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A NOTE ON FOURIER COEFFICIENTS OF FUNCTIONS OF GENERALIZED WIENER CLASS

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Abstract. Let f denote a 2π periodic function in $L[0,2\pi]$, and $\hat{f}(n)$, $n \in Z$, be its Fourier coefficients. For a function f of the generalized Wiener class $\bigwedge\mathrm{BV}(p(n)\uparrow\infty)$ we have proved that

$$\hat{f}(n) = O\biggl(1/\Bigl(\sum_{i=1}^{|n|}\frac{1}{\lambda_i}\Bigr)^{1/p(k(n))}\biggr).$$

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Let f be a 2π periodic function in $L[0,2\pi]$, and $\hat{f}(n)$, $n \in Z$, be its Fourier coefficients. R. Siddiqi [5] extended the classical result "$f \in \mathrm{BV}[0,2\pi]$ implies its Fourier coefficients $\hat{f}(n) = O\bigl(\frac{1}{|n|}\bigr)$" for the Wiener class. He proved that "$f \in \mathrm{BV}(p,[0,2\pi])$ $(1 \le p < \infty)$ implies $\hat{f}(n) = O\bigl(\frac{1}{|n|^{1/p}}\bigr)$". The concept of Wiener class was generalized by H. Kita and K. Yoneda [1] as generalized Wiener class $\mathrm{BV}(p(n)\uparrow\infty)$ and also by Shiba [4] as p-$\bigwedge$-bounded variations $(\bigwedge\mathrm{BV}(p))$. From these two generalizations, one can define a more generalized class $\bigwedge\mathrm{BV}(p(n)\uparrow\infty)$ as follows.

Definition. Given a subinterval I of $[0,2\pi]$, a sequence $\bigwedge = \{\lambda_m\}$ $(m = 1,2,\dots)$ of non-decreasing positive real numbers λ_m such that $\sum\frac{1}{\lambda_m}$ diverges and $1 \le p(n)\uparrow\infty$ as $n\to\infty$, we say that $f \in \bigwedge\mathrm{BV}(p(n)\uparrow\infty, I)$ (that is, f is a function of $p(n)$-$\bigwedge$-bounded variation over (I)) if

$$V_\Lambda(f,p(n),I) = \sup_{n\ge1}\sup_{\{I_m\}}\{V_\Lambda(\{I_m\},f,p(n),I) : \rho\{I_m\} > 2\pi/2^n\} < \infty,$$

where

$$V_\Lambda(\{I_m\},f,p(n),I) = \biggl(\sum_m\frac{|f(a_m)-f(b_m)|^{p(n)}}{\lambda_m}\biggr)^{1/p(n)},$$
$$\rho\{I_m\} = \inf_m|a_m-b_m|$$

and $\{I_m\}$ is a sequence of nonoverlapping subintervals $I_m = [a_m,b_m] \subset I = [a,b]$.

Note that if $p(n) = p$ for all n, one gets the class $\bigwedge\mathrm{BV}(p,I)$; if $\lambda_m \equiv 1$ for all m, one gets the class $\mathrm{BV}(p(n)\uparrow\infty)$; if $p(n) = 1$ for all n and $\lambda_m \equiv 1$ for all

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m, one gets the class $\mathrm{BV}(I)$; if $p(n) = 1$ for all n and $\lambda_m \equiv m$ for all m, one gets the class Harmonic $\mathrm{BV}(I)$.

Schramm and Waterman [3] estimated the Fourier coefficients of function of $\bigwedge \mathrm{BV}(p)$. They proved the following theorem.

Theorem A. *If* $f \in \bigwedge \mathrm{BV}(p, [0, 2\pi])$ $(1 \le p < \infty)$, *then*

$$\hat{f}(n) = O\left(1 \Big/ \left(\sum_{i=1}^{|n|} \frac{1}{\lambda_i}\right)^{1/p}\right).$$

For the generalized Wiener class T. Akhobadze [6] proved the following theorem.

Theorem B. *If* $f \in \mathrm{BV}(p(n) \uparrow \infty,\ [0, 2\pi])$, *then* $\hat{f}(n) = O(1/|n|^{1/p(k(n))})$, *where* $k(n)$ *is an integer for which*

$$1 + \log_2 |n| < k(n) \le 2 + \log_2 |n|. \tag{1}$$

Here we have extended these two results and estimated the order of the magnitude of the Fourier coefficients of functions of $\bigwedge \mathrm{BV}(p(n) \uparrow \infty)$.

Theorem. *Let* $f \in \bigwedge \mathrm{BV}(p(n) \uparrow \infty, [0, 2\pi])$, *then*

$$\hat{f}(n) = O\left(1 \Big/ \left(\sum_{i=1}^{|n|} \frac{1}{\lambda_i}\right)^{1/p(k(n))}\right),$$

where $k(n)$ *is an integer satisfying* (1).

