Georgian Mathematical Journal
Volume 11 (2004), Number 4, 681–690

EQUICONTINUITY AND QUASI-UNIFORMITIES

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Abstract. For topological spaces X, Y with a fixed compatible quasi-uniformity $\mathcal{Q}$ in Y and for a family $(f_i)_{i\in I}$ of mappings from X to Y, the notions of even continuity in the sense of Kelley, topological equicontinuity in the sense of Royden and $\mathcal{Q}$-equicontinuity (i.e., equicontinuity with respect to the topology of X and $\mathcal{Q}$) are compared. It is shown that $\mathcal{Q}$-equicontinuity implies even continuity, and if $\mathcal{Q}$ is locally symmetric, it implies topological equicontinuity too. It turns out that these notions are equivalent provided $\mathcal{Q}$ is a uniformity compatible with a compact topology, but the equivalence may fail even for a locally symmetric quasi-uniformity $\mathcal{Q}$ compatible with a compact metrizable topology.

2000 Mathematics Subject Classification: Primary 54C35; Secondary 54E15.
Key words and phrases: Topological space, quasi-uniform space, even continuity, topological equicontinuity, uniform equicontinuity.

1. Introduction

There exists a natural notion of equicontinuity for a set of mappings from a topological space (X, τ) to a uniform or quasi-uniform space $(Y, \mathcal{Q})$, which we call $(\tau, \mathcal{Q})$-equicontinuity. On the other hand, there are several notions of equicontinuity type which require only the presence of a topology η in the second space without any reference to a uniformity or quasi-uniformity compatible with η. The first of them is the notion of (τ, η)-even continuity [5]. A related notion of (τ, η)-topological equicontinuity was introduced in [11].

Our note originates with the following question: since, as a rule, there are many uniformities or quasi-uniformities compatible with a given topology, is it possible to find among them such a quasi-uniformity $\mathcal{Q}$ which reduces the checking of (τ, η)-even continuity or (τ, η)-topological equicontinuity to $(\tau, \mathcal{Q})$-equicontinuity?

First of all we show that $(\tau, \mathcal{Q})$-equicontinuity always implies $(\tau, \mathcal{T}_{\mathcal{Q}})$-even continuity (Proposition 3.3), and if $\mathcal{Q}$ is a locally symmetric quasi-uniformity, then $(\tau, \mathcal{Q})$-equicontinuity implies $(\tau, \mathcal{T}_{\mathcal{Q}})$-topological equicontinuity too (Proposition 3.5). It appears that, rather unexpectedly, the local symmerty is essential for the validity of the latter conclusion (Remark 3.6(1)). Using Propositions 3.3 and 3.5, we give the following partial positive answer to the above-mentioned question: if $\mathcal{Q}$ is a *uniformity* compatible with a compact topology, then $(\tau, \mathcal{T}_{\mathcal{Q}})$-even continuity, $(\tau, \mathcal{Q})$-equicontinuity and $(\tau, \mathcal{T}_{\mathcal{Q}})$-topological equicontinuity are

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

equivalent (Theorem 3.7). Finally, it is established that in Theorem 3.7 the uniformity cannot be replaced by locally symmetric quasi-uniformity compatible with a compact metrizable topology (Proposition 3.8).

2. Basic Definitions

2.1. **Quasi-uniformities.** For a topological space (X, τ) we denote

(np) by $\mathcal{N}_\tau(x)$ the collection of all τ-neighbourhoods of a point $x \in X$,
(ns) by $\mathcal{N}_\tau(K)$ the collection of all τ-neighbourhoods of a set $K \subset X$, and
(nd) by $\mathcal{D}_\tau$ the collection of all $\tau \times \tau$-neighbourhoods of the diagonal

$$\Delta_X := \{(x,x) \in X \times X \,|\, x \in X\}.$$

In the sequel, for a set X and a subset $G \subset X$ S_G will stand for the subset of $X \times X$ defined by the equality

$$S_G := (G \times G) \cup (X \setminus G) \times X.$$

If $\mathcal{P}$ is a set of subsets of $X \times X$, then, by definition, $\mathcal{P}^{-1} = \{P^{-1} : P \in \mathcal{P}\}$, where

$$P^{-1} := \{(y,x) \in X \times X \,|\, (x,y) \in P\} \text{ for } \subset X \times X.$$

A relation $P \subset X \times X$ is called *symmetric* if $P^{-1} = P$.

For a set X and two relations $U, V \subset X \times X$, as usual, we set

$$U \circ V := \{(x,y) \in X \times X \,|\, \exists z \in X \text{ such that } (x,z) \in V, (z,y) \in U\}.$$

All the terms and concepts, which are not defined below, are taken from [5].

