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A NOETHER THEOREM ON UNIMPROVABLE CONSERVATION LAWS FOR VECTOR-VALUED OPTIMIZATION PROBLEMS IN CONTROL THEORY

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Dedicated to the memory of Almaskhan Gugushvili

Abstract. We obtain a version of Noether's invariance theorem for optimal control problems with a finite number of cost functionals. The result is obtained by formulating E. Noether's result for optimal control problems subject to isoperimetric constraints, and then using the unimprovable (Pareto) notion of optimality. It was A. Gugushvili who drew the author's attention to a result of this kind that was posed as an open mathematical question of a great interest in applications of control engineering.

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1. Introduction

E. Noether's theorem, which relates symmetries and conservation laws, describes the fundamental fact that "invariance with respect to some group of parameter transformations gives rise to the existence of conservation laws". A typical application of conservation laws is to lower the order of systems. They are also a useful tool for many other reasons, e.g., they allow one to prove regularity of minimizers in the calculus of variations and optimal control [10]. Noether's theorem comprises all results on conservation laws known to classical mechanics. Thus, e.g., the invariance relative to translation with respect to time yields conservation of energy, while conservation of linear and angular momenta reflects, respectively, translational and rotational invariance. Noether's theorem is applicable also in quantum mechanics, field theory, electromagnetic theory, and has deep implications in the general theory of relativity. It is useful to explain a myriad of things, from the fusion of hydrogen to the motion of planets orbiting the sun [7]. Moreover, it turns out that Noether's theorem is much more than a theorem: it is a *principle* which can be formulated as a theorem in many different contexts, under many different assumptions. It is possible, e.g., to formulate the classical Noether's theorem of the calculus of variations for bigger classes of nonsmooth admissible functions [11], in a more general context of optimal control [1, 2, 8], or to obtain discrete-time versions [9]. For an account of Noether's symmetry principle in the context of optimal control, the use of conservation laws to integrate and decrease the order

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of equations given by the Pontryagin maximum principle [5], and for practical examples such as the problem of synchronization of difficult control systems we refer the reader to [2]. Here we are interested in generalizing the previous results to cover optimal control problems which, in place of a single cost functional, have a vector-valued functional to minimize. For the introduction to problems of optimal control with multiple objectives we refer the reader to Salukvadze's book [6]. Multiobjective optimal control attracts more and more attention, and is the source of many open questions [3]. The motivation for the present study was a challenge proposed to the author by A. Gugushvili on November 18, 2003. A. Gugushvili wanted to generalize the symmetry and conservation laws to multiobjective problems of optimal control: *"We would like to develop E. Noether's theory for multicriteria optimal control systems. If you have any ideas and work on these problems, please, let us know."* Theorem 4.2 is, to the best of our knowledge, the first attempt in this direction.

2. Optimal Control with Isoperimetric Constraints

It is well known that necessary optimality conditions for optimal control problems subject to isoperimetric constraints, are also necessary for unimprovable (Pareto) optimality in the problem with a vector-valued cost (cf., e.g., [4, Ch. 17], [6, p. 22]). Consider a nonlinear control system,

$$\dot{x}(t) = \varphi\left(t, x(t), u(t)\right) \tag{1}$$

of n differential equations, subject to k isoperimetric equality constraints,

$$\int_a^b g_i\left(t, x(t), u(t)\right) \mathrm{d}t = \xi_i \,, \quad i = 1, \ldots, k \,, \tag{2}$$

m isoperimetric inequality constraints,

$$\int_a^b g_j\left(t, x(t), u(t)\right) \leq \xi_j \,, \quad j = k+1, \ldots, k+m \,, \tag{3}$$

and $2n$ boundary conditions

$$x(a) = \alpha \,, \quad x(b) = \beta \,. \tag{4}$$

The problem consists in finding a piecewise-continuous control function $u(\cdot) = (u_1(\cdot), \ldots, u_r(\cdot))$ taking values on a given set $\Omega \subseteq \mathbb{R}^r$, and the corresponding state trajectory $x(\cdot) = (x_1(\cdot), \ldots, x_n(\cdot))$, satisfying (1), (2), (3), and (4), and minimizing (or maximizing) the (scalar) integral cost functional

