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ON GENERALIZED STEP-FUNCTIONS AND SUPERPOSITION OPERATORS

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Abstract. For a given σ-ideal of sets, the notion of a generalized step-function is introduced and investigated in connection with the problem of sup-measurability of certain functions of two variables, regarded as superposition operators.

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Let R denote the real line and let $\Phi : R \times R \to R$ be a function of two variables. Then this Φ can be treated as a superposition operator defined as follows: for any function $f : R \to R$, we put

$$(\Phi(f))(x) = \Phi(x, f(x)) \ \ (x \in R).$$

Sometimes, Φ is also called a Nemytskii superposition operator.

Let λ denote the standard Lebesgue measure on R. In many cases, it is important to know whether a given superposition operator Φ preserves the class $L(R)$ of all real-valued Lebesgue measurable functions on R (i.e., $\Phi(f)$ is λ-measurable whenever f is λ-measurable). There are various sufficient conditions under which Φ maps $L(R)$ into itself.

In particular, if Φ is λ-measurable with respect to the first variable and continuous with respect to the second variable (the so-called Carathéodory classical conditions), then Φ preserves $L(R)$ or, in short, Φ is sup-measurable. In such a case, Φ is also λ_2-measurable where $\lambda_2 = \lambda \times \lambda$ stands for the two-dimensional Lebesgue measure on the plane R^2.

Other conditions for the sup-measurability of Φ can be found, e.g., in [1].

Under some additional set-theoretical axioms, there exist sup-measurable Φ which are not λ_2-measurable (see, for instance, [2], [3], [4], [5]). On the other hand, as Shelah and Roslanowski have recently announced, the statement "all sup-measurable operators Φ are λ_2-measurable" is consistent with ZFC theory.

In this paper, we are focused on the following problem: give a characterization of all those functions $f \in L(R)$ for which there exists a superposition operator Φ having rather good descriptive properties and such that $\Phi(f)$ does not belong to $L(R)$. In order to solve this problem, we need several auxiliary notions and propositions. First of all let us recall the following classical statement from descriptive set theory.

Lemma 1. *Let E be a Polish topological space, E' be a metric space, and let $h : E \to E'$ be a continuous mapping whose range* $\mathrm{ran}(h)$ *is uncountable. Then*

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there exists a set $C \subset E$ homeomorphic to the Cantor discontinuum such that the restriction $h \mid C$ is injective (hence $h \mid C$ is a homeomorphism between C and $h(C)$).

For the proof of this lemma see, e.g., [6].

Recall that $f \in L(R)$ is a step-function if $\mathrm{card}(\mathrm{ran}(f)) \leq \omega$, i.e., the range of f is at most countable. We shall say that $f \in L(R)$ is a generalized step-function if there exists at least one step-function $g \in L(R)$ such that f and g are equivalent (with respect to the measure λ).

Lemma 2. *If $f \in L(R)$, then the following two assertions are equivalent:*

1) f is not a generalized step-function;

2) there exists a set $Y \subset R$ with $\lambda^*(Y) > 0$ such that the restriction $f \mid Y$ is injective.

Proof. The implication 2) ⇒ 1) is trivial. Let us prove the implication 1) ⇒ 2). Suppose that $f \in L(R)$ satisfies 1). Let us denote

$$T_0 = \{t \in \mathrm{ran}(f) : \lambda(f^{-1}(t)) > 0\}.$$

Evidently, we have $\mathrm{card}(T_0) \leq \omega$. Since f is not a generalized step-function, we also have $\lambda(R \setminus f^{-1}(T_0)) > 0$. Moreover, applying the classical Luzin theorem to f, we claim that there exists a closed set $P \subset R \setminus f^{-1}(T_0)$ with $\lambda(P) > 0$ for which the restriction $f \mid P$ is continuous and $\mathrm{card}(\mathrm{ran}(f \mid P)) > \omega$.

Let us put $h = f \mid P$ and $T = \mathrm{ran}(h)$. Then $\lambda(h^{-1}(t)) = 0$ for each $t \in T$. Denote by α the least ordinal number of cardinality continuum and let $(P_\xi)_{\xi<\alpha}$ be an injective family of all closed subsets of P having strictly positive λ-measure. Construct, by using the method of transfinite recursion, a family $\{y_\xi : \xi < \alpha\}$ of points of P. Namely, take an ordinal $\xi < \alpha$ and suppose that the partial family $\{y_\zeta : \zeta < \xi\}$ has already been defined. Keeping in mind Lemma 1, it is not difficult to check that

$$P_\xi \setminus \cup\{h^{-1}(h(y_\zeta)) : \zeta < \xi\} \neq \varnothing.$$

Hence there exists a point y belonging to $P_\xi \setminus \cup\{h^{-1}(h(y_\zeta)) : \zeta < \xi\}$.