A uniformity in a nonempty set X is a nonempty set $\mathcal{P}$ of subsets of $X \times X$ with the following properties:

(RF) $\mathcal{P}$ is a filter such that $\Delta_X \subset P$ for every $P \in \mathcal{P}$,
(DF) for any $P \in \mathcal{P}$ there exists $R \in \mathcal{P}$ such that $R \circ R \subset P$.
(SF) $\mathcal{P}^{-1} = \mathcal{P}$.

A quasi-uniformity in a non-empty set X is a non-empty set $\mathcal{P}$ of subsets of $X \times X$ with the properties (RF) and (DF).

If $\mathcal{P}$ is a quasi-uniformity, then $\mathcal{P}^{-1}$ is also a quasi-uniformity which is called the *conjugate* of $\mathcal{P}$. Therefore, a quasi-uniformity $\mathcal{P}$ is a uniformity provided it satisfies the *symmetry* condition (SF). Any member P of a given quasi-uniformity or uniformity is called *an entourage*.

Let $\mathcal{P}$ be a quasi-uniformity in X; a subfamily $\mathcal{P}_0 \subset \mathcal{P}$ is called:

— *a base* for $\mathcal{P}$ if every member of $\mathcal{P}$ contains some member of $\mathcal{P}_0$,

— *a subbase* for $\mathcal{P}$ if the family of finite intersections of members of $\mathcal{P}_0$ is a base for $\mathcal{P}$.

Every quasi-uniformity $\mathcal{P}$ induces in X the topology $\mathcal{T}_\mathcal{P}$ for which

$$\{P[x] \,|\, P \in \mathcal{P}\} = \mathcal{N}_{\mathcal{T}_\mathcal{P}}(x) \ \forall x \in X.$$

A topology τ in X is *compatible* with a quasi-uniformity $\mathcal{P}$ if $\tau = \mathcal{T}_\mathcal{P}$.

For a given topology τ in X the collection $\{S_G \,|\, G \in \tau\}$ is a subbase of a quasi-uniformity in X called *the Pervin quasi-uniformity* and which we denote by $\mathcal{Q}_{per}(\tau)$.

Several important properties of the Pervin's quasi-uniformity are listed in the next proposition.

Proposition 2.2. *Let X be a nonempty set and τ be a topology in X.*

(1) *(Pervin) $\mathcal{Q}_{per}(\tau)$ is compatible with τ.*

(2) *If τ is a $\mathfrak{T}_1$-topology, then $(\mathcal{Q}_{per}(\tau))^{-1}$ is compatible with the discrete topology in X.*

(3) *If τ is a $\mathfrak{T}_0$-topology, then $\mathcal{Q}_{per}(\tau)$ is a uniformity if and only if τ is a discrete topology.*

Proof. (1) See [10, p. 15, Theorem 1.19].

(2) Fix $x_0 \in X$ and let us find $G \in \tau$ such that if $P := (S_G)^{-1}$, then $P[x_0] = \{x_0\}$. Since τ is a $\mathfrak{T}_1$-topology, we have $G = (X \setminus \{x_0\}) \in \tau$. Clearly, $P[x_0] = \{x_0\}$.

(3) see [3, (2.35)]. □

For a given topology τ in X there exists a compatible uniformity if and only if (X, τ) is a completely regular topological space (see, e.g., [5, p. 188, Corollary 6.17]). It is also known that if τ is a compact regular topology in X, then $\mathcal{D}_\tau$ is a uniformity compatible with τ, and if two uniformities $\mathcal{U}_1, \mathcal{U}_2$ in a set X are compatible with a given compact topology τ, then $\mathcal{U}_1 = \mathcal{U}_2$ [5, p. 198, Corollary 6. 30].

A pair $(X, \mathcal{P})$, where X is a set and $\mathcal{P}$ is a (quasi-)uniformity is called a (quasi-)uniform space.

Every quasi-uniform space $(X, \mathcal{P})$ is endowed with the topology $\mathcal{T}_\mathcal{P}$ and hence is also treated as a topological space.

A quasi-uniform space $(Y, \mathcal{Q})$ as well as a quasi-uniformity $\mathcal{Q}$ are called *Lebesgue* [4] if for each open cover $\mathfrak{O}$ of $(Y, \mathcal{T}_\mathcal{Q})$ there is an entourage $Q \in \mathcal{Q}$ such that the family $(Q[y])_{y \in Y}$ refines $\mathfrak{O}$.

