Georgian Mathematical Journal
Volume 12 (2005), Number 3, 493–504

SET-THEORETIC PROPERTIES OF SCHMIDT'S IDEAL

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Abstract. We study some set-theoretic properties of Schmidt's σ-ideal on $\mathbb{R}$, emphasizing its analogies and dissimilarities with both the classical σ-ideals on $\mathbb{R}$ of Lebesgue measure zero sets and of Baire first category sets. We highlight the strict analogy between Schmidt's ideal on $\mathbb{R}$ and Mycielski's ideal on $2^{\mathbb{N}}$.

2000 Mathematics Subject Classification: 03E50, 11K60, 28A05, 91A44.
Key words and phrases: σ-ideals, (α, β)-games, Mycielski σ-ideal, Borel sets, countable chain condition, Steinhaus property, Ruziewicz property, Sierpiński–Erdős maps, badly approximable numbers.

1. Introduction

Let X stand for either $\mathbb{R}$ or the Cantor space $2^{\mathbb{N}}$ (i.e., the space of functions from the set $\mathbb{N}$ of nonnegative integers into $\{0, 1\}$, equipped with the natural topology, product measure and additive group structure).

Recall that a σ-ideal $\mathcal{J}$ on X is said to be *proper* if $X \notin \mathcal{J}$; *uniform* if $\{x\} \in \mathcal{J}$ for all $x \in X$; *translation invariant* if $x + J \in \mathcal{J}$ whenever $x \in X$ and $J \in \mathcal{J}$. We say that two σ-ideals $\mathcal{J}_0$ and $\mathcal{J}_1$ on X are *orthogonal*, and write $\mathcal{J}_0 \perp \mathcal{J}_1$ if $X = J_0 \cup J_1$ for some $J_0 \in \mathcal{J}_0$ and $J_1 \in \mathcal{J}_1$. A *Sierpiński–Erdős* $(\mathcal{J}_0, \mathcal{J}_1)$*-map* is an involutive map $f : X \to X$ such that $f(J_0) \in \mathcal{J}_1$ iff $J_0 \in \mathcal{J}_0$.

Throughout the paper, $\mathcal{B}$ and $\mathcal{A}$ will denote, respectively, the Borel σ-algebra of X and the family of analytic subsets of X. If $\mathcal{J}$ is a σ-ideal on X, by $\sigma(\mathcal{B} \cup \mathcal{J})$ we mean the σ-algebra of X generated by $\mathcal{B}$ and $\mathcal{J}$. It consists of the sets $A = B \triangle J$, where $B \in \mathcal{B}$ and $J \in \mathcal{J}$. We say that $\mathcal{J}$ is *Borel generated* if for every $J \in \mathcal{J}$ there exists $B \in \mathcal{B} \cap \mathcal{J}$ containing J; that has the *countable chain condition* (c.c.c.) if any disjoint family of sets in $\mathcal{B} \setminus \mathcal{J}$ is countable; that has the *Steinhaus property* if 0 belongs to the topological interior of $B - B$ for each $B \in \mathcal{B} \setminus \mathcal{J}$. A σ-ideal $\mathcal{J}$ on $\mathbb{R}$ has the *Ruziewicz property* if for every $B \in \mathcal{B} \setminus \mathcal{J}$ and every $A \subseteq \mathbb{R}$ with $|A| < \aleph_0$ there exists an affine copy of A contained in B. Moreover, $\mathcal{J}$ is *invariant under diffeomorphisms* if for any $J \in \mathcal{J}$, any open interval I, and any diffeomorphism $f : I \to \mathbb{R}$ there holds $f(I \cap J) \in \mathcal{J}$ (by a diffeomorphism $f : I \to \mathbb{R}$ we mean a C^1-mapping such that $f'(x) \neq 0$ for every $x \in I$).

The most important examples of proper, uniform, translation invariant, and Borel generated σ-ideals on X are those of measure (i.e., the family of Lebesgue measure zero sets) and of category (i.e., the family of Baire first category sets). We denote these ideals by $\mathcal{L}$ and $\mathcal{K}$, respectively, and by $\mathcal{E}$ the σ-ideal generated by closed measure zero sets. It is well known that $\mathcal{E} \subsetneq \mathcal{K} \cap \mathcal{L}$; that $\mathcal{L} \perp \mathcal{K}$; and

ISSN 1072-947X / \$8.00 /

that $\mathcal{A} \subseteq \sigma(\mathcal{B} \cup \mathcal{L}) \cap \sigma(\mathcal{B} \cup \mathcal{K})$. Further, $\mathcal{L}$, $\mathcal{K}$ and $\mathcal{L} \cap \mathcal{K}$ have the c.c.c. and the Steinhaus property. In the real line case, $\mathcal{L}$, $\mathcal{K}$ and $\mathcal{L} \cap \mathcal{K}$ are also invariant under diffeomorphisms and have the Ruziewicz property (see, e.g., [22]).

Under CH (or MA) there exists a Sierpiński–Erdős $(\mathcal{L}, \mathcal{K})$-map $f : X \to X$ (see [16], Theorem 19.3). Being an isomorphism between $\mathcal{L}$ and $\mathcal{K}$, such a mapping clearly preserves all their purely set-theoretic properties. However, according to [3] and [11] it cannot be additive, thus it does not preserve the properties involving the algebraic structure of the space.

