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FUNCTIONAL DIFFERENTIAL INEQUALITIES WITH UNBOUNDED DELAY

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Abstract. We prove that a function of several variables satisfying a functional differential inequality with unbounded delay can be estimated by a solution of a suitable initial problem for an ordinary functional differential equation. As a consequence of the comparison theorem we obtain a Perron-type uniqueness result and a result on continuous dependence of solutions on given functions for partial functional differential equations with unbounded delay. We consider classical solutions on the Haar pyramid.

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1. Differential Inequalities on the Haar Pyramid

For any metric spaces A and B we denote by $C(A,B)$ the class of all continuous functions defined on A and taking values in B. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Suppose that $g \in C([0,a],\mathbb{R}^n_+)$, $\mathbb{R}_+ = [0,+\infty)$, $a > 0$, and $g = (g_1,\dots,g_n)$. Write

$$h(t) = \int_0^t g(\tau)d\tau, \quad h = (h_1,\dots,h_n), \quad t \in [0,a].$$

Let H be the Haar pyramid

$$H = \big\{(t,x) = (t,x_1,\dots,x_n) \in \mathbb{R}^{1+n}: \quad t \in [0,a], \quad -b + h(t) \le x \le b - h(t)\big\}$$

and $E = \mathbb{R}_- \times [-b,b]$, where $b = (b_1,\dots,b_n) \in \mathbb{R}^n_+$ and $\mathbb{R}_- = (-\infty,0]$. We assume that $b > h(a)$. Note that if $g(t) = M$ for $t \in [0,a]$ where $M = (M_1,\dots,M_n) \in \mathbb{R}^n_+$, then we have a classical Haar pyramid considered in [6], [9], [10].

Let Y be the space of initial functions $\varphi : E \to \mathbb{R}$. We assume that Y is a linear space with the norm $\|\cdot\|_Y$ and that $(Y,\|\cdot\|_Y)$ is a Banach space. For $0 \le t \le a$ we put $H_t = H \cap ([0,t]\times\mathbb{R}^n)$. Let $\|\cdot\|_t$ be the supremum norm in the space $C(H_t,\mathbb{R})$.

For each t, $0 < t \le a$, we consider the space Y_t consisting of functions $z : E \cup H_t \to \mathbb{R}$ such that $z|_E \in Y$. We assume that Y_t is a linear space with the norm $\|\cdot\|_{Y_t}$. Write $Y_* = Y_a$ and $\|\cdot\|_* = \|\cdot\|_{Y_a}$. Let $\Omega = H \times Y_* \times \mathbb{R}^n$

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and assume that $f : \Omega \to \mathbb{R}$ and $\varphi \in Y$ are given functions. We consider the functional differential equation

$$\partial_t z(t,x) = f(t,x,z,\partial_x z(t,x)) \tag{1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{for} \quad (t,x) \in E, \tag{2}$$

where $\partial_x z = (\partial_{x_1} z, \dots, \partial_{x_n} z)$. A function $z : E \cup H \to \mathbb{R}$ is said to be a classical solution of (1), (2) if and only if

1) the function $z|_H$ is continuous and the partial derivatives $\partial_t z(t,x)$, $\partial_x z(t,x)$ exist for $(t,x) \in H \cap ((0,a] \times \mathbb{R}^n)$,
2) $z|_E \in Y$, z satisfies equation (1) on $H \cap ((0,a] \times \mathbb{R}^n)$ and condition (2) holds.

We assume that f satisfies the following Volterra condition: if $z, \bar{z} \in Y_*$, $(t,x,q) \in H \times \mathbb{R}^n$ and $z(\tau,y) = \bar{z}(\tau,y)$ for $(\tau,y) \in E \cup H_t$ then $f(t,x,z,q) = f(t,x,\bar{z},q)$. Note that the Volterra condition means that the value of f at the point $(t,x,z,q) \in \Omega$ depends on (t,x,q) and on the restriction of z to the set $E \cup H_t$ only.

It is well known that differential equations with deviated variables and a large class of integral functional problems can be obtained from (1) by specializing the operator f.

The aim of this paper is to give uniqueness results for functional differential problems with unbounded delay. Methods of differential inequalities are the basic tools in investigations of solutions of initial or initial boundary value problems for nonlinear differential or differential functional equations with partial derivatives. Uniqueness theorems require assumptions about an estimation of the right-hand side increase and about the regularity of solutions. Uniqueness criteria are obtained as a consequence of suitable comparison theorems for differential or functional differential inequalities.

Equations with partial derivatives of first order have the following property: the problem of the existence of their classical or generalized solutions is strictly connected with the problem of solving ordinary differential equations. Ordinary differential inequalities find numerous applications in the theory of first order partial differential equations or systems. Such problems as estimates of solutions of partial equations, estimates of the domain of their solutions, estimates of the difference between two solutions, criteria of uniqueness, are classical examples but not the only ones ([6], [9], [10]).

A similar role in the theory of differential functional equations with partial derivatives is played by differential functional inequalities with ordinary derivatives. The monograph [5] contains an exposition of recent developments of hyperbolic functional differential inequalities and applications.

Now we present relations between the problem of uniqueness of classical solutions of nonlinear partial differential equations with initial conditions and some properties of ordinary differential equations.

Let $\tilde{H}$ be the Haar pyramid

$$\tilde{H} = \{(t,x) \in \mathbb{R}^{1+n} : \quad t \in [0,a], \quad -b + Mt \le x \le b - Mt\}, \tag{3}$$

where $a > 0$, $b = (b_1, \dots, b_n)$, $b_i > 0$ for $1 \le i \le n$ and $M = (M_1, \dots, M_n) \in \mathbb{R}^n_+$. We assume that $b > Ma$. Suppose that $\tilde{f} : \tilde{H} \times \mathbb{R} \times \mathbb{R}^n$ and $\phi : [-b, b] \to \mathbb{R}$ are given functions. Consider the Cauchy problem

$$\partial_t z(t,x) = \tilde{f}(t, x, z(t,x), \partial_x z(t,x)), \tag{4}$$

$$z(0,x) = \phi(x) \quad \text{for} \quad x \in [-b, b], \tag{5}$$

A function $z : \tilde{H} \to \mathbb{R}$ is called the function of class D if $z \in C(\tilde{H}, \mathbb{R})$, the derivatives $\partial_t z$ and $\partial_x z$ exist in $\tilde{H} \cap ((0,a] \times R^n)$ and z possesses the total differential on $\partial\tilde{H} \cap ((0,a) \times \mathbb{R}^n)$ where $\partial\tilde{H}$ is the boundary of $\tilde{H}$. We consider solutions of (4), (5) which are of the class D. We report a result of J. Szarski ([9], Th. 42.1) which is a generalization of Haar's theorem ([2], see also [3] Ch. 6, Th. 10.1).

