Entire solutions of ODE with polynomial coef-
ficients, recursion formulas for coefficients, Newton sum rules for reciprocal
of zeros, Bell polynomials.

1. Introduction

The problem of finding the Newton sum rules of polynomial solutions of
an ordinary differential equation with polynomial coefficients in terms of the
coefficients of the same equation was considered in several articles [1], [2], [3], [4],
[5]. In particular, in [2] Buendia, Dehesa and Galvez were able to represent the
coefficients of the relevant polynomial solutions in terms of the above-mentioned
coefficients by proving the following result:

Proposition 1.1. Consider the polynomial eigenfunctions

\[ P_N(x) = \text{const} \sum_{k=0}^{N} (-1)^k \alpha_k x^{N-k} \]

of a linear differential operator of order \( n \)

\[ \sum_{i=0}^{n} g_i(x) f^{(i)}(x) = 0, \quad (1.1) \]

where the coefficients \( g_i(x) \) are polynomials of degree \( c_i \), defined by

\[ g_i(x) = \sum_{j=0}^{c_i} a_{ij} x^j. \quad (1.2) \]

Assume that \( P_N(x) = \text{const} \prod_{l=1}^{N} (x - x_l) \), where all \( x_l \) are different.
Then the coefficients of $P_N(x)$ can be computed recursively in terms of coefficients (1.2) of differential operators by means of the formula:

$$a_{i+q}^{(i)} = -\sum_{k=1}^{s} (-1)^k \alpha_{s-k} \sum_{i=0}^{n} \frac{(N-s+k)!}{(N-s+i)!} a_{i+q-k}^{(i)}; \quad (1.3)$$

where

$$q := \max\{c_i - i; \quad i = 0, 1, \ldots, n\}. \quad (1.4)$$

If $c_i \leq i$ ($i = 0, 1, \ldots, n$), then the differential operator (1.1) is called of hypergeometric type.

In the present article the above recursive formula is extended to the case of entire functions satisfying the differential equation (1.1). Some examples are given in Section 3.

This gives a possibility to find explicit formulas representing the Newton sum rules of the reciprocal of the zeros of the considered entire function in terms of coefficients (1.2). We do not give these formulas but in the concluding section we give the expression of the Newton sum rules of the reciprocal of the zeros of a class of entire functions in terms of their coefficients using Bell polynomials.

2. Extension of the Recursive Formula

We consider now an entire function

$$f(x) = \text{const} \sum_{k=0}^{\infty} (-1)^k \alpha_k x^k \quad (2.1)$$

satisfying the differential equation (1.1), with polynomial coefficients (1.2). Proposition 1.1 can be extended as follows:

**Proposition 2.1.** The coefficients of $f(x)$ are expressed in terms of coefficients (1.2) by means of the recurrent formulas

$$a_{s}^{(i)} = -\sum_{k=0}^{s} (-1)^k \alpha_{s-k} \sum_{i=0}^{n} \frac{(s-k)!}{(s-i)!} a_{i+k-q}^{(i)}; \quad (2.2)$$

where

$$q := \max\{i - c_i; \quad i = 0, 1, \ldots, n\}. \quad (2.3)$$

**Proof.** Assuming the normalization such that $\text{const} = 1$, it is sufficient to substitute the derivative

$$f^{(i)}(x) = \sum_{k=i}^{\infty} (-1)^k \frac{k!}{(k-i)!} \alpha_k x^{k-i}$$

into equation (1.1), thus obtaining

$$\sum_{i=0}^{n} \sum_{j=0}^{\infty} \sum_{k=i}^{\infty} (-1)^k a_{j}^{(i)} \alpha_{k} \frac{k!}{(k-i)!} x^{k+j-i} = 0. \quad (2.4)$$
By using the identity principle of power series, from equation (2.4) we obtain
\[ \sum_{k=0}^{s} (-1)^{s-k} \alpha_{s-k} \sum_{i=0}^{n} \frac{(s-k)!}{(s-k-i)!} a_{i+k-q}^{(i)} = 0. \] (2.5)
This latter equation gives our result. \( \square \)