Since the beginning of set-theoretic Real Analysis, a significant part of the mathematical literature has been devoted to the investigation of analogies (and differences as well) between $\mathcal{L}$ and $\mathcal{K}$. The notion of σ-ideal is in fact essential for the abstract formalization of the "smallness" of subsets of X. Moreover, the orthogonality relation between σ-ideals on X efficaciously expresses how certain different concepts of smallness for sets can be, in some cases, even antithetical.

In certain applications, the ideals of measure and category turn out not to be suitable ones. In these situations one may try to use another *workable* σ-ideal on X alternative to $\mathcal{L}$ and $\mathcal{K}$. Here "workable" stands for "having some of the good descriptive, algebraic and geometric properties shared by $\mathcal{L}$ and $\mathcal{K}$".

In 1966, Schmidt introduced in [21] the so-called (α, β)-games and studied, from a number-theoretic viewpoint, the σ-ideal of losing sets related to them. The definitions of Schmidt's games and σ-ideal are postponed until the next section.

Inspired by the work of Schmidt, Mycielski introduced in [15] a σ-ideal $\mathcal{M}$ on $2^{\mathbb{N}}$. His work inspired many authors to study ideals in the Cantor space related to games (to list only some: [2], [6], [8], [17], [18], [19], [20]). We limit ourselves to stating roughly that $\mathcal{M}$ consists of "losing" sets associated with certain games with perfect information played by two competitors.

Theorem 1.1 (Mycielski)**.** *The σ-ideal $\mathcal{M}$ on $2^{\mathbb{N}}$ is proper, uniform, translation invariant, and G_δ-generated. Moreover, $\mathcal{M}$ is orthogonal to $\mathcal{E}$ (a fortiori, to both $\mathcal{L}$ and $\mathcal{K}$).*

See [15], Propositions 1, 2, 5, and Theorems 3, 4, 10. While Theorem 1.1 collects some of analogies between $\mathcal{M}$ and both $\mathcal{L}$ and $\mathcal{K}$, the next theorem illustrates a few dissimilarities:

Theorem 1.2 (Mycielski, Rosłanowski)**.** *$\mathcal{M}$ has neither the c.c.c. nor the Steinhaus property. Moreover, $\mathcal{A} \nsubseteq \sigma(\mathcal{B} \cup \mathcal{M})$ (hence, $\sigma(\mathcal{B} \cup \mathcal{M})$ is not closed under the Souslin operation).*

The first statement is due to Mycielski [15], Propositions 11 and 12; the second one to Rosłanowski [18], Theorem 2.4.

A natural question then arises: what can be said about the real line?

In our paper (following [23]) we study an ideal $\mathcal{S}$ being a minor – but seemingly necessary for obtaining good descriptive results – a modification of the original Schmidt's σ-ideal (for this reason also $\mathcal{S}$ will be referred to as Schmidt's ideal). It turns out that $\mathcal{S}$ plays in $\mathbb{R}$ the same role as $\mathcal{M}$ does in $2^{\mathbb{N}}$. Moreover, from the point of view of number-theoretic applications, with which Schmidt's paper is mainly concerned, $\mathcal{S}$ proves to be useful in much the same way.

One of the aims of this paper is to stress a very close analogy between Schmidt's and Mycielski's σ-ideals. We wish to clarify, however, that such a parallelism is far from being immediate: because of the "discrete" nature of games defined in the Cantor space, Mycielski's games are easier to deal with (this is what Mycielski writes at the beginning of his paper). On the other hand, Schmidt's (α, β)-games are necessarily more involved and require a different approach. As an example, the reader is invited to compare Theorem 3.2 below and the corresponding Theorem 10 in [15].

Let us now summarize the main results of our paper. After collecting in Theorem 2.2 a few necessary Schmidt's results about $\mathcal{S}$, in the third section we prove that $\mathcal{S}$ is G_δ-generated (this sharpens Theorem 12 in [23]). We also prove in Theorem 3.3 that $\mathcal{S}$ is invariant under diffeomorphisms (our ad hoc proof is simpler than that of Theorem 1 in [21]). Observe the exact correspondence between Theorem 1.1 (holding for $\mathcal{M}$ on $2^{\mathbb{N}}$) and Theorems 2.2–3.3 (holding for $\mathcal{S}$ on $\mathbb{R}$).

In the fourth section we present some consequences of Theorems 2.2–3.3. In particular, in Corollary 4.3 we derive the existence of a purely transcendental uncountable subfield of $\mathbb{R}$ all of whose irrational numbers are badly approximable. It is worth remarking that the existence of such a field cannot be deduced by appealing to the celebrated Mycielski's theorems [13], [14], for the set of badly approximable numbers is neither residual nor of full Lebesgue measure (it is, instead, both of the first category and Lebesgue measure zero). Proposition 4.4 establishes, under CH, the existence of Sierpiński–Erdős $(\mathcal{S}, \mathcal{J})$-maps $f : \mathbb{R} \to \mathbb{R}$, where $\mathcal{J}$ is equal to any one of the following: $\mathcal{L}$, $\mathcal{K}$, $\mathcal{L} \cap \mathcal{K}$, $\mathcal{E}$.