Theorem 1.1. *Suppose that*

1) *the functions* $\tilde{f} : \tilde{H} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *and* $\phi : [-b, b] \to \mathbb{R}$ *are continuous,*
2) *there is a function* $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*
   (i) σ *is continuous and* $\sigma(t, 0) = 0$ *for* $t \in [0,a]$,
   (ii) *the maximal solution of the Cauchy problem*

$$\eta'(t) = \sigma(t, \eta(t)), \quad \eta(0) = 0, \tag{6}$$

   is $\bar{\eta}(t) = 0$ *for* $t \in [0,a]$,
   (iii) *the estimate*

$$\left|\tilde{f}(t,x,p,q) - \tilde{f}(t,x,\bar{p},\bar{q}\right| \le \sigma(t, |p - \bar{p}|) + \sum_{i=1}^{n} M_i |q_i - \bar{q}_i|$$

   holds on $\tilde{H} \times \mathbb{R} \times \mathbb{R}^n$.

Under these assumptions the Cauchy problem (4), (5) *admits at most one solution* $z : \tilde{H} \to \mathbb{R}$ *of the class* D.

The above theorem is proved by using the method of differential inequalities.

Now we formulate an extension of the above uniqueness result to functional differential equations. We formulate the problem. Suppose that $\tilde{H}$ is the Haar pyramid defined by (3) and $\tilde{E} = [-b_0, 0] \times [-b, b] \subset \mathbb{R}^{1+n}$ with $b_0 \ge 0$. Given the functions $F : \tilde{H} \times C(\tilde{E} \cup \tilde{H}, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ and $\Phi : \tilde{E} \to \mathbb{R}$, we consider the Cauchy problem

$$\partial_t z(t,x) = F(t, x, z, \partial_x z(t,x)), \tag{7}$$

$$z(t,x) = \Phi(t,x) \quad \text{on} \quad \tilde{E}. \tag{8}$$

A function $z : \tilde{E} \cup \tilde{H} \to \mathbb{R}$ is called a function of the class D^* if $z \in C(\tilde{E} \cup \tilde{H}, \mathbb{R})$ and the function $z\,|_{\tilde{H}}$ is of the class D on $\tilde{H}$. We consider solutions of (7), (8) which are of the class D^*. We assume that F satisfies the Volterra condition. We need the operator $V : C(\tilde{E} \cup \tilde{H}, \mathbb{R}) \to C([-b_0, a], \mathbb{R}_+)$ defined by

$$V[z](t) = \begin{cases} \max\big\{|z(t,x)| : x \in [-b+Mt, b-Mt]\big\} & \text{for} \quad t \in [0,a], \\ \max\big\{|z(t,x)| : x \in [-b,b]\big\} & \text{for} \quad t \in [-b_0, 0]. \end{cases}$$

Note that the continuity of $V[z]$ follows from Theorem 33.1 in [9].

A criterion that implies the uniqueness of solutions of (7), (8) is the following Perron type condition.

Theorem 1.2. *Suppose that*

1) *the functions $F : \tilde{H} \times C(\tilde{E} \cup \tilde{H}, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$, $\Phi : \tilde{E} \to \mathbb{R}$ are continuous and F satisfies the Volterra condition,*
2) *there is a function $\sigma : [0,a] \times C([-b_0, a], \mathbb{R}_+) \to \mathbb{R}_+$ such that*
   (i) *σ is continuous and satisfies the Volterra condition,*
   (ii) *σ is nondecreasing with respect to the functional variable and $\sigma(t, \theta) = 0$ for $t \in [0,a]$ where $\theta(\tau) = 0$ for $\tau \in [-b_0, a]$,*
   (iii) *the maximal solution of the Cauchy problem*

$$\eta'(t) = \sigma(t, \eta), \quad \eta(t) = 0 \quad \textit{for} \quad t \in [-b_0, 0], \tag{9}$$

   is $\bar{\eta}(t) = 0$ for $t \in [-b_0, a]$,
   (iv) *the estimate*

$$\big|F(t,x,z,q) - F(t,x,\bar{z},\bar{q})\big| \le \sigma\big(t, V[z-\bar{z}]\big) + \sum_{i=1}^{n} M_i |q_i - \bar{q}_i|$$

   holds on $\tilde{H} \times C(\tilde{E} \cup \tilde{H}, \mathbb{R}) \times \mathbb{R}$.

Under these assumptions the Cauchy problem (7), (8) *admits at most one solution $z : \tilde{E} \cup \tilde{H} \to \mathbb{R}$ of the class D^*.*

The above result is proved by using theorems on functional differential inequalities ([5], Ch. 1).

Thus we see that in the uniqueness theory for first order partial differential functional equations, ordinary differential functional equations are considered as comparison problems of the Perron type. Uniqueness results with comparison functions of the Kamke type can be found in [5].

We wish to emphasize that functional differential comparison problems are the main tool in this research. Adequate examples are given in [5], Ch. 1.

The classical theory of partial differential inequalities is described extensively in the monographs [6], [9], [10]. Functional differential inequalities and applications are discussed in [5].

The aim of this paper is to construct comparison problems for partial functional differential equations with unbounded delay. We show that relations between classical equations (1) and (6) given in Theorem 1.1 and relations between functional differential equations (7) and (9) given in Theorem 1.2 can be extended to functional differential equations with unbounded delay. We prove that under natural assumptions on given functions there is a maximal solution of an adequate initial problem. We show that a function satisfying a functional differential inequality can be estimated by a maximal solution of a suitable initial problem.

The paper is organized as follows. In Section 2 we consider ordinary functional differential equations with unbounded delay. We prove that under natural assumptions on given functions there is a maximal solution of an adequate initial problem. We show that a function satisfying a functional differential inequality can be estimated by a maximal solution on a suitable initial problem.

In Section 3 we give a comparison result for partial functional differential inequalities with unbounded delay. We prove that a function of several variables satisfying an adequate functional differential inequality can be estimated by a maximal solution of a suitable initial problem for an ordinary functional differential equation. As a consequence of the comparison theorem we obtain a Perron type uniqueness result.

Examples are presented in the last part of the paper.

Existence results for nonlinear functional differential equations with unbounded delay can be found in [8].

The theory of ordinary functional differential equations with unbounded delay has rich literature. It is not our aim to show a full review of the papers dealing with the problem. We mention only those references which contain such reviews. These are the monographs [4], [7] and the survey paper [1].