Proposition 2.3 (Lebesgue's covering lemma; [4, Proposition 5.1])**.** *If $(Y, \mathcal{Q})$ is a quasi-uniform space such that $(Y, \mathcal{T}_\mathcal{Q})$ is a compact topological space, then $(Y, \mathcal{Q})$ is a Lebesgue quasi-uniform space.*

A quasi-uniform space $(Y, \mathcal{Q})$ as well as the quasi-uniformity $\mathcal{Q}$ are called:

- *point-symmetric* ([4], [8, p. 887]) if for every $y \in Y$ and every $Q \in \mathcal{Q}$ there is a symmetric entourage $S \in \mathcal{Q}$ such that $S[y] \subset Q[y]$;
- *small-set symmetric* if $(Y, \mathcal{Q}^{-1})$ is point-symmetric [8, p. 887];
- *locally symmetric* if for every $y \in Y$ and every $Q \in \mathcal{Q}$ there is a symmetric entourage $S \in \mathcal{Q}$ such that $S \circ S[y] \subset Q[y]$.

Proposition 2.4. *Let $(Y, \mathcal{Q})$ be a quasi-uniform space.*

(a) [4, Proposition 2.21] *$(Y, \mathcal{Q})$ is point-symmetric if and only if $\mathcal{T}_\mathcal{Q} \subset \mathcal{T}_{\mathcal{Q}^{-1}}$.*

(b) ([10, Theorem 3.17], [4, Proposition 2.23]) *If $\mathcal{Q}$ is a locally symmetric quasi-uniformity, then $\mathcal{T}_\mathcal{Q}$ is a regular topology.*

(c_1) [4, Corollary of Proposition 5.2] *If $\mathcal{Q}$ is a Lebesgue quasi-uniformity such that $\mathcal{T}_\mathcal{Q}$ is a regular topology, then $\mathcal{Q}$ is a locally symmetric.*

(c_2) [4, Proposition 2.26] *If $\mathcal{T}_{\mathcal{Q}}$ is a compact regular topology, then $\mathcal{Q}$ is a locally symmetric quasi-uniformity.*

2.5. **Equicontinuities.** Let X, Y be sets, I a nonempty index set, and $\mathbb{F} := (f_i)_{i\in I}$ a family of mappings from X to Y; if $x_0 \in X$ is a point and $A \subset X$, $W \subset Y$, then

$$\mathbb{F}(x_0, W) := \{j \in I \mid f_j(x_0) \in W\}, \quad \mathbb{F}(A, W) := \{j \in I \mid f_j(A) \subset W\},$$
$$\mathbb{F}^{\natural}(A, W) := \{j \in I \mid f_j(A) \cap W \neq \varnothing\}.$$

Note that $\mathbb{F}(x_0, W) = \mathbb{F}(\{x_0\}, W) = \mathbb{F}^{\natural}(\{x_0\}, W)$ and if $x_0 \in A$, then $\mathbb{F}(x_0, W) \subset \mathbb{F}^{\natural}(A, W)$.

A given family of mappings $\mathbb{F} := (f_i)_{i\in I}$ from a topological space (X, τ) to a topological space (Y, η) is called:

- *evenly continuous at $x_0 \in X$ and $y_0 \in Y$* if for every $B \in \mathcal{N}_\eta(y_0)$ there are $A \in \mathcal{N}_\tau(x_0)$ and $W \in \mathcal{N}_\eta(y_0)$ such that $\mathbb{F}(x_0, W) \subset \mathbb{F}(A, B)$.
- *evenly continuous at $x_0 \in X$* if $\mathbb{F}$ is evenly continuous at x_0 and y for each $y \in Y$.
- *evenly continuous* if for every $x \in X$ we have that $\mathbb{F}$ is evenly continuous at x.

Moreover, $\mathbb{F}$ is called:

- *topologically equicontinuous at $x_0 \in X$ and $y_0 \in Y$* if for every $B \in \mathcal{N}_\eta(y_0)$ there are $A \in \mathcal{N}_\tau(x_0)$ and $W \in \mathcal{N}_\eta(y_0)$ such that $\mathbb{F}^{\natural}(A, W) \subset \mathbb{F}(A, B)$.
- *topologically equicontinuous at $x_0 \in X$* if $\mathbb{F}$ is topologically equicontinuous at x_0 and y for each $y \in Y$.
- *topologically equicontinuous* if for every $x \in X$ we have that $\mathbb{F}$ is topologically equicontinuous at x .

Let us also recall the usual notions of equicontinuity and uniform equicontinuity.