3. Examples

Example 1. The modified Bessel equation. Consider the modified Bessel equation:
\[ x \frac{d^2y}{dx^2} + \frac{3 \, dy}{dx} + y = 0. \] (3.1)
In this equation the coefficients in (1.2) are given by
\[ a_2^{(2)} = 0, \quad a_1^{(2)} = 1, \quad a_0^{(2)} = 0, \]
\[ a_2^{(1)} = 0, \quad a_1^{(1)} = 0, \quad a_0^{(1)} = \frac{3}{2}, \]
\[ a_2^{(0)} = 0, \quad a_1^{(0)} = 0, \quad a_0^{(0)} = 1. \] (3.2)
Furthermore, \( q = 1. \) The normalized entire solution is given by
\[ y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma \left( k + \frac{3}{2} \right) k!} x^k. \] (3.3)
Recalling that \( \Gamma \left( k + \frac{3}{2} \right) = \sqrt{\pi} \frac{(2k+1)!!}{2^{k+1}}, \) we find
\[ y(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{2^k}{k!(2k+1)!!} x^k. \] (3.4)
Indeed, by using the recurrence formula (2.2) we find
\[ \alpha_0 = 1, \quad \alpha_1 = -\frac{2}{3}, \quad \alpha_2 = \frac{2^2}{3 \cdot 5 \cdot 2!}, \quad \alpha_3 = -\frac{2^3}{3 \cdot 5 \cdot 7 \cdot 3!}, \quad \ldots. \] (3.5)

Example 2. The Airy equation. Consider the Airy equation
\[ \frac{d^2y}{dx^2} - xy = 0. \] (3.6)
In this equation the coefficients in (1.2) are given by
\[ a_2^{(2)} = 0, \quad a_1^{(2)} = 0, \quad a_0^{(2)} = 1, \]
\[ a_2^{(1)} = 0, \quad a_1^{(1)} = 0, \quad a_0^{(1)} = 0, \]
\[ a_2^{(0)} = 0, \quad a_1^{(0)} = -1, \quad a_0^{(0)} = 0. \] (3.7)
Furthermore, \( q = 2. \) The normalized entire solution is given by
\[ y(x) = 2 \sum_{k=0}^{\infty} \frac{1}{3^{2k+\frac{2}{3}} k! \Gamma \left( k + \frac{2}{3} \right)} x^{3k}. \] (3.8)
Indeed, by using the recurrence formula (2.2) we find
\[
\alpha_0 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = 0, \\
\alpha_3 = \frac{1}{3 \cdot 2 \cdot 1!}, \quad \alpha_4 = 0, \quad \alpha_5 = 0, \\
\alpha_6 = \frac{1}{3^2 \cdot 2 \cdot 5 \cdot 2!}, \quad \alpha_7 = 0, \quad \alpha_8 = 0, \\
\alpha_9 = \frac{1}{3^3 \cdot 2 \cdot 5 \cdot 8 \cdot 3!}, \quad \alpha_{10} = 0, \quad \alpha_{11} = 0, \\
\ldots
\]