Finally, in the last section we collect some of the differences between $\mathcal{S}$ and both $\mathcal{L}$ and $\mathcal{K}$. More precisely, by means of a suitable Cantor-like construction we prove in Theorem 5.1 that $\mathcal{S}$ has neither the c.c.c., nor the Steinhaus property, nor the Ruziewicz property. Moreover, appealing to a theorem of Balcerzak [1] we show that $\mathcal{A} \nsubseteq \sigma(\mathcal{B} \cup \mathcal{S})$. Once more, we stress the precise correspondence between Theorem 1.2 and our Theorem 5.1.

2. Schmidt's (α, β)-Games and σ-Ideal $\mathcal{S}$

From now on, we let $Q := (0, \frac{1}{2}) \cap \mathbb{Q}$. By $\mathcal{I}_{\mathbb{R}}$ we denote the set of all nontrivial compact subintervals of $\mathbb{R}$. For $I \in \mathcal{I}_{\mathbb{R}}$, $\ell(I)$ is the length of I. If $\delta \in (0, 1)$, then

$$B^\delta(I) := \{C \in \mathcal{I}_{\mathbb{R}} : C \subseteq I \text{ and } \ell(C) = \delta \ell(I)\}.$$

Take $\alpha, \beta \in Q$ and $A \subseteq \mathbb{R}$. The Schmidt (α, β)-game relative to A between the two players Adam and Eve is defined as follows. Adam selects $A_0 \in \mathcal{I}_{\mathbb{R}}$. Then Eve chooses $E_0 \in B^\alpha(A_0)$. Adam selects in his turn $A_1 \in B^\beta(E_0)$, and so on. In general, the rules of the game can be summarized as below:

$$A_0 \in \mathcal{I}_{\mathbb{R}};$$
$$E_n \in B^\alpha(A_n) \text{ for all } n \in \mathbb{N};$$
$$A_{n+1} \in B^\beta(E_n) \text{ for all } n \in \mathbb{N}.$$

Clearly,

$$\bigcap_{n=0}^{\infty} E_n = \bigcap_{n=0}^{\infty} A_n = \{x\}$$

for a unique $x \in \mathbb{R}$ (called the result of the game). If $x \notin A$, we say that Eve wins the (α, β)-game. An (α, β)-strategy relatively to A (for Eve) is a map $\sigma_A = \sigma_A(\alpha, \beta)$ associating with every finite sequence of Adam's legal moves $A_0, \ldots, A_n$ an interval $E_n = \sigma_A(\alpha, \beta; n; A_0, \ldots, A_n) \in B^\alpha(A_n)$. We say that an (α, β)-strategy σ_A relatively to A is winning if for any legal sequence $(A_n)_{n=0}^\infty$ of Adam's moves it holds

$$\bigcap_{n=0}^{\infty} E_n = \bigcap_{n=0}^{\infty} A_n \nsubseteq A.$$

According to [23], we say that A is (α, β)-*losing*, and that its complement $E := \mathbb{R} \setminus A$ is (α, β)-*winning*, in case there exists a winning (α, β)-strategy relative to A (for Eve). We say that A is *losing* if it is (α, β)-losing for every $\alpha, \beta \in Q$. Consistently, we then say that E is *winning*.

By $\mathcal{S}$ we denote the family of all losing subsets of $\mathbb{R}$. We call $\mathcal{S}$ *Schmidt's σ-ideal*. Our definition makes sense, for it is shown in [21] (for all real parameters) and in [23] (for the rational ones) that the family of losing sets is indeed a σ-ideal on $\mathbb{R}$.

Remark 2.1. In Schmidt's original paper all real parameters $\alpha \in (0, \frac{1}{2})$ and $\beta \in (0, 1)$ are considered. Following [23], here we focus exclusively on rational ones. Observe that in this way we define an ideal possibly larger than the original one (unfortunately we do not know whether this inclusion is strict).

Recall that a real number x is said to be *badly approximable* if $|x - \frac{p}{q}| > \frac{c}{q^2}$ for some $c > 0$ and all rationals $\frac{p}{q}$ (as a few examples of badly approximable numbers we mention the "golden ratio" $g := \frac{1+\sqrt{5}}{2}$ and, more generally, all the quadratic irrationals).

Theorem 2.2 (Schmidt). *$\mathcal{S}$ is a proper, uniform and translation invariant σ-ideal on $\mathbb{R}$. Moreover, $\mathcal{S} \perp \mathcal{E}$.*

Proof. It follows from Theorem 2 in [21] (see also [23], Theorem 10) that $\mathcal{S}$ is a σ-ideal. Obviously $\mathcal{S}$ is proper, uniform and translation invariant.

Further, we infer from Theorem 3 in [21] that the class of badly approximable numbers is winning. On the other hand, the very same set is immediately seen to be an F_σ-set of the first category and, according to Khintchine's theorem ([9], Theorems 23 and 29), it has Lebesgue measure zero. Hence $\mathcal{S} \perp \mathcal{E}$. □

3. "Nice" Properties of $\mathcal{S}$

It is shown in [23] that $\mathcal{S}$ is coanalytic generated. Here we strengthen that result by showing that $\mathcal{S}$ is G_δ-generated. We begin with an obvious lemma, whose verification is omitted.