If (X, τ) is a topological space and $(Y, \mathcal{Q})$ is a quasi-uniform space, then $\mathbb{F}$ is called:

- $(\tau, \mathcal{Q})$-*equicontinuous at $x_0 \in X$* if $\forall Q \in \mathcal{Q}, \quad \exists A \in \mathcal{N}_\tau(x_0)$ such that
$$f_i(A) \subset Q[f_i(x_0)] \qquad \forall i \in I.$$
- $(\tau, \mathcal{Q})$-*equicontinuous* if $\mathbb{F}$ is $(\tau, \mathcal{Q})$-equicontinuous at every $x \in X$.

If $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ are both quasi-uniform spaces, then $\mathbb{F}$ is called:

- $(\mathcal{P}, \mathcal{Q})$-*uniformly equicontinuous* if $\forall Q \in \mathcal{Q}, \quad \exists P \in \mathcal{P}$ such that
$$f_i \times f_i(P) \subset Q \quad \forall i \in I,$$
where for a given $f : X \to Y$ the map $f \times f : X \times X \to Y \times Y$ at a point (x_1, x_2) is defined by the equality $f \times f(x_1, x_2) = (f(x_1), f(x_2))$.

The notion of even continuity, resp., topological equicontinuity is taken from [5, Chapter 7], resp., from [11, Part Three, §14.2 (p. 362)]. The notions of equicontinuiy and uniform equicontinuity are obvious (see [1], [10]). Since the definition of the latter notion in the case of quasi-uniform spaces is absolutely the same as in the case of uniform spaces, we shall follow [7, p. 191] and use the term "uniformly equicontinuous family" instead of the term "quasi-uniformly equicontinuous family".

Remark 2.6. Let (X,τ), (Y,η) be topological spaces, $x_0 \in X$ a point and $f : X \to Y$ a continuous mapping at x_0.

(1) The singleton $\mathbb{F} = \{f\}$ is (τ,η)-evenly continuous at x_0.

(2) The singleton $\mathbb{F} = \{f\}$ is (τ,η)-topologically equicontinuous at x_0 provided (Y,η) is either a regular or a Hausdorff space.

(3) The conclusion of (2) may not be true (even for a compact non-Hausdorff $\mathfrak{T}_1$-space).

Proof of (1). Take an arbitrary $y_0 \in Y$ and an open $B \in \mathcal{N}_\eta(y_0)$. If $f(x_0) \notin B$, then $\mathbb{F}(x_0, B) = \varnothing$ and the condition of even continuity at x_0 and y_0 is satisfied trivially for every $A \in \mathcal{N}_\tau(x_0)$ with $B = W$. If $f(x_0) \in B$, then, since B is open, $B \in \mathcal{N}_\eta(f(x_0))$; hence, by the continuity of f at x_0, we have that $A := f^{-1}(B) \in \mathcal{N}_\tau(x_0)$ and the condition of even continuity at x_0 and y_0 is satisfied now for A and $B = W$.

Proof of (2). Take arbitrarily $y_0 \in Y$. If $f(x_0) = y_0$, the continuity of f at x_0 implies trivially that $\mathbb{F}$ is topologically equicontinuous at x_0 and y_0.

Let $f(x_0) \neq y_0$ *and assume that* (Y,η) *is regular.* Take an open $B \in \mathcal{N}_\eta(y_0)$.

If $f(x_0) \in B$, then, by the continuity of f at x_0, $A := f^{-1}(B) \in \mathcal{N}_\tau(x_0)$ and the condition of topological equicontinuity at x_0 and y_0 is satisfied for A and $B = W$.

If $f(x_0) \notin B$, take *a closed* $W \in \mathcal{N}_\eta(y_0)$ with $W \subset B$ (here we need the regularity of (Y,η)). Then $B_1 = Y \setminus W \in \mathcal{N}_\eta(f(x_0))$; by the continuity of f at x_0 we have that $A_1 := f^{-1}(B_1) \in \mathcal{N}_\tau(x_0)$, and the condition of topological equicontinuity at x_0 and y_0 is satisfied trivially for A_1 and W (because $\mathbb{F}^\natural(A_1, W) = \varnothing \subset \mathbb{F}(A_1, B)$).

Let $f(x_0) \neq y_0$ *and assume that* (Y,η) *is Hausdorff.* Take an arbitrary $B \in \mathcal{N}_\eta(y_0)$. Since (Y,η) is Hausdorff, there are $W \in \mathcal{N}_\eta(y_0)$ and $B_0 \in \mathcal{N}_\eta(f(x_0))$ with $B_0 \cap W = \varnothing$. Then by the continuity of f at x_0 we have that $A := f^{-1}(B_0) \in \mathcal{N}_\tau(x_0)$ and the condition of topological equicontinuity at x_0 and y_0 is satisfied trivially for A and W (because $\mathbb{F}^\natural(A, W) = \varnothing \subset \mathbb{F}(A, B)$).