2. Maximal Solutions of Initial Problems

For a function $\eta : (-\infty, a] \to \mathbb{R}$ and for a point $t \in (-\infty, a]$ we define a function $\eta_t : \mathbb{R}_- \to \mathbb{R}$ as follows: $\eta_t(\tau) = \eta(t+\tau)$, $\tau \in \mathbb{R}_-$. The function η_t is the restriction of η to the set $(-\infty, t]$ and this restriction is shifted to $\mathbb{R}_-$.

The phase space X for ordinary functional differential equations with unbounded delay is a linear space with the norm $\|\cdot\|_X$ consisting of functions mapping $\mathbb{R}_-$ into $\mathbb{R}$. The fundamental axioms assumed on X are the following.

Assumption $H[X]$**.** Suppose that $(X, \|\cdot\|)$ is a Banach space and

1) if $\eta : (-\infty, a] \to \mathbb{R}$, $a > 0$, is a function such that $\eta_0 \in X$ and $\eta\,|_{[0,a]}$ is continuous, then
   (i) $\eta_t \in X$ for $t \in [0, a]$,
   (ii) there exist constants $L, K, M \in \mathbb{R}_+$ (independent of η) such that

$$|\eta(t)| \le L\|\eta_t\|_X \tag{10}$$

   and

$$\|\eta_t\|_X \le K \max\big\{|\eta(\tau)| : \quad \tau \in [0,t]\big\} + M\|x_0\|_X, \tag{11}$$

   where $t \in [0, a]$,
2) for $\eta : (-\infty, a] \to \mathbb{R}$ the mapping $t \to \eta_t$ is a continuous function on $[0, a]$.

Let

$$X_+ = \{w \in X : w(t) \geq 0 \quad \text{for} \quad t \in \mathbb{R}_-\}. \tag{12}$$

Suppose that $\sigma : [0,a] \times X_+ \to \mathbb{R}_+$ and $\lambda \in X_+$ are given functions. We consider the Cauchy problem

$$\eta'(t) = \sigma(t, \eta_t), \tag{13}$$

$$\eta(t) = \lambda(t) \quad \text{for} \quad t \in \mathbb{R}_-. \tag{14}$$

We say that the function $\sigma : [0,a] \times X_+ \to \mathbb{R}_+$ satisfies the monotonicity condition W_+ if for any $(t,w), (t,\bar{w}) \in [0,a] \times X_+$ such that $w(s) \leq \bar{w}(s)$ for $s \in \mathbb{R}_-$ and $w(0) = \bar{w}(0)$ we have $\sigma(t,w) \leq \sigma(t,\bar{w})$.

Given $w \in X$, let $sw : (-\infty, a] \to \mathbb{R}$ be a function defined by

$$sw(t) = w(t) \quad \text{for} \quad t \in \mathbb{R}_- \quad \text{and} \quad sw(t) = w(0) \quad \text{for} \quad t \in [0,a].$$

Theorem 2.1. *Suppose that*

1) *assumption $H[X]$ is fulfilled and $\lambda \in X_+$,*
2) *the function $\sigma : [0,a] \times X_+ \to \mathbb{R}_+$ is continuous and satisfies the monotonicity condition W_+.*

Then there is $\alpha \in (0,a]$ such that the Cauchy problem (13), (14) *has the maximal solution on the interval $(-\infty, \alpha]$.*

Proof. There exists $\bar{M} \in \mathbb{R}_+$ such that

$$\sigma(t, (s\lambda)_t) \leq \bar{M} \quad \text{for} \quad t \in [0,a].$$

Moreover, there exists $\bar{b} > 0$ such that for any $w \in X_+$ and $t \in [0,a]$ the condition $\|w - (s\lambda)_t\|_X \leq \bar{b}$ implies that

$$\big|\sigma(t,w) - \sigma(t,(s\lambda)_t)\big| \leq 1.$$

Suppose, contrary to our claim, that for any $k \in \mathbb{N}$ there is a function $w^{(k)} \in X_+$ and a point $t_k \in [0,a]$ such that

$$\big\|w^{(k)} - (s\lambda)_{t_k}\big\|_X < 1/k \quad \text{and} \quad \big|\sigma(t_k, w^{(k)}) - \sigma(t_k, (s\lambda)_{t_k})\big| > 1. \tag{15}$$

We choose a subsequence $\{t_{k_p}\} \subset \{t_k\}$ which is convergent to some $\bar{t} \in [0,a]$. This, together with (15), contradicts the continuity of σ at the point $(\bar{t}, (s\lambda)_{\bar{t}})$.

Therefore for any $w \in X_+$ such that $\|w - (s\lambda)_t\|_X \leq \bar{b}$ for $t \in [0,a]$, we have

$$\sigma(t,w) \leq \bar{M} + 1.$$

Let $h_\varepsilon \in X_+$ be defined by

$$h_\varepsilon(t) = \begin{cases} 0 & \text{for} \quad t \in (-\infty, -\varepsilon], \\ \varepsilon + t & \text{for} \quad t \in (-\varepsilon, 0]. \end{cases} \tag{16}$$

For any ε satisfying

$$0 \leq \varepsilon \leq \bar{b}/2 \quad \text{and} \quad K\varepsilon + M\|h_\varepsilon\|_X \leq \bar{b}/2 \tag{17}$$

we consider the Cauchy problem

$$\eta'(t) = \sigma(t, \eta_t) + \varepsilon, \tag{18}$$

$$\eta(t) = \lambda(t) + h_\varepsilon(t) \quad \text{for} \quad t \in \mathbb{R}_-. \tag{19}$$

Let $\lambda_\varepsilon = \lambda + h_\varepsilon$ and let $w \in X$ be such a function that $\|w - (s\lambda_\varepsilon)_t\|_X \le \bar{b}/2$. Then

$$\big\|w - (s\lambda)_t\big\|_X \le \bar{b}/2 + K\varepsilon + M\|h_\varepsilon\|_X \le \bar{b}$$

for $t \in [0, a]$ and consequently

$$\sigma(t, w) + \varepsilon \le \bar{M} + 1 + \bar{b}/2.$$

Write

$$\alpha = \min\left\{a, \frac{\bar{b}/2}{K(\bar{M} + 1 + \bar{b}/2)}\right\}.$$

Let us denote by U the set of all functions $u : (-\infty, \alpha] \to \mathbb{R}_+$ such that

$$u(t) = \lambda_\varepsilon(t) \quad \text{for} \quad t \in \mathbb{R}_-,$$
$$|u(t) - u(\bar{t})| \le (\bar{M} + 1 + \bar{b}/2)|t - \bar{t}| \quad \text{for} \quad t, \bar{t} \in [0, \alpha].$$

Let T be the operator defined on U in the following way

$$T[\eta](t) = \lambda_\varepsilon(t) \quad \text{for} \quad t \in \mathbb{R}_-,$$
$$T[\eta](t) = \lambda_\varepsilon(0) + \int_0^t [\sigma(\tau, \eta_\tau) + \varepsilon] d\tau \quad \text{for} \quad t \in [0, \alpha].$$

