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MONOTONICITY AND THE ASYMPTOTIC ESTIMATE OF BLEIMANN BUTZER AND HAHN OPERATORS BASED ON q-INTEGERS

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Abstract. In this study, we obtain a Voronovskaja type asymptotic estimate for q-BBH operators. Second purpose of this paper is to obtain the monotonicity properties of q-BBH operators.

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1. Introduction

In [4], Bleimann, Butzer and Hahn introduced the following positive linear operators

$$L_n(f;x) = (1+x)^{-n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k \tag{1.1}$$

for $x \geq 0$.

Using the test functions $\left(\frac{x}{1+x}\right)^{\nu}, \nu = 0,1,2$, Jayasri and Sitaraman [8] obtained direct and inverse results for the operators (1.1). Later in [7], Gadjiev and Çakar established a Bohman–Korovkin type theorem and investigated the approximation properties of the BBH operators with the help of the same test functions $\left(\frac{x}{1+x}\right)^{\nu}$. Some generalizations of BBH operators are investigated by Agratini [1], [2] and the first author in [6].

In [10] and [11], Phillips constructed a generalization for the classical Bernstein polynomials based on q-integers.

We start by recalling some definitions about q-integers denoted by $[\,\cdot\,]$.

For any non-negative integer r, the q-integer of the number r is defined by

$$[r] = \begin{cases} \frac{1-q^r}{1-q} & \text{if } q \neq 1, \\ r & \text{if } q = 1, \end{cases} \tag{1.2}$$

where q is a positive real number. The q-factorial is defined as

$$[r]! = \begin{cases} [1][2]\cdots[r] & \text{if } r = 1,2,\dots, \\ 1 & \text{if } r = 0, \end{cases}$$

and the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}\,.$$

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It is obvious that q-binomial coefficients reduce to the ordinary ones when $q=1$.

Let us recall the Euler identity

$$\prod_{s=0}^{n-1}(1+q^s x)=\sum_{k=0}^{n} q^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n \\ k \end{array}\right] x^k. \tag{1.3}$$

It is clear that, this identity becomes an ordinary binomial expansion of $(1+x)^n$ when $q=1$.

In the light of these explanations, in a recent paper [3], we have introduced and investigated some Bohman–Korovkin type approximation properties and order of approximation in terms of a modulus of continuity for the following positive linear operators:

$$L_n(f;q;x)=\frac{1}{\ell_n(x)}\sum_{k=0}^{n} f\left(\frac{[k]}{[n-k+1]q^k}\right) q^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n \\ k \end{array}\right] x^k, \tag{1.4}$$

where

$$\ell_n(x)=\prod_{s=0}^{n-1}(1+q^s x).$$

Observe that the operators (1.4) reduce to the classical BBH operators when $q=1$.

In (1.4), if we take $f\left(\frac{[k]}{[n-k+1]}\right)$ instead of $f\left(\frac{[k]}{[n-k+1]q^k}\right)$ then we arrive naturally at a generalization of q-BBH operators. But in this case it is impossible to obtain explicit formulae for the second moment of (1.4) and in this situation a Voronovskaja type asymptotic estimate cannot be obtained.

In [3], we have derived the following equalities for the operators (1.4):

$$\begin{aligned} & L_n(1;q;x)=1, \\ & L_n\left(\frac{t}{1+t};q;x\right)=\frac{[n]}{[n+1]}\frac{x}{1+x}, \\ & L_n\left(\frac{t^2}{(1+t)^2};q;x\right)=\frac{[n][n-1]}{[n+1]^2}q^2\frac{x^2}{(1+x)(1+qx)}+\frac{[n]}{[n+1]^2}\frac{x}{1+x}. \end{aligned} \tag{1.5}$$

The aim of this paper is to obtain a Voronovskaja type asymptotic estimate and the monotonicity properties for the operators defined by (1.4).

2. A Voronovskaja Type Asymptotic Estimate

A Voronovskaja type theorem for the rate of convergence of q-Bernstein polynomials is given by Phillips [11].

In this section, we obtain a Voronovskaja type asymptotic estimate of the operators (1.4) for the test functions $\left(\frac{x}{1+x}\right)^{\nu}$, $\nu=0,1,2$.