We put $y_\xi = y$. By proceeding in the same manner as above, the required family of points $\{y_\xi : \xi < \alpha\}$ will be constructed. Denote now $Y = \{y_\xi : \xi < \alpha\}$. It follows directly from our construction that Y is a partial selector of the disjoint family of sets $\{h^{-1}(t) : t \in T\}$. This implies that the restriction $h \mid Y$ (consequently, the restriction $f \mid Y$) is injective. Moreover, since $P_\xi \cap Y \neq \varnothing$ for each $\xi < \alpha$, we easily infer that $\lambda^*(Y) = \lambda(P) > 0$. This completes the proof of Lemma 2. □

Lemma 3. *If $f \in L(R)$ is not a generalized step-function, then there exists a λ-nonmeasurable set $X \subset R$ for which the restriction $f \mid X$ is injective.*

Proof. According to Lemma 2, there exists a set $Y \subset R$ with $\lambda^*(Y) > 0$ such that $f \mid Y$ is an injection. If Y is not measurable in the Lebesgue sense, then we are done. Suppose now that $Y \in \mathrm{dom}(\lambda)$ and hence $\lambda(Y) > 0$. It is well known (see, e.g., [7] or [8]) that Y contains a nonmeasurable subset with respect to λ.

Take any such subset and denote it by X. Clearly, $f \mid X$ is an injection and the proof is completed. □

Theorem 1. *Let $f \in L(R)$ and suppose that f is not a generalized step-function. Then there exists a superposition operator $\Phi : R \times R \to R$ satisfying the following relations:*

1) $\mathrm{ran}(\Phi) = \{0, 1\}$*;*
2) *for any $x \in R$, the partial function $\Phi(x, \cdot)$ is lower semi-continuous;*
3) *for any $y \in R$, the partial function $\Phi(\cdot, y)$ is lower semi-continuous;*
4) *Φ is a λ_2-measurable operator;*
5) *the function $\Phi(f)$ is not λ-measurable.*

Proof. According to Lemma 3, there exists a λ-nonmeasurable set $X \subset R$ for which the restriction $f \mid X$ is injective. Define the required superposition operator Φ as follows:

$$\Phi(x, y) = 0 \quad (x \in X, \ \ y = f(x)),$$
$$\Phi(x, y) = 1 \quad (x \in R \setminus X, \ \ y = f(x)),$$
$$\Phi(x, y) = 1 \quad (x \in R, \ \ y \in R, \ \ y \neq f(x)).$$

For this Φ, relations 1), 2) and 3) are verified directly. Further, since the graph of f is a λ_2-measure zero subset of R^2, we claim that Φ is equivalent to 1 and, consequently, Φ is λ_2-measurable. Finally, we have

$$\Phi(x, y) = 0 \Leftrightarrow (x \in X \ \& \ y = f(x))$$

whence it follows that $(\Phi(f))^{-1}(0) = X$ and therefore $\Phi(f)$ is not λ-measurable. This ends the proof. □

Theorem 2. *Let $f \in L(R)$ and suppose that f is not a generalized step-function. Then there exists a superposition operator $\Psi : R \times R \to R$ such that:*

1) $\mathrm{ran}(\Psi) = \{1, 2\}$*;*
2) *for any $x \in R$, the partial function $\Psi(x, \cdot)$ is lower semi-continuous;*
3) *for any $y \in R$, the partial function $\Psi(\cdot, y)$ is lower semi-continuous;*
4) *Ψ is a λ_2-nonmeasurable operator;*
5) *the function $\Psi(f)$ is λ-nonmeasurable.*

Proof. By using the method of transfinite recursion and applying the standard argument (cf. [9]), we can define an injective function $g : R \to R$ whose graph is λ_2-thick in R^2 and does not intersect the graph of f. Let Φ_g denote the characteristic function of the graph of g (regarded as a subset of R^2). We put

$$\Psi = \Phi + 1 - \Phi_g \, ,$$

where Φ is the superposition operator of Theorem 1. It is easy to verify that Ψ is the required superposition operator, i.e., Ψ satisfies all the relations 1)–5) above. □

Remark 1. If a superposition operator $\Phi : R \times R \to R$ is λ-measurable with respect to the first variable, then $\Phi(f)$ is λ-measurable for every generalized step-function $f \in L(R)$. We thus see (in view of Theorem 1) that the generalized step-functions are exactly those functions $f \in L(R)$ for which any superposition operator Φ measurable with respect to the first variable yields measurable $\Phi(f)$.