Lemma 3.1. *Let $K, \beta \in (0,1)$ and $I \in \mathcal{I}_\mathbb{R}$. Then there is a finite subset $\widehat{B}^{K\beta}(I)$ of $B^{K\beta}(I)$ with the property that any $C \in B^\beta(I)$ includes an element of $\widehat{B}^{K\beta}(I)$.* □

Theorem 3.2. *$\mathcal{S}$ is G_δ-generated.*

Proof. We will show that the complement E of any losing set A includes a winning F_σ-set W. Since $|Q| = \aleph_0$, it suffices to show that for fixed $\alpha, \beta \in Q$ the set E contains an (α, β)-winning F_σ-set $W(\alpha, \beta)$, and then to take $W := \bigcup_{\alpha,\beta\in Q} W(\alpha, \beta)$.

Let us fix $\alpha, \beta \in Q$. We shall define an (α, β)-strategy $\sigma_{W(\alpha,\beta)}$ (for Eve) and an F_σ-set $W(\alpha, \beta) \subseteq E$ being the set of all possible results determined by $\sigma_{W(\alpha,\beta)}$. Clearly, such a set is (α, β)-winning.

To this end, we will define an auxiliary game with parameters $\alpha', \beta' \in Q$ in such a way that

$$A_0 \supseteq A'_0 \supseteq E'_0 \supseteq E_0 \supseteq \cdots \supseteq A_n \supseteq A'_n \supseteq E'_n \supseteq E_n \supseteq \cdots, \tag{3.1}$$

where the sequences $(A_n, E_n)_{n=0}^\infty$ and $(A'_n, E'_n)_{n=0}^\infty$ form legal gameplays in the (α, β)-game and in the (α', β')-game, respectively. In constructing the strategy $\sigma_{W(\alpha,\beta)}$ we will use Eve's winning strategy σ'_A in the (α', β')-game as an oracle. Observe that by (3.1) the result of the (α, β)-game is a result of the associated (α', β')-game. Thus, in order to ensure that $\sigma_{W(\alpha,\beta)}$ is a winning strategy (that is, $W(\alpha, \beta) \subseteq E$) we only need to check that in the associated (α', β')-game Eve follows her winning strategy σ'_A.

We will also take care of the following conditions concerning the (α', β')-game:

(1) A'_0 is an interval with *rational* endpoints;
(2) for all $n \in \mathbb{N} \setminus \{0\}$, A'_n belongs to a *finite* set depending only on A'_0.

Notice that, in view of (2), A'_0 determines a compact set of all possible results of the game. Thus, by (1), the set of all possible results of the game is an F_σ-set.

We will choose constants $k, K \in \mathbb{R}$ in such a way that for all $n \in \mathbb{N}$

$$\frac{\ell(A'_n)}{\ell(A_n)} = K \quad \text{and} \quad \frac{\ell(E_n)}{\ell(E'_n)} = k. \tag{3.2}$$

Let us choose rational α' and β' such that

$$\alpha < \alpha' < \frac{1}{2} \quad \text{and} \quad \alpha'\beta' = \alpha\beta. \tag{3.3}$$

Now we can describe Eve's strategy $\sigma_W(\alpha, \beta)$. Suppose that Adam's first move is the compact interval A_0. Eve finds a compact subinterval A'_0 with rational endpoints of length $K\ell(A_0)$ for some real constant $K \in (\frac{\alpha}{\alpha'}, 1)$. Treating A'_0 as Adam's first move in the (α', β')-game, she follows her strategy σ'_A to select an interval E'_0 of length $\alpha'\ell(A'_0)$. Finally, she chooses E_0 being the "leftmost" subinterval of E'_0 of length $k\ell(E'_0)$, where k is the real constant determined by the following

$$kK = \frac{\alpha}{\alpha'}. \tag{3.4}$$

It remains to check that E_0 is indeed a legal move in the (α, β)-game. In fact, we can now verify that our choice of constants k, K guarantees legality of both games also in subsequent moves. Indeed, if A_{n+1} and E'_n are legal moves in the (α, β)-game and in the (α', β')-game respectively, then we get, by (3.2) and (3.4),

$$\frac{\ell(E_n)}{\ell(A_n)} = \frac{\ell(E_n)}{\ell(E'_n)} \cdot \frac{\ell(E'_n)}{\ell(A'_n)} \cdot \frac{\ell(A'_n)}{\ell(A_n)} = k\alpha' K = \frac{\alpha\alpha'}{\alpha'} = \alpha$$

and, by (3.3),

$$\frac{\ell(A'_{n+1})}{\ell(E'_n)} = \frac{\ell(A'_{n+1})}{\ell(A_{n+1})} \cdot \frac{\ell(A_{n+1})}{\ell(E_n)} \cdot \frac{\ell(E_n)}{\ell(E'_n)} = K\beta k = \frac{\alpha\beta}{\alpha'} = \frac{\alpha'\beta'}{\alpha'} = \beta'.$$

Now suppose that Eve's $(n+1)$-th move E_n has been just played. Adam's next move A_{n+1} in the (α, β)-game is supposed to be a subinterval of E_n of length $\beta\ell(E_n)$. In view of Lemma 3.1, Eve finds a finite nonempty family $\widehat{B}^{K\beta}(E_n)$ of subintervals of E_n of length $K\beta\ell(E_n)$ (so being Adam's legal moves in the (α', β')-game!) such that any possible Adam's choice of A_{n+1} contains one of them.