First, let us give the following lemmas for the first and the second central moment of the operators (1.4):

Lemma 2.1. *For the first central moment of the operators* (1.4), *we have*

$$\mu_{n,1}\left(\frac{x}{1+x}\right)=\frac{1}{[n+1]}\frac{-xq^n}{(1+x)} \tag{2.1}$$

where

$$\mu_{n,1}\left(\frac{x}{1+x}\right)=L_n\left(\frac{t}{1+t}-\frac{x}{1+x};q;x\right).$$

Proof. By (1.5), the proof is obvious. □

Lemma 2.2. *For the second central moment of the operators* (1.4), *we have*

$$\begin{aligned}\mu_{n,2}\left(\frac{x}{1+x}\right)=\frac{1}{[n+1]}\Bigg(&\left(\frac{x}{1+x}\right)^2\Bigg(\frac{-[n]^2q(1-q)}{(1+qx)[n+1]}+\frac{[n](q-2)}{(1+qx)[n+1]}+\\&+\frac{1}{(1+qx)[n+1]}+\frac{qx[n]^2(1-q)^2}{(1+qx)[n+1]}+\frac{qx([n](2q-3)+1)}{(1+qx)[n+1]}\Bigg)\\&+\frac{[n]}{[n+1]}\frac{x}{1+x}\Bigg),\end{aligned} \tag{2.2}$$

where

$$\mu_{n,2}(\frac{x}{1+x})=L_n\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2;q;x\right).$$

Proof. It is clear that we can write

$$q^k[n-k+1]=[n+1]-[k],\ q[k-1]=[k]-1. \tag{2.3}$$

If we use (2.3) and (1.5) in the equality

$$\begin{aligned}\mu_{n,2}\left(\frac{x}{1+x}\right)=L_n\left(\left(\frac{t}{1+t}\right)^2;q;x\right)-2\,\frac{x}{1+x}\,L_n\left(\frac{t}{1+t};q;x\right)\\+\left(\frac{x}{1+x}\right)^2L_n(1;q;x),\end{aligned}$$

after simple calculations, we get (2.2). □

Now consider a sequence of positive numbers $q=(q_n)$ such that

$$\lim_{n\to\infty}q_n=1,\quad \lim_{n\to\infty}q_n^n=a, \tag{2.4}$$

where $a>0$, $a\neq 1$. In the rest of this section $[\,\cdot\,]$ stands for a q_n-integer. We have

$$\lim_{n\to\infty}[n]=\lim_{n\to\infty}[n+1]=\infty\ \text{ and }\ \lim_{n\to\infty}\frac{[n]}{[n+1]}=1. \tag{2.5}$$

Lemma 2.3. *If the conditions in* (2.4) *hold, then we have*

$$\lim_{n\to\infty}[n+1]\,\mu_{n,1}\left(\frac{x}{1+x}\right)=\frac{-ax}{(1+x)}, \tag{2.6}$$

and

$$\lim_{n\to\infty}[n+1]\,\mu_{n,2}\left(\frac{x}{1+x}\right)=\frac{(a-2)x^2+x}{(1+x)^3} \tag{2.7}$$

for $q=(q_n)$.

Proof. By using (2.4) and (2.5) in (2.1) we have (2.6).

On the other hand, using (2.4) and (2.5), we can easily show that the following limits are satisfied:

$$\lim_{n\to\infty}\frac{-[n]^2q_n(1-q_n)}{(1+q_nx)[n+1]}=\frac{a-1}{1+x},\quad \lim_{n\to\infty}\frac{[n](q_n-2)}{(1+q_nx)[n+1]}=\frac{-1}{1+x},$$
$$\lim_{n\to\infty}\frac{1}{(1+q_nx)[n+1]}=0,\quad \lim_{n\to\infty}\frac{q_nx[n]^2(1-q_n)^2}{(1+q_nx)[n+1]}=0,$$
$$\lim_{n\to\infty}\frac{q_nx([n](2q_n-3)+1)}{(1+q_nx)[n+1]}=\frac{-x}{1+x},\quad \lim_{n\to\infty}\frac{[n]}{[n+1]}\frac{x}{1+x}=\frac{x}{1+x}.$$

If we use these equalities in (2.2), then we obtain (2.7). □

Now, we can give the following first main result.

Theorem 2.4. *Suppose that the first and the second derivatives of* $f(\frac{x}{1+x})$ *exist for* $x\geq 0$, *then we have*

$$\lim_{n\to\infty}[n+1]\left(L_n\left(f\left(\frac{t}{1+t}\right);q_n;x\right)-f\left(\frac{x}{1+x}\right)\right)$$
$$=\frac{-ax}{(1+x)}f'\left(\frac{x}{1+x}\right)+\frac{(a-2)x^2+x}{2(1+x)^3}f''\left(\frac{x}{1+x}\right), \tag{2.8}$$

where the sequence (q_n) *satisfies conditions* (2.4).

Proof. This is analogous to the Voronovskaja's proof (see [9, p. 22]).