Remark. Here, $p(n) = p$ for all n, reduces the class $\bigwedge \mathrm{BV}(p(n) \uparrow \infty)$ to the class $\bigwedge \mathrm{BV}(p)$, and $\left(\sum_{i=1}^{|n|} \frac{1}{\lambda_i}\right)^{1/p(k(n))}$ reduces to $\left(\sum_{i=1}^{|n|} \frac{1}{\lambda_i}\right)^{1/p}$, that is we get Theorem A as a particular case. Similarly, $\lambda_m \equiv 1$ for all m, reduces the class $\bigwedge \mathrm{BV}(p(n) \uparrow \infty)$ to the class $\mathrm{BV}(p(n) \uparrow \infty)$ and we get Theorem B as a particular case. Thus the theorem generalizes a non-lacunary analogue of our earlier result [2, Theorem 5].

Proof of the Theorem. We know that

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx,$$

$$\hat{f}(n) = \frac{-1}{2\pi} \int_0^{2\pi} f\left(x + \frac{\pi}{n}\right) e^{-inx} dx,$$

$$\hat{f}(n) = \frac{-1}{2\pi} \int_0^{2\pi} \left(T_{\frac{\pi}{n}} f\right)(x) e^{-inx} dx, \quad \text{where} \quad \left(T_{\frac{\pi}{n}} f\right)(x) = f\left(x + \frac{\pi}{n}\right).$$

Then

$$|\hat{f}(n)| = \frac{1}{4\pi}\left|\int_0^{2\pi} \left(f(x) - \left(T_{\frac{\pi}{n}} f\right)(x)\right) e^{-inx} dx\right|. \tag{1.1}$$

Because of the periodicity of $f(x)$, we have for any positive integer j

$$\left|\int_0^{2\pi} \left(T_{\frac{j\pi}{n}} f - T_{\frac{(j-1)\pi}{n}} f\right)(x) dx\right| = \left|\int_0^{2\pi} \left(f(x) - \left(T_{\frac{\pi}{n}} f\right)(x)\right) dx\right|,$$

this together with (1.1) implies

$$|\hat{f}(n)| \leq \frac{1}{4\pi}\int_0^{2\pi} \left|\left(T_{\frac{j\pi}{n}} f - T_{\frac{(j-1)\pi}{n}} f\right)(x)\right| dx. \tag{1.2}$$

For a given natural number $|n|$, let $k(n)$ be an integer such that

$$\frac{\pi}{2|n|} \leq \frac{2\pi}{2^{k(n)}} \leq \frac{\pi}{|n|},$$

i.e., (1) is true. Let $q(k(n))$ be such that $\frac{1}{p(k(n))} + \frac{1}{q(k(n))} = 1$, then by the Hölder inequality, from (1.2) we get

$$\begin{aligned} |\hat{f}(n)| &\leq \frac{1}{2\pi}\left(\int_0^{2\pi} \left|\left(T_{\frac{j\pi}{n}} f - T_{\frac{(j-1)\pi}{n}} f\right)(x)\right|^{p(k(n))} dx\right)^{1/p(k(n))} \left(\int_0^{2\pi} 1 dx\right)^{1/q(k(n))} \\ &= \left(\frac{1}{2\pi}\right)^{1/p(k(n))} \left(\int_0^{2\pi} \left|\left(T_{\frac{j\pi}{n}} f - T_{\frac{(j-1)\pi}{n}} f\right)(x)\right|^{p(k(n))} dx\right)^{1/p(k(n))}. \end{aligned}$$

Then

$$|\hat{f}(n)|^{p(k(n))} \leq \frac{1}{2\pi}\left(\int_0^{2\pi} \left|\left(T_{\frac{j\pi}{n}} f - T_{\frac{(j-1)\pi}{n}} f\right)(x)\right|^{p(k(n))} dx\right). \tag{1.3}$$

Dividing both sides of equation (1.3) by λ_j and then performing summation from $j = 1$ to $|n|$, we get

$$|\hat{f}(n)|^{p(k(n))}\left(\sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right) \leq \frac{1}{2\pi}\left(\int_0^{2\pi} \sum_{j=1}^{|n|} \frac{\left|\left(T_{\frac{j\pi}{n}} f - T_{\frac{(j-1)\pi}{n}} f\right)(x)\right|^{p(k(n))}}{\lambda_j} dx\right).$$

Hence $|\hat{f}(n)|^{p(k(n))} \leq \frac{V_\wedge(f,p(n),[0,2\pi])}{\left(\sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right)}$. This proves the theorem. □

References

1. H. Kita and K. Yoneda, A generalization of bounded variation. *Acta Math. Hungar.* **56**(1990), No. 3-4, 229–238.
2. J. R. Patadia and R. G. Vyas, Fourier series with small gaps and functions of generalised variations. *J. Math. Anal. Appl.* **182**(1994), No. 1, 113–126.
3. M. Schramm and D. Waterman, On the magnitude of Fourier coefficients. *Proc. Amer. Math. Soc.* **85**(1982), No. 3, 407–410.
4. M. Shiba, On absolute convergence of Fourier series of function of class $\Lambda - \mathrm{BV}^{(p)}$. *Sci. Rep. Fac. Ed. Fukushima Univ.* No. 30 (1980), 7–10.
5. R. Siddiqi, The order of Fourier coefficients of function of higher variation. *Proc. Japan Acad.* **48**(1972), 569–572.
6. T. Akhobadze, On Fourier coefficients of functions of generalized Wiener class. *Georgian Math. J.* **7**(2000), No. 1, 1–10.

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