Proof of (3). Endow the set $\mathbf{Z}$ of integers with the cofinite topology τ and let $X = Y = \mathbf{Z}$. Consider the mapping $x \mapsto f(x) := x + 1$. Then f is continuous everywhere, but $\{f\}$ is not topologically equicontinuous at 0 and 0.

Indeed, the continuity of f is evident. Let now $B := \mathbf{Z}\setminus\{1\}$. Then $B \in \mathcal{N}_\tau(0)$. Observe that since τ is a cofinite topology, for every $A \in \mathcal{N}_\tau(0)$ and $W \in \mathcal{N}_\tau(0)$ we have that $f(A) \cap W \neq \varnothing$, but there does not exist $A \in \mathcal{N}_\tau(0)$ with $f(A) \subset B$. Hence $\{f\}$ is not topologically equicontinuous at 0 and 0. □

For a group $(X,+)$ and an element $a \in X$ we denote by r_a *the right translation map* $x \mapsto x + a$, while ι stands for *the group inversion map* $x \mapsto -x$.

It is clear from the definitions that if a family $\mathbb{F} := (f_i)_{i\in I}$ of mappings from (X, τ) to a (Y, η) is topologically equicontinuous (at $x \in X$ and $y \in Y$), then it is evenly continuous at ($x \in X$ and $y \in Y$). The next statement provides, in particular, the examples of evenly continuous families which are not topologically equicontinuous.

Proposition 2.7. *Let $(X, +)$ be an Abelian group and τ a topology in X.*

(a) *The family $(r_a)_{a\in X}$ is (τ, τ)-evenly continuous if and only if $+$ is $(\tau\times\tau, \tau)$-continuous.*

(b) *The family $(r_a)_{a\in X}$ is (τ, τ)-topologically equicontinuous if and only if $(X, +, \tau)$ is a topological group (i.e., $+$ is $(\tau \times \tau, \tau)$-continuous and ι is (τ, τ)-continuous as well).*

(c) *If $X = \mathbf{R}$ is the real line with the usual addition and σ is the Sorgenfrey topology in $\mathbf{R}$, then the family $(r_a)_{a\in X}$ is (σ, σ)-evenly continuous and is not (σ, σ)-topologically equicontinuous.*

Proof. (a) is a particular case of [2, Theorem 5.2].

(b) is a particular case of [2, Theorem 6.2].

(c) follows from (a) and (b) (because $+$ is $(\sigma\times\sigma, \sigma)$-continuous, but $(\mathbf{R}, +, \sigma)$ is not a topological group). □

3. Comparison of Different Types of Equicontinuities

For the sake of completeness we begin our consideration with the following known result.

Proposition 3.1. *Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be quasi-uniform spaces, $\mathbb{F} := (f_i)_{i\in I}$ a family of mappings from X to Y.*

(a) *If $\mathbb{F}$ is $(\mathcal{P}, \mathcal{Q})$-uniformly equicontinuous, then $\mathbb{F}$ is $(\mathcal{T}_{\mathcal{P}}, \mathcal{Q})$-equicontinuous.*

(b) [7, p. 210, Proposition 9] *If $(X, \mathcal{P})$ is a Lebesgue quasi-uniform space, $(Y, \mathcal{Q})$ is a uniform space and $\mathbb{F}$ is $(\mathcal{T}_{\mathcal{P}}, \mathcal{Q})$-equicontinuous, then $\mathbb{F}$ is $(\mathcal{P}, \mathcal{Q})$-uniformly equicontinuous.*

(c) [9] *If $(X, \mathcal{P})$ is a Lebesgue quasi-uniform space, $(Y, \mathcal{Q})$ is a small-set symmetric quasi-uniform space and $f : X \to Y$ is a $(\mathcal{T}_{\mathcal{P}}, \mathcal{T}_{\mathcal{Q}})$-continuous mapping, then f is $(\mathcal{P}, \mathcal{Q})$-uniformly continuous.*

Remark 3.2. (1) In Proposition 3.1(a) we can assert the $(\mathcal{T}_{\mathcal{P}}, \mathcal{T}_{\mathcal{Q}})$-even continuity (see Proposition 3.3), but not the $(\mathcal{T}_{\mathcal{P}}, \mathcal{T}_{\mathcal{Q}})$-topological equicontinuity (see Remark 3.6(1)).