Using Taylor's formula, we can write

$$f(\alpha)=f(t)+(\alpha-t)f'(t)+(\alpha-t)^2\left(\frac{1}{2}f''(t)+\eta(\alpha-t)\right), \tag{2.9}$$

where $|\eta(h)|\leq H$ for all h and converges to zero as $h\to 0$. By taking $\alpha=\frac{[k]}{[n+1]}$ and $t=\frac{x}{1+x}$ in (2.9) we get

$$f\left(\frac{[k]}{[n+1]}\right)=f\left(\frac{x}{1+x}\right)+\left(\frac{[k]}{[n+1]}-\frac{x}{1+x}\right)f'\left(\frac{x}{1+x}\right)$$
$$+\left(\frac{[k]}{[n+1]}-\frac{x}{1+x}\right)^2\left(\frac{1}{2}f''\left(\frac{x}{1+x}\right)+\eta\left(\frac{[k]}{[n+1]}-\frac{x}{1+x}\right)\right). \tag{2.10}$$

Applying the equality (2.10) to the operator (1.4), we have

$$L_n\left(f\left(\frac{t}{1+t}\right);q_n;x\right)-f\left(\frac{x}{1+x}\right)$$
$$=f'\left(\frac{x}{1+x}\right)\mu_{n,1}\left(\frac{x}{1+x}\right)+\frac{1}{2}f''\left(\frac{x}{1+x}\right)\mu_{n,2}\left(\frac{x}{1+x}\right)$$

$$+ L_n\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2\eta\left(\frac{t}{1+t}-\frac{x}{1+x}\right);q_n;x\right).$$

Since $\eta(h)\to 0$ for $h\to 0$, if we use (2.7), then we get

$$L_n\left(f\left(\frac{t}{1+t}\right);q_n;x\right)-f\left(\frac{x}{1+x}\right)=f'\left(\frac{x}{1+x}\right)\mu_{n,1}\left(\frac{x}{1+x}\right) + \frac{1}{2}f''\left(\frac{x}{1+x}\right)\mu_{n,2}\left(\frac{x}{1+x}\right)+\frac{\mathcal{O}(\varepsilon_n)}{[n+1]}. \tag{2.11}$$

Multiplying both sides of (2.11) by $[n+1]$ and using (2.6) and (2.7), we obtain desired result. This completes the proof of the theorem. □

From Theorem 2.1 it follows that if one or two of the following statements

(i) $f'\left(\frac{x}{1+x}\right)\neq 0$ and $x\neq 0$,

(ii) $f''\left(\frac{x}{1+x}\right)\neq 0$, $x\neq 0$ and $x\neq\frac{1}{2-a}$

are valid, then we can say that the rate of convergence of $L_n(f(\frac{t}{1+t});q_n;x)$ to $f(\frac{x}{1+x})$ is exactly of order q_n-1.

3. Monotonicity Properties

The first results on the monotonicity properties of classical Bernstein polynomials were obtained by Temple [13]. Later, Stancu [12] obtained nice results on the monotonicity of the sequence formed by first order derivatives of Bernstein polynomials.

Also, some useful monotonicity properties for Szász operators and Meyer–König and Zeller operators were obtained by Cheney and Sharma [5]. They proved the following two theorems for Meyer–König and Zeller operators M_n and also for Szász operators S_n which are defined by

$$M_n(f;x)=(1-x)^{n+1}\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{\nu+n}\right)\binom{\nu+n}{\nu}x^{\nu}$$

and

$$S_n(f;x)=e^{-nx}\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{n}\right)\frac{(nx)^{\nu}}{\nu!},$$

respectively.

Theorem A ([5]). *If f is convex, then $M_n(f;x)$ is decreasing in n, unless f is linear (in which case $M_n(f;x)=M_{n+1}(f;x)$ for all n).*

Theorem B ([5]). *If f is convex, then $S_n(f;x)$ is decreasing in n, unless f is linear (in which case $S_n(f;x)=S_{n+1}(f;x)$ for all n).*

Below we will give a theorem of this type for operators defined by (1.4). Note that, our result remains valid not only for convex functions but also for linear functions. Firstly, let us give the following lemmas.