For $\eta \in U$ and $t \in [0, \alpha]$ we obtain

$$\begin{aligned}\big\|\eta_t - (s\lambda_\varepsilon)_t\big\|_X &\le K \max\big\{\big|u(\tau) - (s\lambda_\varepsilon)(\tau)\big| : \tau \in [0, t]\big\} \\ &\le K(\bar{M} + 1 + \bar{b}/2)t \le \bar{b}/2 \quad t \in [0, \alpha].\end{aligned}$$

This gives

$$\sigma(t, \eta_t) + \varepsilon \le \bar{M} + 1 + \bar{b}/2, \quad t \in [0, \alpha],$$

and

$$\big|T[\eta](t) - T[\eta](\bar{t})\big| \le (\bar{M} + 1 + \bar{b}/2)|t - \bar{t}| \quad \text{for} \quad t, \bar{t} \in [0, \alpha].$$

We thus get $T : U \to U$. It follows from the Schauder fixed point theorem that T has a fixed point in U. Therefore problem (18), (19) has a solution on $(-\infty, \alpha]$ for ε satisfying (17). Notice that α is independent of ε.

Suppose that $\varepsilon, \bar{\varepsilon}$ satisfy (17) and $\varepsilon < \bar{\varepsilon}$. Let $w(\,\cdot\,, \varepsilon)$ and $w(\,\cdot\,, \bar{\varepsilon})$ be solutions of (18), (19), with ε and $\bar{\varepsilon}$ respectively. Using the method of differential inequalities we obtain $w(t, \varepsilon) < w(t, \bar{\varepsilon})$ for $t \in [0, \alpha]$. Let $\{\varepsilon_m\}$ be a sequence satisfying the conditions

(i) for each m, the number ε_m satisfies (17),
(ii) $0 < \varepsilon_{m+1} < \varepsilon_m$ for all $m \in \mathbb{N}$ and $\lim_{m \to +\infty} \varepsilon_m = 0$.

The sequence $\{w(\cdot,\varepsilon_m)|_{[0,a]}\}$ forms a family of equicontinuous and uniformly bounded functions. Then there is $w:(-\infty,\alpha]\to\mathbb{R}_+$ such that $w(t)=\lambda(t)$ on $\mathbb{R}_-$ and

$$w(t)=\lim_{m\to+\infty} w(t,\varepsilon_m) \quad \text{uniformly on} \quad [0,\alpha].$$

One can see that w is a solution of (13), (14). Moreover for any solution $\bar{u}$ of problem (13), (14) we have that

$$\bar{u}(t)<w(t,\varepsilon_m) \quad \text{for} \quad t\in[0,\alpha], \quad m\in\mathbb{N}.$$

Thus $\bar{u}(t)\le w(t)$ for $t\in[0,\alpha]$, which yields that w is the maximal solution of (13) (14) on $(-\infty,\alpha]$.

Now we formulate a comparison result for equations with unbounded delay. Let us denote by D_- the left-hand lower Dini derivative. □

Lemma 2.2. *Suppose that*

1) *Assumption* $H[X]$ *is fulfilled and* $\lambda\in X_+$*,*
2) *the function* $\sigma:[0,a]\times X_+\to\mathbb{R}_+$ *is continuous and satisfies the monotonicity condition* W_+*,*
3) $w(\cdot,\lambda)$ *is the maximal solution of* (13), (14) *on* $(-\infty,a]$*,*
4) *the function* $\bar{\eta}:(-\infty,a]\to\mathbb{R}_+$ *satisfies the conditions*
   (i) $\bar{\eta}_0\in X$ *and* $\bar{\eta}\,|_{[0,a]}$ *is continuous,*
   (ii) *the functional differential inequality*
   $$D_-\bar{\eta}(t)\le\sigma(t,\bar{\eta}_t), \tag{20}$$
   holds for $t\in S_+$ *where*
   $$S_+=\big\{t\in[0,a]:\bar{\eta}(t)>w(t,\lambda)\big\}$$
   (iii) $\bar{\eta}(t)\le\lambda(t)$ *for* $t\in\mathbb{R}_-$*.*

Then

$$\bar{\eta}(t)\le w(t,\lambda) \quad \textit{for} \quad t\in[0,a]. \tag{21}$$

Proof. First we shall prove that there exists $0<\alpha\le 0$ such that $\bar{\eta}(t)\le w(t,\lambda)$ for $t\in[0,\alpha]$. Let $\bar{M}>0$, $\bar{b}>0$ be such constants that

$$\sigma(t,(s\lambda)_t)\le\bar{M} \quad \text{for} \quad t\in[0,a]$$

and

$$\big|\sigma(t,w)-\sigma(t,(s\lambda)_t\big|\le 1$$

for $w\in X_+$ and $\|w-(s\lambda)_t\|_X\le\bar{b}$. Consider $\varepsilon>0$ satisfying (17). Then the Cauchy problem (18), (19) has a solution $w_\varepsilon(\cdot,\lambda)$ on the interval $[0,\alpha]$, where

$$\alpha=\min\left\{a,\frac{\bar{b}/2}{K(\bar{M}+1+\bar{b}/2)}\right\}.$$

We claim that $\bar{\eta}(t)<w_\varepsilon(t,\lambda)$ for $t\in[0,\alpha]$. If this assertion is false, then there exists $t_0\in[0,\alpha]$ such that $\bar{\eta}(t)<w_\varepsilon(t,\lambda)$ for $t\in[0,t_0)$ and $\bar{\eta}(t_0)=w_\varepsilon(t_0,\lambda)$. This implies that $t_0\in S_+$ and

$$D_-\bar{\eta}(t_0)\ge w_\varepsilon'(t_0,\lambda)=\sigma(t_0,(w_\varepsilon(\cdot,\lambda))_{t_0})+\varepsilon$$

and consequently

$$D_-\bar{\eta}'(t_0) > \sigma(t_0, \bar{\eta}_{t_0}).$$

This contradicts our assumption that $\bar{\eta}$ satisfies (20) at the point t_0.