(2) Until now Proposition 3.1(c) has been the best generalization of Heine-Cantor's theorem. In the case of a compact quasi-uniform space $(X, \mathcal{P})$ it was obtained in [6, Proposition 1]. It seems to be unknown whether or not an arbitrary equicontinuous family $\mathbb{F}$ from a Lebesgue quasi-uniform space to a small-set symmetric quasi-uniform space is uniformly equicontinuous.

In the next proposition, the item (b) is a local version of [5, Theorem 7.23], where it is proved by using of a gauge of pseudometrics. For the sake of self-containedness we present this statement with a direct (i.e., pseudometric free) proof.

Proposition 3.3. *Let (X,τ) be a topological space, $x_0 \in X$ a point, $(Y,\mathcal{Q})$ a quasi-uniform space, and $\mathbb{F} := (f_i)_{i\in I}$ a family of mappings from X to Y.*

(a) *If $\mathbb{F}$ is $(\tau,\mathcal{Q})$-equicontinuous at x_0, then it is $(\tau,\mathcal{T}_{\mathcal{Q}})$-evenly continuous at x_0.*

(b) [5, p. 237, Theorem 7.23] *If $\mathbb{F}$ is $(\tau,\mathcal{T}_{\mathcal{Q}})$-evenly continuous at x_0, the set $\mathbb{F}(x_0) = \{f_i(x_0) \,|\, i \in I\}$ has the $\mathcal{T}_{\mathcal{Q}}$-compact closure and $\mathcal{Q}$ is a uniformity, then $\mathbb{F}$ is $(\tau,\mathcal{Q})$-equicontinuous at x_0.*

Proof. (a) Take an arbitrary $y \in Y$ and $B \in \mathcal{N}_{\mathcal{T}_{\mathcal{Q}}}(y)$. There are $Q \in \mathcal{Q}$ and $R \in \mathcal{Q}$ such that $Q[y] \subset B$ and $R \circ R \subset Q$. By the $(\tau,\mathcal{Q})$-equicontinuity of $\mathbb{F}$ at x_0, there exists $A \in \mathcal{N}_\tau(x_0)$ such that $f_i(A) \subset R[f_i(x_0)] \quad \forall i \in I$. Set $W := R[y] \in \mathcal{N}_{\mathcal{T}_{\mathcal{Q}}}(y)$ and let $i \in \mathbb{F}(x_0, W)$. Then $(y, f_i(x_0)) \in R$. From this, since $(f_i(x_0), f_i(x)) \in R \; \forall x \in A$, we get $(y, f_i(x)) \in R \circ R \subset Q \quad \forall x \in A$. This means that $i \in \mathbb{F}(A, Q[y]) \subset \mathbb{F}(A,B)$. Consequently, $\mathbb{F}(x_0, W) \subset \mathbb{F}(A,B)$ and hence $\mathbb{F}$ is $(\tau,\mathcal{T}_{\mathcal{Q}})$-evenly continuous at x_0 and y.

(b) Fix $Q \in \mathcal{Q}$ and a symmetric $S \in \mathcal{Q}$ with $S \circ S \subset Q$. Let K be the $\mathcal{T}_{\mathcal{Q}}$-closure of $\mathbb{F}(x_0)$ and $y \in K$. Then $S[y] \in \mathcal{N}_{\mathcal{T}_{\mathcal{Q}}}(y)$. Since $\mathbb{F}$ is $(\tau,\mathcal{Q})$-equicontinuous at x_0 and y, there are $A_y \in \mathcal{N}_\tau(x_0)$ and an open $W_y \in \mathcal{N}_{\mathcal{T}_{\mathcal{Q}}}(y)$ such that $\mathbb{F}(x_0, W_y) \subset \mathbb{F}(A_y, S[y])$. As the family $(W_y)_{y\in K}$ covers K, there are a natural n and the elements $y_1, \ldots, y_n \in K$ with $K \subset \cup_{k=1}^n W_{y_k}$. Set $A := \cap_{k=1}^n A_k$.

Fix an arbitrary $i \in I$. Since $f_i(x_0) \in K$, there is $k \le n$ such that $f_i(x_0) \in W_{y_k}$. Then $i \in \mathbb{F}(A_{y_k}, S[y_k])$. In particular, $i \in \mathbb{F}(A, S[y_k])$, i.e., $f_i(A) \subset S[y_k]$. Therefore for a given $x \in A$ we have $(y_k, f_i(x)) \in S$ and $(y_k, f_i(x_0)) \in S$. This, because S is symmetric, gives

$$(f_i(x_0), f_i(x)) \in S \circ S \subset Q.$$

Hence, $f_i(A) \subset Q[f_i(x_0)]$ and $\mathbb{F}$ is $(\tau,\mathcal{Q})$-equicontinuous at x_0. □

Remark 3.4. We retain the notation of Proposition 3.3.