When Adam plays his $(n+2)$-th move A_{n+1}, Eve's strategy is to find a member A'_{n+1} of $\widehat{B}^{K\beta}(E_n)$ contained in it. Regarding it as Adam's $(n+2)$-th move in the (α', β')-game, she follows her strategy σ'_A in the (α', β')-game to pick a subinterval E'_{n+1} of length $\alpha'\ell(A'_{n+1})$ and shrinks it, choosing its "leftmost" subinterval E_{n+1} of length $k\ell(E'_{n+1})$.

By finiteness of $\widehat{B}^{K\beta}(E_n)$, condition (2) is satisfied. This ends the construction of the strategy. □

We remark that the proof of Theorem 3.2 depends on the countability of Q. Our considering, differently from Schmidt, only rational αs and βs is (we believe) a necessary device. In view of Freiling's result [7], we suppose that $\mathcal{S}$ contains strictly Schmidt's original σ-ideal (i.e., that obtained by admitting all real parameters). We do not know whether the latter is G_δ-generated.

The next theorem establishes that $\mathcal{S}$ is invariant under diffeomorphisms. Our proof is ad hoc and simpler than that of Theorem 1 in [21]. In the proof below, $I[x, \varrho]$ stands for the compact subinterval of $\mathbb{R}$ with centre $x \in \mathbb{R}$ and radius $\varrho > 0$.

Theorem 3.3. *$\mathcal{S}$ is invariant under diffeomorphisms.*

Proof. We have to prove that for any losing set A, any $\alpha, \beta \in Q$, any open interval I and diffeomorphism $f : I \to \mathbb{R}$, the set $f(A \cap I)$ is (α, β)-losing.

Assume, without loss of generality, that Adam's first move A_0 is entirely contained in I. Let k, K be positive rationals such that

$$k \le |f'(x)| \le K \quad \text{for all} \quad x \in f^{-1}(A_0) \tag{3.5}$$

or, equivalently,

$$\frac{1}{K} \le |(f^{-1})'(y)| \le \frac{1}{k} \quad \text{for all} \quad y \in A_0. \tag{3.6}$$

In view of the continuity of f' and by possibly reducing $\ell(A_0)$ we may also assume $\frac{K}{k}\alpha < \frac{1}{2}$. With Adam's $(n+1)$-th move $A_n := I[a_n, \varrho_n]$ we associate $A'_n := I[f^{-1}(a_n), \frac{\varrho_n}{K}]$. Let us now consider a closed interval E'_n accordingly to a winning $(\frac{K}{k}\alpha, \frac{k}{K}\beta)$-strategy relative to $A \cap I$. Thus

$$E'_n := I\left[e'_n, \frac{\alpha\varrho_n}{k}\right] = \sigma_{A\cap I}\left(\frac{K}{k}\alpha, \frac{k}{K}\beta; n; A'_0, \ldots, A'_n\right) \in B^{\frac{K}{k}\alpha}(A'_n).$$

Finally, let $E_n := I[f(e'_n), \alpha\varrho_n]$. In view of (3.5), (3.6), and the Lagrange Theorem from elementary analysis, for all $n \in \mathbb{N}$ we have:

$$E'_n \in B^{\frac{K}{k}\alpha}(A'_n) \text{ implies } E_n \in B^{\alpha}(A_n);$$
$$A_{n+1} \in B^{\beta}(E_n) \text{ implies } A'_{n+1} \in B^{\frac{k}{K}\beta}(E'_n).$$

Consequently, $\bigcap_{n=0}^{\infty} A'_n = \bigcap_{n=0}^{\infty} E'_n = \{\lim_{n\to\infty} e'_n\} \nsubseteq A \cap I$, which gives, the map f being continuous, $\bigcap_{n=0}^{\infty} A_n = \bigcap_{n=0}^{\infty} E_n = \{\lim_{n\to\infty} f(e'_n)\} \nsubseteq f(A\cap I)$. □

4. Some Consequences

Recall that a subset A of $\mathbb{R}$ is said to be *algebraically independent* (over $\mathbb{Q}$) if for any $n \in \mathbb{N}$, any nonnull polynomial $P \in \mathbb{Q}(X_0, \ldots, X_n)$, and any choice of distinct elements $a_0, \ldots, a_n \in A$, it holds $P(a_0, \ldots, a_n) \neq 0$. If A is algebraically independent, then the subfield $\mathbb{Q}(A)$ of $\mathbb{R}$ generated by A is called *purely transcendental*.