Let $\bar{a}$ be the supremum of such numbers α, $0 < \alpha \le a$, that

$$\bar{\eta}(t) \le w(t, \lambda) \quad \text{for} \quad t \in [0, \alpha].$$

We shall prove that $a = \bar{a}$. Assume that $\bar{a} < a$. Then $\bar{\eta}(t) \le w(t, \lambda)$ for $t \in [0, \bar{a}]$ and $\bar{\eta}(\bar{a}) = w(\bar{a}, \lambda)$. Consider the Cauchy problem

$$\eta'(t) = \sigma(t, \eta_t), \tag{22}$$

$$\eta(t) = \bar{\lambda}(t) \quad \text{for} \quad t \in (-\infty, \bar{a}], \tag{23}$$

where

$$\bar{\lambda}(t) = \begin{cases} \lambda(t) & \text{for} \quad t < 0, \\ w(t, \lambda) & \text{for} \quad t \in [0, \bar{a}]. \end{cases}$$

Since $w(\cdot, \lambda)$ is the maximal solution of this problem on $(-\infty, a]$, there exists $\varepsilon_0 > 0$ such that $\bar{u}(t) \le w(t, \lambda)$ for $t \in (-\infty, \bar{a} + \varepsilon_0]$, which contradicts the definition of $\bar{a}$. □

Lemma 2.3. *Suppose that Assumption $H[X]$ is satisfied and*

1) *the function $\sigma : [0, a] \times X_+ \to R_+$ is continuous and satisfies the monotonicity condition W_+,*
2) *$\lambda \in X_+$ and the sequence of functions $\{\lambda_m\}$ satisfies the conditions*
   (i) *$\lambda_m \in X_+$ for $m \in \mathbb{N}$ and $\lambda(t) \le \lambda_m(t)$ for $t \in \mathbb{R}_-$, $m \in \mathbb{N}$,*
   (ii) $\lim_{m\to\infty} \|\lambda - \lambda_m\|_X = 0$,
3) *the sequence on numbers $\{\varepsilon_m\}$ is such that $\varepsilon_m \in \mathbb{R}_+$ for $m \in \mathbb{N}$ and* $\lim_{m\to\infty} \varepsilon_m = 0$.

Then there are $\alpha \in (0, a]$ and $N \in \mathbb{N}$ such that

1) *for each $m \ge N$, the maximal solution $\omega(\,\cdot\,, \lambda_m)$ of the Cauchy problem*

$$\eta'(t) = \sigma(t, \eta_t) + \varepsilon_m, \quad \eta(t) = \lambda_m(t) \quad \textit{for} \quad t \in \mathbb{R}_-, \tag{24}$$

is defined on $(-\infty, \alpha]$,

2) *the maximal solution $\omega(\,\cdot\,, \lambda)$ of problem (9) is defined on $(-\infty, \alpha]$ and*

$$\lim_{m\to\infty} \omega(t, \lambda_m) = \omega(t, \lambda) \quad \textit{uniformly on} \quad [0, \alpha].$$

Proof. There is $\bar{M} \in \mathbb{R}_+$ such that

$$\sigma(t, (s\lambda)_t) + \varepsilon_m \le \bar{M} \quad \text{for} \quad t \in [0, a] \quad m \in \mathbb{N}.$$

Moreover, there is $\bar{b} > 0$ such that for any $w \in X_+$ and $t \in [0, a]$, the condition $\|w - (s\lambda)_t\|_X \le \bar{b}$ implies that

$$\sigma(t, w) - \sigma(t, (s\lambda)_t)| \le 1.$$

Therefore, for $w \in X_+$ such that $\|w - (s\lambda)_t\|_X \le \bar{b}$, $t \in [0, a]$, we have

$$\sigma(t, w) + \varepsilon_m \le \bar{M} + 1, \quad m \in \mathbb{N}.$$

Let $h_\varepsilon \in X_+$ be the function defined by (16) with ε satisfying

$$0 \le \varepsilon \le \frac{\bar b}{3}\,, \quad K\varepsilon + M\|h_\varepsilon\|_X \le \frac{\bar b}{3}\,. \tag{25}$$

Consider the Cauchy problem

$$\eta'(t) = \sigma(t, \eta_t) + \varepsilon_m + \varepsilon, \tag{26}$$

$$\eta(t) = \lambda_m(t) + h_\varepsilon(t) \quad \text{for} \quad t \in \mathbb{R}_-. \tag{27}$$

There is $N \in \mathbb{N}$ such that for $m \ge N$ we have

$$\|(s\lambda_m)_t - (s\lambda)_t\|_X \le \frac{\bar b}{3}\,, \quad t \in [0, a].$$

Write

$$\alpha = \min\left\{ a,\ \frac{\bar b/3}{K(\bar M + 1 + \bar b/3)} \right\}.$$

Let $m \ge N$ be fixed. We denote by U_m the set of all functions $\eta : (-\infty, \alpha] \to \mathbb{R}_+$ such that

$$\eta(t) = \lambda_m(t) + h_\varepsilon(t) \quad \text{for} \quad t \in \mathbb{R}_-,$$

$$|\eta(t) - \eta(\bar t)| \le (\bar M + 1 + \bar b/3)|t - \bar t| \quad \text{for} \quad t, \bar t \in [0, \alpha].$$

Let the operator T_m be defined on U_m in the following way:

$$T_m[\eta](t) = \lambda_m(t) + h_\varepsilon(t) \quad \text{for} \quad t \in \mathbb{R}_-,$$

$$T_m[\eta](t) = \lambda_m(0) + h_\varepsilon(0) + \int_0^t [\,\sigma(\tau, \eta_\tau) + \varepsilon_m + \varepsilon]\, d\tau, \quad t \in [0, \alpha].$$

For $\eta \in U_m$ and $t \in [0, \alpha]$ we have

$$\begin{aligned} \|\eta_t - (s\lambda)_t\|_X &\le \|\eta_t - (s\lambda_m + sh_\varepsilon)_t\|_X \\ &\quad + \|(s\lambda_m + sh_\varepsilon)_t - (s\lambda + sh_\varepsilon)_t\|_X + \|(s\lambda_\varepsilon)_t\|_X \\ &\le K \max\left\{ |\eta(\tau) - (\lambda_m + h_\varepsilon)(\tau)| :\ \tau \in [0, \alpha] \right\} \\ &\quad + \frac{\bar b}{3} + K\varepsilon + M\|h_\varepsilon\|_X \le \bar b. \end{aligned}$$

This gives

$$\sigma(t, \eta_t) + \varepsilon_m + \varepsilon \le \bar M + 1 + \bar b/3$$

and consequently

$$\left|T_m[\eta](t) - T_m[\eta](\bar t)\right| \le (\bar M + 1 + \bar b/3)|t - \bar t| \quad \text{for} \quad t, \bar t \in [0, \alpha].$$

Therefore $T_m : U_m \to U_m$ for each $m \ge N$. Using Schauder's fixed point theorem we get that T_m has a fixed point in U_m and problem (26), (27) has a solution $\omega(\,\cdot\,, \lambda_m, \varepsilon)$ on $(-\infty, \alpha]$ for ε satisfying (25). It is easy to see that there is

$$\omega(t, \lambda_m) = \lim_{\varepsilon \to 0} \omega(t, \lambda_m, \varepsilon), \quad t \in [0, \alpha],$$

and $\omega(\,\cdot\,,\lambda_m)$ is a maximal solution of (24). Note that the constant α is independent of $m \geq N$. In the same way we prove that there is a maximal solution $\omega(\,\cdot\,,\lambda) : (-\infty,\alpha] \to \mathbb{R}_+$ of problem (9)