Lemma 3.1. *q-binomial coefficients satisfy the following equalities*

$$\begin{aligned} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} &= \frac{[n+1][n]}{[n-k][k+1]} \begin{bmatrix} n-1 \\ k \end{bmatrix} \\ \begin{bmatrix} n \\ k \end{bmatrix} &= \frac{[n]}{[n-k]} \begin{bmatrix} n-1 \\ k \end{bmatrix} \\ \begin{bmatrix} n \\ k+1 \end{bmatrix} &= \frac{[n]}{[k+1]} \begin{bmatrix} n-1 \\ k \end{bmatrix}. \end{aligned} \tag{3.1}$$

Proof. The proof is obvious from the definition of a q-binomial coefficient. □

Lemma 3.2. *The equalities*

$$\lambda_1 + \lambda_2 = 1 \tag{3.2}$$

and

$$\frac{[k+1]}{[n-k+1]q^{k+1}} = \lambda_1\alpha_1 + \lambda_2\alpha_2 \tag{3.3}$$

are satisfied for

$$\lambda_1 = \frac{q^{n-k}[k+1]}{[n+1]}, \quad \lambda_2 = \frac{[n-k]}{[n+1]}, \quad \alpha_1 = \frac{[k]}{[n-k+1]q^k} \quad \text{and} \quad \alpha_2 = \frac{[k+1]}{[n-k]q^{k+1}}.$$

Proof. From the definition of q-integers, we have

$$[k+1] = 1 + q + q^2 + \cdots + q^k, \quad [n-k] = 1 + q + q^2 + \cdots + q^{n-k-1}. \tag{3.4}$$

Then we obtain

$$\lambda_1 + \lambda_2 = \frac{q^{n-k}[k+1] + [n-k]}{[n+1]} = \frac{1 + q + q^2 + \cdots + q^n}{[n+1]} = 1.$$

On the other hand, by direct calculations, we have

$$\begin{aligned} \lambda_1\alpha_1 + \lambda_2\alpha_2 &= \frac{q^{n-k}[k+1]}{[n+1]} \frac{[k]}{[n-k+1]q^k} + \frac{[k+1]}{[n+1]q^{k+1}} \\ &= \frac{[k+1]}{[n+1]q^{k+1}} \frac{q^{n-k+1}[k] + [n-k+1]}{[n-k+1]}. \end{aligned} \tag{3.5}$$

If we use similar equalities in (3.4), we have

$$q^{n-k+1}[k] + [n-k+1] = 1 + q + q^2 + \cdots + q^n = [n+1]. \tag{3.6}$$

By using (3.6) in (3.5), we get (3.3) and the proof is completed. □

Now, let us give our second main result.

Theorem 3.3. *If f is convex and decreasing, then $L_n(f;q;x)$ is decreasing in n for all $x > 0$.*

Proof.

$$\begin{aligned} &L_{n+1}(f;q;x) - L_n(f;q;x) \\ &\quad = \frac{1}{\ell_{n+1}(x)} \sum_{k=0}^{n} f\left(\frac{[k]}{[n-k+2]q^k}\right) q^{\frac{k(k-1)}{2}} \begin{bmatrix} n+1 \\ k \end{bmatrix} x^k \end{aligned}$$

$$- \frac{1}{\ell_n(x)} \sum_{k=0}^{n} f\left(\frac{[k]}{[n-k+1]q^k}\right) q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

$$= \frac{1}{\ell_{n+1}(x)} q^{\frac{n(n+1)}{2}} x^{n+1} \left(f\left(\frac{[n+1]}{q^{n+1}}\right) - f\left(\frac{[n]}{q^n}\right)\right)$$

$$+ \frac{1}{\ell_{n+1}(x)} \sum_{k=0}^{n-1} f\left(\frac{[k+1]}{[n-k+1]q^{k+1}}\right) q^{\frac{k(k-1)}{2}} q^k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} x^{k+1}$$

$$- \frac{1}{\ell_{n+1}(x)} \sum_{k=0}^{n-1} f\left(\frac{[k]}{[n-k+1]q^k}\right) q^{\frac{k(k-1)}{2}} q^n \begin{bmatrix} n \\ k \end{bmatrix} x^{k+1}$$

$$- \frac{1}{\ell_{n+1}(x)} \sum_{k=0}^{n-1} f\left(\frac{[k+1]}{[n-k]q^{k+1}}\right) q^{\frac{k(k-1)}{2}} q^k \begin{bmatrix} n \\ k+1 \end{bmatrix} x^{k+1}.$$