Remark 2. If a superposition operator $\Phi : R \times R \to R$ is continuous with respect to the first variable and lower semi-continuous (more generally, Borel) with respect to the second variable, then Φ is a Borel mapping from R^2 into R, hence Φ is also sup-measurable.

Remark 3. Theorems 1 and 2 admit direct analogues for functions possessing the Baire property (detailed information about this property can be found in [6] and [8]). These analogues can be proved by the same scheme as for Lebesgue measurable functions. Only one essential moment should be mentioned. Namely, the proofs of Theorems 1 and 2 are based on the classical Luzin theorem concerning the structure of λ-measurable functions. Since we cannot apply the Luzin theorem to functions possessing the Baire property, we must replace this theorem by an appropriate similar statement. Such a statement is well known in general topology (see [6]) and is formulated as follows.

Let E_1 be a topological space, E_2 be a topological space with a countable base and let $f : E_1 \to E_2$ be a mapping possessing the Baire property. Then there exists a first category set $Z \subset E_1$ such that the restriction $f \mid (E_1 \setminus Z)$ is continuous.

We may assume, without loss of generality, that Z is an F_σ-subset of E_1, hence $E_1 \setminus Z$ is a G_δ-set in E_1. If the original space E_1 is Polish, then $E_1 \setminus Z$ is also Polish (by virtue of the Alexandrov theorem). Consequently, if E_1 is a Polish space and $E_2 = R$, we are able to apply Lemma 1 to the continuous function $f \mid (E_1 \setminus Z)$.

Under some additional set-theoretical axioms, Lemma 2 admits a significant generalization. Let us consider some abstract version of this lemma.

Fix an uncountable set E and a proper σ-ideal $\mathcal{I}$ of subsets of E, containing all singletons in E. We shall say that $g : E \to R$ is a step-function if $\mathrm{card}(\mathrm{ran}(g)) \leq \omega$. We shall say that $f : E \to R$ is a generalized step-function with respect to $\mathcal{I}$ if there exists at least one step-function $g : E \to R$ for which we have

$$\big\{x \in E : f(x) \neq g(x)\big\} \in \mathcal{I}\,,$$

i.e., f and g are $\mathcal{I}$-equivalent functions. Recall that a family of sets $\mathcal{B} \subset \mathcal{I}$ forms a base of $\mathcal{I}$ if, for any set $Y \in \mathcal{I}$, there exists a set $Z \in \mathcal{B}$ such that $Y \subset Z$.

The following statement is valid.

Theorem 3. *Let* $\mathrm{card}(E) = \omega_1$, *let* $\mathcal{I}$ *be a proper* σ*-ideal of subsets of* E, *containing all singletons in* E *and possessing a base whose cardinality does not exceed* ω_1, *and let* $f : E \to R$ *be a function. Then the following two assertions are equivalent:*

1) f *is not a generalized step-function with respect to* $\mathcal{I}$*;*

2) *there exists a set* $X \subset E$ *such that* $X \notin \mathcal{I}$ *and the restriction* $f \mid X$ *is injective.*

Proof. The implication 2) $\Rightarrow$ 1) is evident. Let us establish the validity of the implication 1) $\Rightarrow$ 2). Suppose that f satisfies 1) and introduce the following two sets:

$$T_0 = \{t \in \mathrm{ran}(f) : f^{-1}(t) \notin \mathcal{I}\}, \quad T_1 = \{t \in \mathrm{ran}(f) : f^{-1}(t) \in \mathcal{I}\}.$$

According to our assumption, there exists a base $\{B_\xi : \xi < \omega_1\}$ of the given σ-ideal $\mathcal{I}$. Only two cases are possible.

1. $\mathrm{card}(T_0) = \omega_1$. In this case we may write $T_0 = \{t_\xi : \xi < \omega_1\}$ where $t_\xi \neq t_\zeta$ for all $\xi < \omega_1$, $\zeta < \omega_1$, $\xi \neq \zeta$. Consider the family of sets $\{f^{-1}(t_\xi) \setminus B_\xi : \xi < \omega_1\}$. Obviously, $f^{-1}(t_\xi) \setminus B_\xi \neq \varnothing$ for each ordinal $\xi < \omega_1$. Let $x_\xi \in f^{-1}(t_\xi) \setminus B_\xi$ for any $\xi < \omega_1$, and let $X = \{x_\xi : \xi < \omega_1\}$. From the definition of X it immediately follows that the restriction $f \mid X$ is an injection. Moreover, we have $X \setminus B_\xi \neq \varnothing$ whenever $\xi < \omega_1$. The latter circumstance implies at once that the set X does not belong to $\mathcal{I}$.