The functions

$$\left\{\, \omega(\,\cdot\,,\lambda_N)|_{[0,\alpha]},\, \omega(\,\cdot\,,\lambda_{N+1})|_{[0,\alpha]},\, \dots \,\right\}$$

are uniformly continuous and equibounded. The uniqueness of the maximal solution $\omega(\,\cdot\,,\lambda)$ shows that

$$\lim_{m\to\infty} \omega(t,\lambda_m) = \omega(t,\lambda) \quad \text{uniformly on} \quad [0,\alpha].$$

This proves the theorem. □

3. Comparison Theorem for Partial Functional Differential Equations

We prove that a function $u : E \cup H \to \mathbb{R}$ satisfying on E an initial inequality and an adequate functional differential inequality on H can be estimated by the maximal solution of problem (13), (14). We give also applications of the comparison result.

Let

$$\partial_0^+ H = \bigcup_{i=1}^{n} \left\{ (t,x) \in H : 0 < t \leq a, x_i = b_i - h_i(t) \right\},$$

$$\partial_0^- H = \bigcup_{i=1}^{n} \left\{ (t,x) \in H : 0 < t \leq a, x_i = -b_i + h_i(t) \right\}$$

and $\partial_0 H = \partial_0^+ H \cup \partial_0^- H$. Moreover, we define

$$S_t = \begin{cases} [-b,b] & \text{for} \quad t \in \mathbb{R}_-, \\ [-b+h(t), b-h(t)] & \text{for} \quad t \in [0,a]. \end{cases}$$

For a point $t \in [0,a]$ and for a function $z : E \cup H_t \to \mathbb{R}$ we define a function $W_t[z] : \mathbb{R}_- \to \mathbb{R}_+$ as follows:

$$W_t[u](\tau) = \sup \left\{ |u(t+\tau,x)| : x \in S_{t+\tau} \right\}, \quad \tau \in \mathbb{R}_-.$$

Assumption $H[X,Y]$. Suppose that for each function $\varphi \in Y$ the function $W_0\varphi$ is such that $W_0[\varphi] \in X$.

We say that the function $u : E \cup H \to \mathbb{R}$ is of the class D if

1) $u\,|_E \in Y$ and $u\,|_H$ is continuous,
2) the partial derivatives $\partial_t u(t,x)$, $\partial_x u(t,x)$ exist for $(t,x) \in H \cap ((0,a] \times R^n)$ and u is differentiable on $\partial_0 H$.

Theorem 3.1. *Suppose that*

1) *Assumptions* $H[X]$, $H[X,Y]$ *are fulfilled,* $\lambda \in X_+$,
2) *the function* $\sigma : [0,a] \times X_+ \to \mathbb{R}_+$ *is continuous and satisfies the monotonicity condition* W_+,
3) $w(\,\cdot\,,\lambda)$ *is the maximal solution of problem* (13) (14) *and* $w(\cdot,\lambda)$ *exists on* $(-\infty,a]$,

4) *$u : E \cup H \to \mathbb{R}$ is of the class D on $E \cup H$ and*

$$|u(t,x)| \le \lambda(t) \quad \textit{for} \quad (t,x) \in E,$$

5) *for every $(t,x) \in H$, $t > 0$, the following functional differential inequality holds:*

$$|\partial_t u(t,x)| \le \sigma(t, W_t[u]) + \sum_{i=1}^{n} g_i(t) |\partial_{x_i} u(t,x)|.$$

Then $|u(t,x)| \le \omega(t,\lambda)$ for $(t,x) \in H$.

Proof. Let $\eta(t) = \max\{|u(t,x)| : x \in S_t\}$ for $t \in (-\infty, a]$. Then η is continuous on $[0,a]$. Moreover, $\eta_t = W_t[u]$ for $t \in [0,a]$. We define

$$I_+ = \big\{ t \in (0,a] : \eta(t) > \omega(t,\lambda) \big\}.$$

We shall prove that

$$D_-\eta(t) \le \sigma(t, \eta_t) \quad \text{for} \quad t \in I_+. \tag{28}$$

Let $t \in I_+$. There exists $x \in S_t$ such that $\eta(t) = |u(t,x)|$. Consider the case where $\eta(t) = u(t,x)$. Let

$$\begin{aligned} J_+[t,x] &= \big\{ i \in \{1,\dots,n\} : x_i = b_i - h_i(t) \big\}, \\ J_+[t,x] &= \big\{ i \in \{1,\dots,n\} : x_i = -b_i + h_i(t) \big\}, \\ J_0[t,x] &= \{1,\dots,n\} \setminus (J_+[t,x] \cup J_-[t,x]). \end{aligned}$$

It follows that $\eta(t) > 0$ and

$$\begin{aligned} \partial_{x_i} u(t,x) &\ge 0 \quad \text{for} \quad i \in J_+[t,x], \\ \partial_{x_i} u(t,x) &\le 0 \quad \text{for} \quad i \in J_-[t,x], \\ \partial_{x_i} u(t,x) &= 0 \quad \text{for} \quad i \in J_0[t,x]. \end{aligned}$$

We define a function $\gamma : [0,a] \to \mathbb{R}^n$, $\gamma = (\gamma_1, \dots, \gamma_n)$ as follows:

$$\gamma_i(\tau) = \begin{cases} b_i - h_i(\tau) & \text{for} \quad i \in J_+[t,x], \\ -b_i + h_i(\tau) & \text{for} \quad i \in J_-[t,x], \\ t & \text{for} \quad i \in J_0[t,x], \end{cases}$$

where $\tau \in [0,a]$. One can see that $\gamma(t) = x$. Let $\Gamma(\tau) = u(\tau, \gamma(\tau))$ for $\tau \in [0,t]$. We thus get $\Gamma(\tau) \le \eta(\tau)$ for $\tau \in [0,t]$ and $\Gamma(t) = \eta(t)$, and consequently $D_-\eta(t) \le \Gamma'(t)$. According to the assumptions 4), 5), we have

$$\begin{aligned} D_-\eta(t) &\le \partial_t u(t,x) + \sum_{i=0}^{n} \partial_{x_i} u(t,x) \gamma_i'(t) \\ &\le \sigma(t,\eta_t) + \sum_{i=0}^{n} g_i(t) |\partial_{x_i} u(t,x)| - \sum_{i \in J_+[t,x]} \partial_{x_i} u(t,x) h_i'(t) \\ &\quad + \sum_{i \in J_-[t,x]} \partial_{x_i} u(t,x) h_i'(t) \le \sigma(t,\eta_t). \end{aligned}$$

We have proved (28) in the case $\eta(t) = u(t,x)$. In a similar way we prove (28) when $\eta = -u(t,x)$. Therefore in virtue of Lemma 2.2 the proof of the theorem is complete. □

Now we give applications of the comparison result.