Two celebrated theorems of Mycielski (see [13] and [14]) state that if $Y \subseteq \mathbb{R}$ is of full Lebesgue measure or residual in $\mathbb{R}$, then there exists an algebraically independent perfect set P such that the field $\mathbb{Q}(P)$ generated by P — easily seen to be an F_σ-set — is contained in $Y \cup \mathbb{Q}$. In particular, there is a purely transcendental uncountable subfield of $\mathbb{R}$ all of whose irrational elements are included in (and, alternatively, excluded from) the set of Liouville numbers ([16], Chapter 2). For the sake of completeness, we recall that an irrational number x is said to be *Liouville* if for each $n \in \mathbb{N} \setminus \{0\}$ there exists a rational $\frac{p}{q}$, with $q \in \mathbb{N} \setminus \{1\}$, such that $|x - \frac{p}{q}| < \frac{1}{q^n}$.

However, Mycielski's theorems cannot be applied to prove the existence of purely transcendental uncountable subfields of $\mathbb{R}$ contained, up to $\mathbb{Q}$, in the class of badly approximable numbers (recall Theorem 2.2). Resorting to Theorems 2.2 and 3.3, here we show that such fields do indeed exist. Admittedly, our result is weaker than those of Mycielski, for it provides no information on descriptive complexity.

Lemma 4.1. *Let $\mathcal{J}$ be a σ-ideal on $\mathbb{R}$ that is uniform and invariant under diffeomorphisms, and f a nonconstant rational real function. If $J \in \mathcal{J}$, then $f^{-1}(J) \in \mathcal{J}$.*

Proof. Put $f := \frac{P}{Q}$, where P and Q are coprime polynomials not both constant. Let D be the domain of f (it coincides with $\mathbb{R}$, up to a finite set). We may assume $J \subseteq f(D)$. Further, let C denote the (finite) set of roots of f'. Clearly, there exists $n \in \mathbb{N}$ such that $D \setminus C = \bigcup_{i=0}^{n} I_i$, the I_is being disjoint open intervals. For every $i \in \{0, \dots, n\}$ let $f_i := f_{|I_i}$ (note that the maps $f_i^{-1} : f_i(I_i) \to \mathbb{R}$ are diffeomorphisms). We have

$$f^{-1}(J) = f^{-1}(J \cap f(D)) = f^{-1}(J \cap f(C)) \cup \bigcup_{i=0}^{n} f_i^{-1}(J \cap f_i(I_i)).$$

To end the proof, it is sufficient to observe that $|f^{-1}(J \cap f(C))| < \aleph_0$ and that for all $i \in \{0, \dots, n\}$ it holds $f_i^{-1}(J \cap f_i(I_i)) \in \mathcal{J}$, by assumption. □

Theorem 4.2. *Let $\mathcal{J}$ be a proper and uniform σ-ideal that is invariant under diffeomorphisms. If $J \in \mathcal{J}$, then there exists a purely transcendental uncountable subfield of $\mathbb{R}$ included in $(\mathbb{R} \setminus J) \cup \mathbb{Q}$.*

Proof. By transfinite induction. Suppose that A is algebraically independent, countable and such that $\mathbb{Q}(A) \subseteq (\mathbb{R} \setminus J) \cup \mathbb{Q}$. As $\mathcal{J}$ is proper and uniform, by Lemma 4.1 there is $x \in \mathbb{R} \setminus \mathrm{alg}_{\mathbb{R}} \mathbb{Q}(A)$ such that $x \notin \bigcup_{f \in \mathcal{F}_A} f^{-1}(J)$, where $\mathrm{alg}_{\mathbb{R}} \mathbb{Q}(A)$ and $\mathcal{F}_A$ stand respectively for the (necessarily countable) algebraic closure of $\mathbb{Q}(A)$ in $\mathbb{R}$ and the (necessarily countable) family of all nonconstant rational functions with coefficients in the field $\mathbb{Q}(A)$. By construction, $\mathbb{Q}(A \cup \{x\})$ is purely transcendental, includes $\mathbb{Q}(A)$ strictly, and is included in $(\mathbb{R} \setminus J) \cup \mathbb{Q}$. □

From Theorems 2.2 and 3.3 we infer the following particular case of Theorem 4.2.

Corollary 4.3. *There exists a purely transcendental uncountable subfield of $\mathbb{R}$ all of whose irrational numbers are badly approximable.*

The next proposition, an application of Theorems 2.2 and 3.2, concerns the existence of many maps a la Sierpiński–Erdős:

Proposition 4.4. *Let $\mathcal{J}$ be any one of the following: $\mathcal{L}$, $\mathcal{K}$, $\mathcal{L} \cap \mathcal{K}$, $\mathcal{E}$. If* CH *holds, then there exists a Sierpiński–Erdős $(\mathcal{S}, \mathcal{J})$-map $f : \mathbb{R} \to \mathbb{R}$.*

Proof. It suffices to apply Theorem 7.7 from [5]. □

Remark 4.5. Observe that the very same proof of Theorem 4.4 applies to $\mathcal{M}$ on $2^{\mathbb{N}}$. In other words, under CH there exists a Sierpiński–Erdős $(\mathcal{M}, \mathcal{J})$-map $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, $\mathcal{J}$ being any one of the following: $\mathcal{L}$, $\mathcal{K}$, $\mathcal{L} \cap \mathcal{K}$, $\mathcal{E}$.