(1) It is easy to see that if the family $\mathbb{F} = (f_i)_{i\in I}$ is such that for some $y_0 \in Y$ we have $f_i(x_0) = y_0, \; \forall i \in I$, then the even continuity of $\mathbb{F}$ at x_0 and y_0 implies the $(\tau,\mathcal{Q})$-equicontinuity of $\mathbb{F}$ at x_0.

(2) In Proposition 3.3(a) it cannot be asserted that $\mathbb{F}$ is $(\tau,\mathcal{T}_{\mathcal{Q}})$-topologically equicontinuous at x_0 (see Proposition 3.5 and Remark 3.6).

Proposition 3.5. *Let (X,τ) be a topological space, $x_0 \in X$ a point, $(Y,\mathcal{Q})$ a locally symmetric quasi-uniform space, and $\mathbb{F} := (f_i)_{i\in I}$ a family of mappings from X to Y.*

If $\mathbb{F}$ is $(\tau,\mathcal{Q})$-equicontinuous at x_0, then it is $(\tau,\mathcal{T}_{\mathcal{Q}})$-topologically equicontinuous at x_0.

Proof. Take an arbitrary $y \in Y$ and $B \in \mathcal{N}_{\mathcal{T}_{\mathcal{Q}}}(y)$. There is $Q \in \mathcal{Q}$ such that $Q[y] \subset B$. By the local symmetry there are symmetric $S, S_1 \in \mathcal{Q}$ such that $S_1 \circ S_1[y] \subset Q[y]$ and $S \circ S[y] \subset S_1[y]$. We can suppose that $S \subset S_1$. Then $S \circ S \circ S[y] \subset S \circ S_1[y] \subset S_1 \circ S_1[y] \subset Q[y]$.

By the $(\tau,\mathcal{Q})$-equicontinuity of $\mathbb{F}$ at x_0, we can find $A \in \mathcal{N}_\tau(x_0)$ such that $f_i(A) \subset S[f_i(x_0)] \quad \forall i \in I$. Set $W = S[y] \in \mathcal{N}_{\mathcal{T}_{\mathcal{Q}}}(y)$, take $i \in \mathbb{F}^\natural(A, W)$ and let us see that then $i \in \mathbb{F}(A,B)$.

Since $f_i(A)\cap W \neq \varnothing$, we have a $x_1 \in A$ such that $(y, f_i(x_1)) \in S$. But we also have $(f_i(x_1), f_i(x_0)) \in S$ and $(f_i(x_0), f_i(x)) \in S \quad \forall x \in A$. Then $(y, f_i(x)) \in S \circ S \circ S \quad \forall x \in A$ and $f_i(A) \in S\circ S\circ S[y] \subset Q[y]$. This means that $i \in \mathbb{F}(A, Q[y]) \subset \mathbb{F}(A, B)$. Consequently, $\mathbb{F}^{\natural}(A, W) \subset \mathbb{F}(A, B)$ and hence $\mathbb{F}$ is $(\tau, \mathcal{T}_{\mathcal{Q}})$-topologically continuous at x_0 and y. □

Remark 3.6. (1) Proposition 3.5 may not be true without the assumption of the local symmetry for $(Y, \mathcal{Q})$.

Indeed, let $X = Y = \mathbf{R}$ and $\mathcal{S}$ be the quasi-uniformity in $\mathbf{R}$ having a base $\{Q_\varepsilon | \varepsilon \in]0, 1[\}$, where $Q_\varepsilon := \{(x, y) \in \mathbf{R^2} \,|\, y - x \in [0, \varepsilon[\}$. Then $\sigma = \mathcal{T}_{\mathcal{S}}$ is the Sorgenfrey topology in $\mathbf{R}$. It is easy to see that the family $(r_a)_{a\in\mathbf{R}}$ is $(\mathcal{S}, \mathcal{S})$-*uniformly equicontinuous*; hence it is $(\sigma, \mathcal{S})$-equicontinuous as well (see Proposition 3.1(a)). However, according to Proposition 2.7(c), the family $(r_a)_{a\in\mathbf{R}}$ is not (σ, σ)-topologically equicontinuous.