Assumption $H[f,\sigma]$**.** Suppose that

1) the function $f : \Omega \to R$ is continuous and satisfies the Volterra condition,
2) there is $\sigma : [0,a] \times X_+ \to \mathbb{R}_+$ such that
   (i) σ is continuous and satisfies the monotonicity condition W_+,
   (ii) the maximal solution of problem (13), (14) with $\lambda(t) = 0$ on $\mathbb{R}_-$ is $\bar{\eta}(t) = 0$ for $t \in (-\infty, a]$
   (iii) the estimate

$$\big|f(t,x,z,q) - f(t,x,\bar{z},\bar{q})\big| \le \sigma\big(t, W_t[z-\bar{z}]\big) + \sum_{i=1}^{n} g_i(t)|q_i - \bar{q}_i| \tag{29}$$

   is satisfied on Ω.

An immediate consequence of Theorem 3.1 is the following uniqueness result.

Theorem 3.2. *If Assumptions $H[X]$, $H[X,Y]$ and $H[f,\sigma]$ are satisfied then the Cauchy problem* (1), (2)*, where $\varphi \in Y$, admits at most one solution of the class D.*

Proof. If u and $\tilde{u}$ are solutions of (1), (2) on $E \cup H$ and they are of the class D, then the function $u - \tilde{u}$ satisfies all the assumptions of Theorem 3.1 with $\lambda = 0$. This gives $u = \tilde{u}$ on H.

Let us consider two problems: (1), (2) and the following one

$$\partial_t z(t,x) = \tilde{f}(t,x,z,\partial_x z(t,x)), \tag{30}$$

$$z(t,x) = \tilde{\varphi}(t,x) \quad \text{for} \quad (t,x) \in E, \tag{31}$$

where $\tilde{f} : \Omega \to \mathbb{R}$ and $\tilde{\varphi} \in Y$ are given functions. We prove a theorem on the estimation of the difference between solutions of (1), (2) and (30), (31). □

Theorem 3.3. *Suppose that Assumptions $H[X]$ and $H[X,Y]$ are satisfied and*

1) *the functions f, $\tilde{f} : \Omega \to R$ are continuous and they satisfy the Volterra condition,*
2) *the functions $u, \tilde{u} : E \cup H \to$ are solutions of problems* (1), (2) *and* (30), (31)*, respectively, and they are of the class D,*
3) *the initial estimate*

$$W_0[\varphi - \tilde{\varphi}](t) \le \lambda(t), \quad t \in \mathbb{R}_-$$

   is satisfied and $\lambda \in X_+$,
4) *there is a function $\sigma : [0,a] \times X_+ \to \mathbb{R}_+$ such that*
   (i) *σ is continuous and satisfies the monotonicity condition W_+,*
   (ii) *the maximal solution $\omega(\,\cdot\,,\lambda)$ of problem* (13), (14) *is defined on $(-\infty, a]$,*

(iii) *the estimate*

$$\left|f(t,x,z,q)-\tilde{f}(t,x,\tilde{z},\tilde{q})\right| \le \sigma(t, W_t[z-\bar{z}]) + \sum_{i=1}^{n} g_i(t)|q_i-\tilde{q}_i|$$

holds on Ω.

Then $|u(t,x)-\tilde{u}(t,x)| \le \omega(t,\lambda)$ *on* E.

Proof. The function $u-\tilde{u}$ satisfies all the assumptions of Theorem 3.1 and the assertion follows. □

Remark 3.4. If we put $\tilde{f}(t,x,z,q)=0$ on Ω and $\tilde{\varphi}(t,x)=0$ on E, then we obtain from Theorem 3.3 an estimate of solutions of problem (1), (2). For $\tilde{f}=f$ we get an estimate of the difference between two solutions of equation (1).

Now we prowe a theorem on continuous dependence of solutions on initial data and on the right-hand sides of equations. Suppose that the sequences of functions $\{f_m\}$ and $\{\varphi_m\}$ are given, where $f_m : \Omega \to \mathbb{R}$ and $\varphi_m \in Y$. For each $m \in \mathbb{N}$ we consider the Cauchy problem

$$\partial_t z(t,x) = f_m(t,x,z,\partial_x z(t,x)), \tag{32}$$

$$z(t,x) = \varphi_m(t,x) \quad \text{on} \quad E. \tag{33}$$

We prove the under natural assumptions on f, f_m and φ, φ_m, solutions of (32), (33) tend to a solution of (1), (2).

Theorem 3.5. *Suppose that Assumptions* $H[X]$, $H[X,Y]$ *and* $H[f,\sigma]$ *are satisfied and*

1) *for each* $m \in \mathbb{N}$ *we have* $f_m \in C(\Omega, R)$ *and* $\varphi_m \in Y$,
2) *there are sequences* $\{\varepsilon_m\}$ *and* $\{\lambda_m\}$, *where* $\varepsilon_m \in \mathbb{R}_+$, $\lambda_m \in X_+$ *and*
   (i) *the estimates*

$$\left|f(t,x,z,q)-f_m(t,x,z,q)\right| \le \varepsilon_m \quad \text{on} \quad \Omega$$

and

$$\left|\varphi(t,x)-\varphi_m(t,x)\right| \le \lambda_m(t) \quad \text{on} \quad E$$

are satisfied for $m \in \mathbb{N}$,
   (ii) *we have*

$$\lim_{m\to\infty} \varepsilon_m = 0, \qquad \lim_{m\to\infty} \|\lambda_m\|_X = 0$$

2) u *and* u_m *are solutions of* (1), (2) *and* (32), (33) *on* $E \cup H$, *respectively, and they are of the class* D.

Then there are $\alpha \in (0,a]$, $N \in \mathbb{N}$ *and a sequence* $\{\delta_m\}$, $\delta_m \in \mathbb{R}_+$ *such that*

$$\left|u(t,x)-u_m(t,x)\right| \le \delta_m \quad \text{for} \quad (t,x) \in H \cap ([0,\alpha] \times R^n \quad \text{and} \quad m \ge N \tag{34}$$

and $\lim_{m\to\infty} \delta_m = 0$.