If we use (3.1) in the last equality, we obtain

$$L_{n+1}(f;q;x) - L_n(f;q;x) = \frac{1}{\ell_{n+1}(x)} q^{\frac{n(n+1)}{2}} x^{n+1} \left(f\left(\frac{[n+1]}{q^{n+1}}\right) - f\left(\frac{[n]}{q^n}\right)\right)$$
$$+ \frac{1}{\ell_{n+1}(x)} \sum_{k=0}^{n-1} \left\{ f\left(\frac{[k+1]}{[n-k+1]q^{k+1}}\right) - \frac{q^{n-k}[k+1]}{[n+1]} f\left(\frac{[k]}{[n-k+1]q^k}\right) \right.$$
$$\left. - \frac{[n-k]}{[n+1]} f\left(\frac{[k+1]}{[n-k]q^{k+1}}\right)\right\} q^k \frac{[n+1][n]}{[n-k][k+1]} \begin{bmatrix} n-1 \\ k \end{bmatrix} x^{k+1}. \quad (3.7)$$

Since $\frac{[n+1]}{q^{n+1}} - \frac{[n]}{q^n} = \frac{1}{q^{n+1}} > 0$ and f is decreasing, we get

$$f\left(\frac{[n+1]}{q^{n+1}}\right) - f\left(\frac{[n]}{q^n}\right) < 0. \quad (3.8)$$

Since f is convex, we obtain

$$f\left(\frac{[k+1]}{[n-k+1]q^{k+1}}\right) - \frac{q^{n-k}[k+1]}{[n+1]} f\left(\frac{[k]}{[n-k+1]q^k}\right)$$
$$- \frac{[n-k]}{[n+1]} f\left(\frac{[k+1]}{[n-k]q^{k+1}}\right) \leq 0 \quad (3.9)$$

by Lemma 3.2.

Using (3.8) and (3.9) in (3.7), we obtain the desired result. □

Next, if f is linear, then we can write

$$f\left(\frac{[k+1]}{[n-k+1]q^{k+1}}\right) - \frac{q^{n-k}[k+1]}{[n+1]} f\left(\frac{[k]}{[n-k+1]q^k}\right)$$
$$- \frac{[n-k]}{[n+1]} f\left(\frac{[k+1]}{[n-k]q^{k+1}}\right) = 0 \quad (3.10)$$

immediately. As a result of (3.10), we obtain

Corollary 3.4. *We have the following monotonicity properties for the operators defined by* (1.4)*:*

(i) *If f is linear and decreasing, then $L_n(f;q;x)$ is decreasing in n for all $x > 0$,*

(ii) *If f is linear and increasing, then $L_n(f;q;x)$ is increasing in n for all $x > 0$.*

References

1. O. Agratini, Approximation properties of a generalization of Bleimann, Butzer and Hahn operators. *Math. Pannon.* **9**(1998), No. 2, 165–171.
2. O. Agratini, A class of Bleimann, Butzer and Hahn type operators. *An. Univ. Timişoara Ser. Mat.-Inform.* **34**(1996), No. 2, 173–180.
3. A. Aral and O. Doğru, Bleimann, Butzer and Hahn operators based on the q-integers, (*Submitted for publication*).
4. G. Bleimann, P. L. Butzer, and L. Hahn, A Bernšteĭn-type operator approximating continuous functions on the semi-axis. *Nederl. Akad. Wetensch. Indag. Math.* **42**(1980), No. 3, 255–262.
5. E. W. Cheney and A. Sharma, Bernstein power series. *Canad. J. Math.* **16**(1964), 241–252.
6. O. Doğru, On Bleimann, Butzer and Hahn type generalization of Balzs operators. *Studia Univ. Babeş-Bolyai Math.* **47**(2002), No. 4, 37–45.
7. A. D. Gadjiev and Ö. Çakar, On uniform approximation by Bleimann, Butzer and Hahn operators on all positive semiaxis. Translated by the authors. *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **19**(1999), No. 5, *Math. Mech.,* 21–26 (2000).
8. C. Jayasri and Y. Sitaraman, Direct and inverse theorems for certain Bernstein-type operators. *Indian J. Pure Appl. Math.* **16**(1985), No. 12, 1495–1511.
9. G. G. Lorentz, Bernstein polynomials. *Mathematical Expositions,* No. 8. *University of Toronto Press, Toronto,* 1953.
10. G. M. Phillips, On generalized Bernstein polynomials. *Numerical analysis,* 263–269, *World Sci. Publishing, River Edge, NJ,* 1996.
11. G. M. Phillips, Bernstein polynomials based on the q-integers. *The heritage of P. L. Chebyshev: a Festschrift in honor of the 70th birthday of T. J. Rivlin. Ann. Numer. Math.* **4**(1997), No. 1-4, 511–518.
12. D. D. Stancu, On the monotonicity of the sequence formed by the first order derivatives of the Bernstein polynomials. *Math. Z.* **98**(1967), 46–51.
13. W. B. Temple, Stieltjes integral representation of convex functions. *Duke Math. J.* **21**(1954), 527–531.

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