(2) Note that $(\mathbf{R}, \sigma)$ is a completely regular Hausdorff space, hence it admits (even) a compatible uniformity, the family $(r_a)_{a\in\mathbf{R}}$ is (σ, σ)-evenly continuous (see Proposition 2.7(c)), but there does not exist in $\mathbf{R}$ a **locally symmetric** quasi-uniformity $\mathcal{Q}$ with $\mathcal{T}_{\mathcal{Q}} = \sigma$, for which the family $(r_a)_{a\in\mathbf{R}}$ would be $(\sigma, \mathcal{Q})$-equicontinuous.

Indeed, if $\mathcal{Q}$ is a locally symmetric quasi-uniformity in $\mathbf{R}$ with $\mathcal{T}_{\mathcal{Q}} = \sigma$, then, by Proposition 3.5, the $(\sigma, \mathcal{Q})$-equicontinuity of the family $(r_a)_{a\in\mathbf{R}}$ would imply its (σ, σ)-topological equicontinuity, which we do not have by Proposition 2.7(c).

The next assertion provides a positive answer to the question from the introduction when the second space is compact regular.

Theorem 3.7. *Let (X, τ) be a topological space, (Y, η) a compact regular topological space, and $\mathcal{Q}$ the unique uniformity in Y compatible with η.*

Then for a point $x_0 \in X$ and a family $\mathbb{F} := (f_i)_{i\in I}$ of mappings from X to Y the following statements are equivalent:

(i) $\mathbb{F}$ *is (τ, η)-evenly continuous at x_0.*

(ii) $\mathbb{F}$ *is $(\tau, \mathcal{Q})$-equicontinuous at x_0.*

(iii) $\mathbb{F}$ *is (τ, η)-topologically equicontinuous at x_0.*

Proof. (i)$\Longrightarrow$(ii) follows from Proposition 3.3(b).

(ii)$\Longrightarrow$(iii) follows from Proposition 3.5 (because any uniformity is locally symmetric).

(iii)$\Longrightarrow$(i) is trivial. □

The next statement shows that in Theorem 3.7 the compatible *uniformity* cannot be replaced by a compatible *locally symmetric quasi-uniformity* (even in the case of compact metrizable topological spaces).

Proposition 3.8. *Let $(X, +, \theta, \tau)$ be a Hausdorff topological group and $\mathcal{Q}_{per}(\tau)$ be the Pervin quasi-uniformity in X.*

(1) *If the family $(r_a)_{a\in X}$ of right translations is $(\tau, \mathcal{Q}_{per}(\tau))$-equicontinuous, then τ is a discrete topology.*

(2) *If* $(X, +, \theta, \tau)$ *is an infinite compact Hausdorff topological group, then the family* $(r_a)_{a \in X}$ *is* (τ, τ)*-topologically equicontinuous, but it is not* $(\tau, \mathcal{Q}_{per}(\tau))$*-equicontionuous* (*although* $\mathcal{Q}_{per}(\tau)$ *is a locally symmetric quasi-uniformity, see Proposition* 2.4(c)).

Proof. (1) We can assume that $X \neq \{\theta\}$. Take $G := X \setminus \{\theta\}$. Then $G \in \tau$ and $G \neq \varnothing$. Since $\{r_a \,|\, a \in X\}$ is $(\tau, \mathcal{Q}_{per}(\tau))$-equicontionuous at θ and the set S_G is an entourage in $\mathcal{Q}_{per}(\tau)$, we obtain that for some symmetric $A \in \mathcal{N}_\tau(\theta)$,

$$A + a = r_a(A) \in S_G[r_a(\theta)] = S_G[a] \;\; \forall a \in X. \tag{3.8.1}$$

Since for $a \in G$ we have $S_G[a] = G$ if and only if $a \in G$, from (3.8.1) we get

$$A + a \subset G \;\; \forall a \in G. \tag{3.8.2}$$

If $A \neq \{\theta\}$, then for some $a \in A \cap G$ we have $A + a \subset G$. Since A is symmetric, $-a \in A$. Therefore $\theta \in A + a \in G$. A contradiction. Consequently, $A = \{\theta\}$ and so, θ is isolated in (X, τ).

(2) The family $\{r_a \,|\, a \in X\}$ is (τ, τ)-topologically equicontinuous in every topological group (see Proposition 2.7(b) or [11, Proposition 4.13]). It cannot be $(\tau, \mathcal{Q}_{per}(\tau))$-equicontinuous by (1) (because an infinite compact topological space cannot be discrete). □

Acknowledgement

The authors are grateful to the referee for the careful reading of the paper and helpful remarks.

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(Received 15.09.2004)

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