$$I[x(\cdot), u(\cdot)] = \int_a^b L(t, x(t), u(t)) \mathrm{d}t \,.$$

This problem is denoted in the sequel by (P_1). Both the initial time a and the terminal time b, $a < b$, are fixed. The boundary values α, $\beta \in \mathbb{R}^n$, and constants ξ_i, $i = 1, \ldots, k+m$, are also given. The functions $L(\cdot,\cdot,\cdot)$, $\varphi(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$

are assumed to be continuously differentiable with respect to all variables. The celebrated Pontryagin's maximum principle [5] gives necessary optimality conditions to be satisfied by the solutions of optimal control problems. The formulation of the maximum principle for problems with isoperimetric constraints can be found, e.g., in [4, §13.12].

Theorem 2.1 (Pontryagin's Maximum Principle for (P_1))**.** *Let $u(t)$, $t \in [a,b]$, be an optimal control for the isoperimetric (scalar) optimal control problem (P_1), and $x(\cdot)$ the corresponding state trajectory. Then there exist a constant $\psi_0 \leq 0$, a continuous costate n-vector function $\psi(\cdot)$ having piecewise-continuous derivatives, and constant multipliers λ_i, $i = 1, \ldots, k+m$, where $(\psi_0, \psi(\cdot), \lambda) \neq 0$, satisfying the pseudo-Hamiltonian system*

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial \psi}\left(t, x(t), u(t), \psi_0, \psi(t), \lambda\right) , \\ \dot{\psi}(t) = -\frac{\partial H}{\partial x}\left(t, x(t), u(t), \psi_0, \psi(t), \lambda\right) , \end{cases}$$

the maximality condition

$$H\left(t, x(t), u(t), \psi_0, \psi(t), \lambda\right) = \max_{u \in \Omega} H\left(t, x(t), u, \psi_0, \psi(t), \lambda\right) ,$$

where the Hamiltonian H is defined by

$$H(t, x, u, \psi_0, \psi, \lambda) = \psi_0 L(t, x, u) + \psi \cdot \varphi(t, x, u) + \lambda \cdot g(t, x, u) . \tag{5}$$

Moreover, $\lambda_j \leq 0$, $j = k+1, \ldots, k+m$, where $\lambda_j = 0$ if

$$\int_a^b g_j\left(t, x(t), u(t)\right) < \xi_j ,$$

and $H(t, x(t), u(t), \psi_0, \psi(t), \lambda)$ is a continuous function of t and, on each interval of continuity of $u(\cdot)$, is differentiable and satisfies the equality

$$\frac{dH}{dt}\left(t, x(t), u(t), \psi_0, \psi(t), \lambda\right) = \frac{\partial H}{\partial t}\left(t, x(t), u(t), \psi_0, \psi(t), \lambda\right) . \tag{6}$$

3. Vector-Valued Optimal Control Problems

When optimal control is used to model a real problem, it is natural that several (conflicting) cost functionals ("objectives") are desired to be taken into account (see [6] for many practical situations). The problem is then to minimize a vector-valued functional with components

$$I_i[x(\cdot), u(\cdot)] = \int_a^b L_i(t, x(t), u(t)) \mathrm{d}t , \quad i = 1, \ldots, N ,$$

subject to the dynamical control system (1), and the boundary conditions (4). We denote this problem by (P).

Definition 3.1. An admissible pair $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is said to be an *unimprovable solution*, *compromise solution*, or a *Pareto solution* for (P) if and only if there does not exist an admissible pair $(x(\cdot), u(\cdot))$ verifying

$$I_i[x(\cdot), u(\cdot)] \leq I_i[\tilde{x}(\cdot), \tilde{u}(\cdot)]\,,$$

for $i \in \{1, \ldots, N\}$, with a strict inequality for at least one I_i.