Proof. It follows that for each $m \in \mathbb{N}$ the function $u-u_m$ satisfies the functional differential inequality

$$\big|\partial_t(u-u_m)(t,x)\big| \le \sigma\big(t, W_t[u-u_m]\big) + \varepsilon_m + \sum_{i=1}^{n} g_i(t)\,\big|\partial_{x_i}(u-u_m)(t,x)\big|,$$

where $(t,x) \in H$ and the initial estimate

$$\big|(\varphi-\varphi_m)(t,x)\big| \le \lambda_m(t), \quad (t,x) \in E$$

holds. Now the assertion of the theorem follows from Lemma 2.3 and Theorem 3.1. □

4. Examples of Function Spaces

We give examples of spaces X and Y satisfying Assumptions $H[X]$ and $H[X,Y]$.

Example 4.1. Let Y be the class of all functions $\varphi : E \to \mathbb{R}$ which are uniformly continuous and bounded on E. For $\varphi \in Y$ we put

$$\|\varphi\|_Y = \sup\big\{\|\varphi(t,x)\| :\ (t,x) \in E\big\}. \tag{35}$$

Let X be the class of all functions $\eta : \mathbb{R}_- \to R$ which are uniformly continuous and bounded on $\mathbb{R}_-$. For $\eta \in X$ we put

$$\|\eta\|_X = \sup\big\{\,|\eta(t)| :\ t \in \mathbb{R}_-\,\big\}. \tag{36}$$

Then, Assumptions $H[X]$ and $H[X,Y]$ are satisfied with all the constants equal to 1.

Example 4.2. Let $\gamma : \mathbb{R}_- \to (0,\infty)$ be a continuous function. Assume also that γ is nonincreasing on $\mathbb{R}_-$. Let Y be the space of all continuous functions $\varphi : E \to \mathbb{R}$ such that

$$\lim_{t\to-\infty} \frac{|\varphi(t,x)|}{\gamma(t)} = 0, \quad x \in [-b,b].$$

Write

$$||\varphi||_Y = \sup\left\{\frac{|\varphi(t,x)|}{\gamma(t)} :\ (t,x) \in E\right\}.$$

Let X be the space of all continuous functions $\eta : \mathbb{R}_- \to \mathbb{R}$ such that

$$\lim_{t\to-\infty} \frac{|\eta(t)|}{\gamma(t)} = 0.$$

For $\eta \in X$ we put

$$\|\eta\|_X = \sup\left\{\frac{|\eta(t)|}{\gamma(t)} :\ t \in \mathbb{R}_-\right\}.$$

Then, Assumptions $H[X]$ and $H[X,Y]$ are satisfied with $K = \frac{1}{\gamma(0)}$, $M = 1$, $L = \gamma(0)$.

Example 4.3. Let Y be the class of all functions $\varphi : E \to \mathbb{R}$ such that $\varphi \in C(E, \mathbb{R})$ and there exists the limit

$$\lim_{t \to -\infty} \varphi(t, x) = \varphi_0(x)$$

uniformly with respect to $x \in [-b, b]$. The norm in the space Y is defined by (35).

Let X be the set of all $\eta : \mathbb{R}_- \to \mathbb{R}$ such that there is the limit $\lim\limits_{t \to -\infty} \eta(t)$. The norm in the space X is defined by (36).

Then, Assumptions $H[X]$ and $H[X, Y]$ are satisfied with all the constants equal to 1.

Example 4.4. Let $r_0 \in \mathbb{R}_+$ and $p \geq 1$ be fixed. We will denote by Y_0 the class of all $\varphi : E \to \mathbb{R}$ such that

(i) φ is continuous on $[-r_0, 0] \times [-b, b]$,

(ii) for each $t \in (-\infty, -r_0]$ the function $\varphi(t, \cdot\,) : [-b, b] \to \mathbb{R}$ is continuous and

$$\int_{-\infty}^{-r_0} \big(S[\varphi](t) \big)^p \, dt \leq \infty,$$

where

$$S[\varphi](t) = \max \big\{ \, |\varphi(t, x)| : \quad x \in [-b, b] \, \big\}, \quad t \in (-\infty, -r_0]. \tag{37}$$

We define the norm in the space Y_0 by

$$\|\varphi\|_{Y_0} = \max \big\{ |\varphi(t, x)| : \ (t, x) \in [-r_0, 0] \times [-b, b] \, \big\} + \Big(\int_{-\infty}^{-r_0} (\, S[\varphi](t) \,)^p \, dt \Big)^{1/p}.$$

Write $Y = \bar{Y}_0$ where $\bar{Y}_0$ is the closure of Y_0 with the above-given norm.

Let X be the space of all $\eta : \mathbb{R}_- \to \mathbb{R}$ such that η is continuous on $[-r_0, 0]$ and

$$\int_{-\infty}^{-r_0} | \, \eta(t) \, |^p \, dt < +\infty.$$

We define the norm in the space X by

$$\|\eta\|_X = \max \big\{ \, |\eta(t)| : \ t \in [-r_0, 0] \, \big\} + \Big(\int_{-\infty}^{-r_0} | \, \eta(t) \, |^p \, dt \Big)^{1/p}.$$

Then Assumptions $H[X]$ and $H[X, Y]$ are satisfied and $K = 1 + a$, $L = 1$, $M = 2$.

Example 4.5. Let Y_0 be the space all functions $\varphi : E \to \mathbb{R}$ which are bounded and $\varphi(t, \cdot) : [-b, b] \to \mathbb{R}$ be continuous for each $t \in \mathbb{R}_-$. We also assume that

$$I[\varphi] = \sup \left\{ \int_{-(n+1)}^{-n} S[\varphi](t)\, dt : \; n \in \mathbb{N} \right\} < +\infty$$

where $S[\varphi]$ is given by (37) with $r_0 = 0$. The norm in the space Y_0 is defined by

$$\|\varphi\|_{Y_0} = \max \left\{ |\varphi(0, x)| : \; x \in [-b, b] \right\} + I[\varphi].$$

Write $Y = \bar{Y}_0$, where $\bar{Y}_0$ is the closure of Y_0 with the above-given norm.

Let X be the class of all functions $\eta : \mathbb{R}_- \to \mathbb{R}$ such that

(i) η is bounded on $\mathbb{R}_-$ and it is continuous on $\{ 0 \}$,
(ii) $I^\star[\eta] < +\infty$, where

$$I^\star[\eta] = \sup \left\{ \int_{-(n+1)}^{-n} |\eta(t)|\, dt : \; n \in \mathbb{N} \right\}.$$

The norm in the space X is defined by $\|\eta\|_X = |\eta(0)| + I^\star[\eta]$.

Then Assumptions $H[X]$ and $H[X, Y]$ are satisfied and $K = 1 + a$, $L = 1$, $M = 2$.

Remark 4.6. All the results of the paper can be extended to weakly coupled functional differential systems.

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