We do not know whether such Sierpiński–Erdős $(\mathcal{S}, \mathcal{J})$-maps can be additive. In particular, we do not know what is the minimal cardinal number κ such that $\mathcal{S}$ is not κ-translatable (an ideal $\mathcal{J}$ is κ-translatable if for every set $A \in \mathcal{J}$ we can find a larger set $B \in \mathcal{J}$ such that any union of κ translates of A is contained in some translate of B). Recall that, in the real line case, $\mathcal{K}$ is $\aleph_0$-translatable [4] but $\mathcal{L}$ is not 2-translatable [11], which in particular shows that no additive Sierpiński–Erdős $(\mathcal{K}, \mathcal{L})$-map can exist.

It is easy to see that there are subsets of $\mathbb{R}$ not belonging to the algebra $\sigma(\mathcal{B}\cup\mathcal{S})$. For instance, by the argument from [10], no Bernstein set belongs to this algebra. We wish to formulate two more results concerning Schmidt's ideal and measurability. In the next two propositions we shall write A for the set of badly approximable numbers, and S for its complement.

Proposition 4.6. *There exists a set $X \in \mathcal{S}$, disjoint from A, such that $X + X \notin \sigma(\mathcal{B}\cup\mathcal{S})$.*

Proof. Since S has full Lebesgue measure, it is easily seen that $S + S = \mathbb{R}$ and therefore $S + S \notin \mathcal{S}$. Taking into account that $S \in \mathcal{S}$ and that $\mathcal{S}$ is uniform and Borel generated, a straightforward application of Corollary 1.7 in [12] yields that there exists $X \subseteq S$ such that $X + X \notin \sigma(\mathcal{B}\cup\mathcal{S})$. □

Proposition 4.7. *There exists a subset X of A such that $X + X$ is not Lebesgue measurable and does not have the Baire property.*

Proof. Since $S \in \mathcal{S}$ and $\mathcal{S}$ is invariant under diffeomorphisms (in particular, under translations and the reflection map $x \mapsto -x$), we easily get $A + A = \mathbb{R}$. By Theorem 1.2 in [12], there exist $X_0, X_1 \subseteq A$ such that $X_0 + X_0$ is not Lebesgue measurable and $X_1 + X_1$ does not have the Baire property. A careful examination of the proof of that theorem reveals that $X := X_0 \cup X_1$ has both the required properties. □

Let us recall the following cardinal coefficients for a given σ-ideal $\mathcal{J}$ on a set X:

$\mathrm{add}(\mathcal{J}) = \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{J} \text{ and } \bigcup\mathcal{H} \notin \mathcal{J}\}$;
$\mathrm{cov}(\mathcal{J}) = \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{J} \text{ and } \bigcup\mathcal{H} = \mathrm{X}\}$;
$\mathrm{non}(\mathcal{J}) = \min\{|\mathrm{Y}| : \mathrm{Y} \subseteq \mathrm{X} \text{ and } \mathrm{Y} \notin \mathcal{J}\}$;
$\mathrm{cof}(\mathcal{J}) = \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{J} \text{ and } \mathcal{H} \text{ generates } \mathcal{J}\}$.

It is shown in [23] that $\mathrm{cof}(\mathcal{S}) \le \mathfrak{c}$ (this obviously follows also from Theorem 3.2). Rosłanowski computed in [18] the coefficients of Mycielski's ideal, showing that $\mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathrm{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mathfrak{c}$. This motivates the following conjecture.

Conjecture 4.8. $\mathrm{add}(\mathcal{S}) = \mathrm{cov}(\mathcal{S}) = \aleph_1$ and $\mathrm{non}(\mathcal{S}) = \mathrm{cof}(\mathcal{S}) = \mathfrak{c}$.

Obviously, our conjecture follows from the Continuum Hypothesis. We now show that it is also consistent with the negation of CH.

Proposition 4.9. *It is relatively consistent with* ZFC *that*

$$\aleph_1 = \mathrm{add}(\mathcal{S}) = \mathrm{cov}(\mathcal{S}) < \mathrm{non}(\mathcal{S}) = \mathrm{cof}(\mathcal{S}) = \mathfrak{c} = \aleph_2.$$

Assuming MA*, we have* $\mathrm{non}(\mathcal{S}) = \mathrm{cof}(\mathcal{S}) = \mathfrak{c}$.

Proof. A theorem of Rothberger (see for instance Theorem 7.3 in [5]) states that if $\mathcal{J}_0$ and $\mathcal{J}_1$ are proper, uniform, translation invariant, and orthogonal σ-ideals on $\mathbb{R}$, then $\aleph_1 \le \mathrm{cov}(\mathcal{J}_0) \le \mathrm{non}(\mathcal{J}_1)$. We therefore infer that $\mathrm{cov}(\mathcal{S}) \le \mathrm{non}(\mathcal{K})$ and $\mathrm{non}(\mathcal{S}) \ge \mathrm{cov}(\mathcal{K})$. Adding $\aleph_2$ Cohen reals to a model of CH, we get $\mathrm{non}(\mathcal{K}) = \aleph_1$ and $\mathrm{cov}(\mathcal{K}) = \aleph_2 = \mathfrak{c}$, so the first statement is true in that model.

It is well known that, under MA, $\mathrm{cov}(\mathcal{K}) = \mathfrak{c}$. Hence we infer the second assertion, again by Rothberger's theorem. □

In the proof of the first part of the proposition one can also use the model obtained by adding $\aleph_2$ random reals (either by the random algebra or by the countable support iteration) to a model of CH. The conclusion is derived by the same argument applied to the measure ideal.