Example 3. A Laguerre-type exponential. Consider the Laguerre-type exponential
\[
y(x) = e_3(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^4}
\] (3.9)
(see [6]), satisfying the differential equation
\[
D_3Le_3(x) := (D + 7xD^2 + 6x^2D^3 + x^3D^4)e_3(x) = e_3(x).
\] (3.10)
In this equation the coefficients in (1.2) are given by
\[
a_4^{(4)} = 0, \quad a_3^{(4)} = 1, \quad a_2^{(4)} = 0, \quad a_1^{(4)} = 0, \quad a_0^{(4)} = 0, \\
a_4^{(3)} = 0, \quad a_3^{(3)} = 0, \quad a_2^{(3)} = 6, \quad a_1^{(3)} = 0, \quad a_0^{(3)} = 0, \\
a_4^{(2)} = 0, \quad a_3^{(2)} = 0, \quad a_2^{(2)} = 0, \quad a_1^{(2)} = 7, \quad a_0^{(2)} = 1, \\
a_4^{(1)} = 0, \quad a_3^{(1)} = 0, \quad a_2^{(1)} = 1, \quad a_1^{(1)} = 0, \quad a_0^{(1)} = 1, \\
a_4^{(0)} = 0, \quad a_3^{(0)} = 0, \quad a_2^{(0)} = 0, \quad a_1^{(0)} = 0, \quad a_0^{(0)} = -1.
\] (3.11)
Furthermore, \( q = 1 \). Indeed, by using the recurrence formula (2.2) we find
\[
\alpha_0 = 1, \quad \alpha_1 = 1, \quad \alpha_2 = \frac{1}{(2!)^4}, \quad \alpha_3 = \frac{1}{(3!)^4}, \quad \alpha_4 = \frac{1}{(4!)^4}, \quad \ldots
\]

4. Newton Sum Rules and Bell Polynomials

It is well known that by the Weierstrass factorization theorem an entire function \( f(z) \) can be represented in terms of the set of its zeros \( \{z_\ell\}_{\ell \in \mathbb{N}}, \ z_\ell \neq 0 \), in the form
\[
f(z) = z^m e^{\varphi(z)} \prod_{\ell=1}^{\infty} \left[ 1 - \frac{z}{z_\ell} \right] \exp \left( \frac{z}{z_\ell} + \frac{z^2}{2z_\ell^2} + \cdots + \frac{z^\ell}{\ell z_\ell^\ell} \right),
\]
where \( \varphi(z) \) is a suitable entire function.
Consider now the entire functions which admit a particular factorization
\[
f(z) = \prod_{\ell=1}^{\infty} \left( 1 - \frac{z}{z_\ell} \right),
\] (4.1)
and denote $\zeta_\ell := \frac{1}{z_\ell}$, $\ell \in \mathbb{N}$, so that relation (4.1) becomes
\[
f(z) = \prod_{\ell=1}^{\infty} (1 - z_\ell) = 1 - \sigma_1 z + \sigma_2 z^2 + \cdots + (-1)^k \sigma_k z^k + \cdots,
\] (4.2)
where the variables $\sigma$ are defined by
\[
\sigma_1 = \sum_i \zeta_i, \quad \sigma_2 = \sum_{i<j} \zeta_i \zeta_j, \quad \ldots, \quad \sigma_k = \sum_{i_1<i_2<\cdots<i_k} \zeta_{i_1} \zeta_{i_2} \cdots \zeta_{i_k}, \quad \ldots.
\]
The relevant *Newton sum rules* are defined by
\[
s_1 = \sum_i \zeta_i, \quad s_2 = \sum_i \zeta_i^2, \quad \ldots, \quad s_k = \sum_i \zeta_i^k, \quad \ldots
\] (4.3)
(cf. [1]).

It is well-known (see, e.g., [7]) that the result related to the following representation formulas of the Newton sum rules by means of Bell polynomials $Y_k(f_1, g_1; f_2, g_2; \ldots ; f_k, g_k)$ hold:
\[
s_k = -\frac{1}{(k-1)!} Y_k \left(1, -\sigma_1; -1, 2! \sigma_2; \ldots ; (-1)^{k-1}(k-1)!, (-1)^k k! \sigma_k \right),
\] (4.4)
for any $k \in \mathbb{N}$.

It is worth to note that the above notation for the Bell polynomials is not the traditional one, used by Riordan, but it was probably introduced for the first time in [8], showing a link with the Fibonacci and Bernoulli numbers. As it is well-known, the Bell polynomials have wide applications (see, e.g., [9] and references therein, [10]).

Therefore, we can conclude that

**Proposition 4.1.** The formula (4.4) gives a representation of the Newton sum rules (4.3) in terms of the coefficients of the Taylor expansion (4.2) of the entire function (4.1).

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**References**


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