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THE NON-VANISHING OF FIRST COHOMOLOGY GROUPS FOR CERTAIN INFINITE-DIMENSIONAL COMPLEX MANIFOLDS

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Abstract. Here, using the ideas of an old paper by S. Dineen (1976), we give large classes of pairs (X, E) such that X is an infinite-dimensional complex space very far from a Banach manifold, E is a holomorphic vector bundle on X and $H^1(X, E)$ is infinite-dimensional.

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Let V be a complex locally convex and Hausdorff topological vector space. A sequence $\{x_n\}_{n\geq 1} \subset V$ is called a nontrivial very strongly convergent sequence if the sequence $\{\lambda_n x_n\}_{n\geq 1}$ converges to $0 \in V$ for all $\lambda_n \in \mathbf{C}$ and $x_n \neq 0$ for all n. For instance, if $V = \mathbf{C}^{\mathbf{N}}$, then the sequence $(1, 0, 0, \dots), (0, 1, 0, \dots), \dots$ is a nontrivial very strongly convergent sequence. By [4], Th. 2.6.13, a Fréchet space contains $\mathbf{C}^{\mathbf{N}}$ if and only if it has no continuous norm. Hence a Fréchet space has a continuous norm if and only if it has a nontrivial strongly convergent sequence. The aim of this short note is to give the following generalization of [1], Prop. 1; we will mostly use the ideas contained in [1].

Theorem. *Let V be a complex locally convex and Hausdorff topological vector space which admits a nontrivial very strongly convergent sequence $\{x_n\}_{n\geq 1}$ and X a reduced and locally integral complex space equipped with a holomorphic map $f : X \to V$ with the following property:*

(α) *for every $P \in X$ there are an open neighborhood A of P in X and an open neighborhood B of $f(P)$ in V such that $f|A$ is a closed embedding of A into B and the analytic set $f(A)$ is the zero-locus of finitely many holomorphic functions on B.*

Let E be a holomorphic vector bundle on X such that $H^0(X, E) \neq 0$. Then $H^1(X, E)$ is an infinite-dimensional $\mathbf{C}$-vector space.

In the statement of Theorem we allow the case in which the fibers of E are infinite-dimensional complex topological vector spaces.

Remark. We use the notation introduced in the statement of Theorem. We also assume that X is integral. Let g be a meromorphic function on X. Then g depends locally only on finitely many variables x_n in the following sense: for every $P \in X$ we take A and B as in the statement of Theorem. Consider $(g|A) \otimes (f|f^{-1}(f(A))$ as a meromorphic function g' on $f(A)$. Then there are

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a neighborhood D of $f(P)$ in B and a meromorphic function g'' on B' such that $g''|f(A) \cap D = g'$. By [1], Lemma 1, there is an integer $N > 0$ such that $\partial g''/\partial x_n \equiv 0$ for every $n \geq N$.

Proof of Theorem. By assumption, X is locally finitely determined in the sense of [3] and hence the set X_{reg} is an open dense subset of X. Fix $s \in H^0(X, E)$, $s \neq 0$. Hence s does not vanish at each point of a dense open subset of X whose complement is an analytic subset of X. In particular, s does not vanish in an open and dense subset of X_{reg}. Fix $P \in X_{reg}$ such that $s(P) \neq 0$. By assumption (α), near P $f(X)$ is a complex submanifold of V with finite codimension and hence its tangent space $T_P f(X)$ at P is a finite codimensional affine linear subspace of V. Deleting finitely many members of the sequence $\{x_n\}_{n \geq 1}$, we may assume that the vector space $T_P f(X) - P$ contains each x_n. Since X is locally integral, to prove the theorem it is sufficient to prove it with the additional assumption that X is integral. Let $\mathcal{M}_X$ be the sheaf of meromorphic functions on X; for the general theory of $\mathcal{M}_X$ when X is not smooth, see [2]. Since $\mathcal{O}_X$ is a subsheaf of $\mathcal{M}_X$, there is an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{M}_X \to \mathcal{M}_X/\mathcal{O}_X \to 0. \tag{1}$$

This is the set-up of the so-called Cousin's first problem or additive Cousin problem. Since E is locally free, tensoring (1) with E we obtain an exact sequence of $\mathcal{O}_X$-sheaves

$$0 \to E \to \mathcal{M}_X \otimes_{\mathcal{O}_X} E \to (\mathcal{M}_X/\mathcal{O}_X) \otimes_{\mathcal{O}_X} E \to 0. \tag{2}$$

Thus to prove Theorem it is sufficient to show that the linear map

$$\rho : H^0(X, \mathcal{M}_X \otimes_{\mathcal{O}_X} E) \to H^0(X, (\mathcal{M}_X/\mathcal{O}_X) \otimes_{\mathcal{O}_X} E)$$

has the infinite-dimensional cokernel. First, we will check that ρ is not surjective. For every integer $n \geq 1$, let $A_n \subset V$ be the linear span of $\{x_1, \dots, x_n\}$. Since A_n is finite-dimensional, it has a topological supplement in V by Hahn–Banach theorem. Construct inductively a decreasing sequence of closed subspaces F_n, $n \geq 1$, of V such that F_n is a topological supplement of A_n. Set $U_2 := \{cx_1 + w : c \in \mathbf{C}, \mathrm{Im}(c) < 11/4 \text{ and } w \in F_1\}$. For each $n > 2$ set $U_n := \{cx_1 + w : c \in \mathbf{C}, n - 3/4 < \mathrm{Im}(c) < n + 3/4 \text{ and } v \in F_1\}$. Set $X_n := f^{-1}(U_n)$. We may define the elements $\alpha_i \in V'$, $i \geq 1$ by the relations $v = \sum_{i=1}^{n} \alpha_i(v) x_i + v_n$ with $v_n \in F_n$ for any $v \in V$. For $n \geq 2$ and any $x \in X$ set $f_n(x) := s(x)\alpha_n(f(x))/(\alpha_1(f(x)) - in)$. As in [1], using Remark we get that this definition gives a nontrivial element of Coker(ρ). Now we will check that $H^1(X, E)$ is infinite-dimensional. Assume $\dim(H^1(X, E)) = k < +\infty$ and fix $z \in V' \backslash \{0\}$ such that the set $\{z(x_n)\}_{n \geq 1} \subset \mathbf{C}$ contains at least $k + 1$ elements. For every $u \in H^0(V, \mathcal{O}_V)$, $f^*(u) \in H^0(X, \mathcal{O}_X)$ and hence the multiplication by $f^*(u)$ induces a linear map $f^*(u)\times : H^1(X, E) \to H^1(X, E)$. For every polynomial $q \in \mathbf{C}[x]$ we have $q(f^*(u)\times) = (f^*(q(u)))\times$. Hence by Hamilton–Cayley there is $q \in \mathbf{C}[x]$ such that $q \neq 0$, $\deg(q) \leq k$ and $f^*(q(z))\times = 0$. Instead of the section s of E use the section $f^*(q(z))s$ to obtain a contradiction. □

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