2. $\mathrm{card}(T_0) \leq \omega$. In this case we obtain $\mathrm{card}(T_1) = \omega_1$ and $f^{-1}(T_1) \notin \mathcal{I}$ (since our f is not a generalized step-function with respect to $\mathcal{I}$). Let us construct, by using the method of transfinite recursion, an ω_1-sequence $\{x_\xi : \xi < \omega_1\}$ of points of $f^{-1}(T_1)$. Suppose that, for an ordinal $\xi < \omega_1$, the partial family of points $\{x_\zeta : \zeta < \xi\}$ has already been defined. Clearly, the set $(\cup\{f^{-1}(f(x_\zeta)) : \zeta < \xi\}) \cup B_\xi$ belongs to $\mathcal{I}$. Therefore,

$$f^{-1}(T_1) \setminus \big((\cup\{f^{-1}(f(x_\zeta)) : \zeta < \xi\}) \cup B_\xi\big) \neq \varnothing.$$

Choose any element x from the above nonempty set and put $x_\xi = x$. Proceeding in this manner, we are able to construct the required ω_1-sequence $\{x_\xi : \xi < \omega_1\}$. Finally, put $X = \{x_\xi : \xi < \omega_1\}$. In view of our construction, X is a partial selector of the disjoint family of sets $\{f^{-1}(t) : t \in T_1\}$. Hence the restriction of f to X is injective. Furthermore, $X \setminus B_\xi \neq \varnothing$ for all ordinals $\xi < \omega_1$, whence it follows that X does not belong to $\mathcal{I}$. □

Remark 4. Assume the Continuum Hypothesis (CH) and take as $\mathcal{I}$ the σ-ideal $\mathcal{L}$ of all Lebesgue measure zero subsets of R. Let $f : R \to R$ be a function distinct from all generalized step-functions with respect to $\mathcal{L}$. Suppose also that the graph of f is a set of λ_2-measure zero. Then it is not difficult to show that, for such an f, there always exists a superposition operator $\Phi : R \times R \to R$ satisfying the relations 1)–5) of Theorem 1. In this connection, let us emphasize that our f does not need to be a λ-measurable function.

Remark 5. Let E be a set, $\mathcal{I}$ be a proper σ-ideal of subsets of E and let $\mathcal{S}$ be a σ-algebra of subsets of E such that $\mathcal{I} \subset \mathcal{S}$. Elements of $\mathcal{S}$ are usually called measurable sets in E and elements of $\mathcal{I}$ are called negligible sets in E. The triple $(E, \mathcal{S}, \mathcal{I})$ is called a measurable space with negligibles (see, e.g., [10]). If $X \subset E$ and $X \notin \mathcal{I}$, then, in general, we cannot assert that X contains at least one subset which does not belong to $\mathcal{S}$. However, in some situations the specific features of a given σ-ideal $\mathcal{I}$ imply that any nonnegligible set in E contains a nonmeasurable subset. For example, assume again that $\mathrm{card}(E) = \omega_1$ and that E is a topological space of second category, whose all singletons are of first category. Let $\mathcal{I} = \mathcal{K}(E)$ denote the σ-ideal of all first category subsets of E and suppose that this σ-ideal possesses a base whose cardinality does not exceed ω_1. Denote also by $\mathcal{B}(E)$ the σ-algebra of all subsets of E having the Baire property (obviously, $\mathcal{K}(E) \subset \mathcal{B}(E)$). Then, for any set $X \subset E$, the following two assertions are equivalent:

(a) $X \notin \mathcal{K}(E)$ (i.e., X is not of first category in E);

(b) there exists a set $Y \subset X$ such that $Y \notin \mathcal{B}(E)$ (i.e., Y does not have the Baire property in E).

The proof of the equivalence of (a) and (b) can be found in [11] where some related results are also presented. Notice once more that this equivalence rests only on the inner properties of the σ-ideal $\mathcal{K}(E)$ and does not touch upon the structure of the σ-algebra $\mathcal{B}(E)$.

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