5. "Nasty" Properties of $\mathcal{S}$

Section 3 has been devoted to some similarities between Schmidt's ideal and those of measure and category. One should not rather expect $\mathcal{S}$ to reveal many similarities with either $\mathcal{L}$ or $\mathcal{K}$. We shall indeed establish in Theorem 5.1 that $\mathcal{S}$ is different from $\mathcal{L}$ and $\mathcal{K}$ in many respects. On the other hand, the very same theorem highlights the strict connection between $\mathcal{S}$, as an ideal on the real line, and Mycielski's ideal $\mathcal{M}$ on the Cantor space.

We begin with some additional notation. Let C denote the set of those reals in $[0,1]$ which admit in their 11-expansions only digits in $\{1,3,5,7,9\}$, i.e.,

$$C := \left\{ \sum_{n=1}^{\infty} \frac{x_n}{11^n} : x_n \in \{1,3,5,7,9\} \right\}.$$

Given $s \in 2^{\mathbb{N}}$, we put

$$C^s := \left\{ \sum_{n=1}^{\infty} \frac{x_n}{11^n} : (x_n \in \{3,7\} \text{ iff } s_{n-1} = 0) \text{ and } (x_n \in \{1,5,9\} \text{ iff } s_{n-1} = 1) \right\}.$$

Hence, a number $x \in [0,1]$ is in C^s iff the $(n+1)$-th digit x_{n+1} in the 11-expansion of x belongs to $\{3,7\}$ or to $\{1,5,9\}$, according to whether $s_n = 0$ or $s_n = 1$.

Theorem 5.1. *Schmidt's σ-ideal $\mathcal{S}$ has neither the c.c.c., nor the Steinhaus property, nor the Ruziewicz property. Moreover, $\mathcal{A} \not\subseteq \sigma(\mathcal{B} \cup \mathcal{S})$ (so, $\sigma(\mathcal{B} \cup \mathcal{S})$ is not closed under the Souslin operation).*

Proof. Let C and C^s, with $s = (s_n)_{n=0}^{\infty} \in 2^{\mathbb{N}}$, be the Cantor-like sets defined above.

Let us first check that for no $s \in 2^{\mathbb{N}}$ the Cantor set C^s is $(\frac{5}{11}, \frac{1}{5})$-losing. In fact, we now show that Adam has a winning strategy to "hit" C^s in the $(\frac{5}{11}, \frac{1}{5})$-game. Adam is going to select his intervals in such a way that his $(n+1)$-th move determines the $(n+1)$-th digit (in 11-expansion) of the result of the game. Let Adam's first move be $A_0 := [0,1]$. No matter how Eve chooses her response $E_0 \in B^{\frac{5}{11}}(A_0)$, Adam can force the first digit to be a member of $\{3,7\}$ or $\{1,5,9\}$ (according to s_0) by an appropriate choice of A_1. The same argument works also for subsequent moves.

As the $\mathfrak{c}$ sets C^s are pairwise disjoint, $\mathcal{S}$ does not have the c.c.c.

To prove that $\mathcal{S}$ does not have the Steinhaus property, it suffices to observe that $C - C$ is of Lebesgue measure zero. This is certainly true, inasmuch as no number in $C - C$ admits in its 11-expansion the two-digits block 99. (We only sketch a proof: let $x, y \in C$ and $z := x - y$. For all $n \in \mathbb{N} \setminus \{0\}$ let x_n, y_n and z_n be the n-th digits of x, y and z, respectively. We can have $z_n = 9$ only if $x_n - y_n = -2$. By elementary arithmetic rule, the equalities $z_n = z_{n+1} = 9$ are incompatible.)

That $\mathcal{S}$ does not have the Ruziewicz property follows from the geometrically evident observation that C cannot contain any affine copy X of the set $\{0, 1, \ldots, 10\}$. (To justify this, we limit ourselves to noticing that if the set X were contained in

$$\left[\frac{1}{11}, \frac{2}{11}\right] \cup \left[\frac{3}{11}, \frac{4}{11}\right] \cup \left[\frac{5}{11}, \frac{6}{11}\right] \cup \left[\frac{7}{11}, \frac{8}{11}\right] \cup \left[\frac{9}{11}, \frac{10}{11}\right],$$

i.e., in the set of those reals in $[0, 1]$ whose first 11-digit belongs to $\{1, 3, 5, 7, 9\}$, then it would be contained in exactly one of the intervals above. By induction, we would conclude that X reduces to a singleton, contrary to our assumption.)

Finally, Theorem 1.2 and Corollary 2.3 in [1] yield $\mathcal{A} \not\subseteq \sigma(\mathcal{B} \cup \mathcal{S})$. □

Acknowledgments

The research was performed when the first author was visiting Institute of Mathematics of the Polish Academy of Sciences. The results included here were presented by the first author at the 33rd Winter School of Abstract Analysis, Section of Topology in Raspenava (Czech Republic) in February 2005, and by the second author at the workshop "Foundations of the Formal Sciences V: Infinite Games" held in Bonn (Germany) in November 2004.

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(Received 16.03.2005; revised 11.04.2005)

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