It turns out that necessary conditions for optimal control problems with isoperimetric constraints are also necessary for the Pareto optimality of optimal control problems with a vector-valued cost. Theorem 3.1 is a simple consequence of Definition 3.1 (cf., e.g., [4, Theorem 17.1]).

Theorem 3.1. *If $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is a Pareto solution of problem (P), then it is a minimizer for isoperimetric optimal control problems with an integral scalar-valued cost*

$$I_i[x(\cdot), u(\cdot)]\,, \quad i \in \{1, \ldots, N\}\,,$$

and isoperimetric constraints

$$I_j[x(\cdot), u(\cdot)] \leq I_j[\tilde{x}(\cdot), \tilde{u}(\cdot)]\,, \quad j = 1, \ldots, N \text{ and } j \neq i\,.$$

From Theorems 3.1 and 2.1 (Pontryagin's maximum principle for problems with isoperimetric constraints) follows the so-called "general theorem of optimal control" (cf. [6, p. 22]).

Theorem 3.2. *If $(x(\cdot), u(\cdot))$ is a Pareto-solution of problem (P), then there exist a continuous costate n-vector function $\psi(\cdot)$ having piecewise-continuous derivatives, and constant multipliers $\lambda = (\lambda_1, \ldots, \lambda_N)$, where $(\psi(\cdot), \lambda) \neq 0$, satisfying the pseudo-Hamiltonian system*

$$\begin{cases} \dot{x}(t) = \frac{\partial \mathcal{H}}{\partial \psi}\left(t, x(t), u(t), \psi(t), \lambda\right)\,, \\ \dot{\psi}(t) = -\frac{\partial \mathcal{H}}{\partial x}\left(t, x(t), u(t), \psi(t), \lambda\right)\,, \end{cases}$$

the maximality condition

$$\mathcal{H}\left(t, x(t), u(t), \psi(t), \lambda\right) = \max_{u \in \Omega} \mathcal{H}\left(t, x(t), u, \psi(t), \lambda\right)\,,$$

where the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}(t, x, u, \psi, \lambda) = \lambda \cdot L(t, x, u) + \psi \cdot \varphi(t, x, u)\,. \tag{7}$$

Moreover, $\lambda_j \leq 0$, $j = 1, \ldots, N$, and $\mathcal{H}(t, x(t), u(t), \psi(t), \lambda)$ is a continuous function of t and, on each interval of continuity of $u(\cdot)$, is differentiable and satisfies the equality

$$\frac{d\mathcal{H}}{dt}\left(t, x(t), u(t), \psi(t), \lambda\right) = \frac{\partial \mathcal{H}}{\partial t}\left(t, x(t), u(t), \psi(t), \lambda\right)\,.$$

4. Main Results: Noether-type Theorems

Theorem 4.1 asserts that the presence of symmetry for optimal control problems involving equality and inequality isoperimetric constraints, implies that their Pontryagin extremals (and solutions) preserve a well-defined quantity (there exists a conservation law associated with each symmetry). The result is formulated, like in the case of the problems of the calculus of variations [11] and for the unconstrained scalar-valued continuous [1, 8] and discrete-time [9] optimal control problems, as an instance of Noether's universal principle.

Definition 4.1. An equation $C\left(t, x(t), u(t), \psi_0, \psi(t), \lambda\right) = \text{constant}$, valid in $t \in [a, b]$ for any quintuple $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot), \lambda)$ satisfying Pontryagin's maximum principle (Theorem 2.1), is called a *conservation law* for problem (P_1).

Theorem 4.1 (Noether's theorem for optimal control problems with isoperimetric constraints)**.** *If $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot), \lambda)$ satisfy the conditions of Theorem* 2.1 *and there exists a C^2-smooth one-parameter group of transformations*

$$h^s : [a, b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \,,$$
$$h^s(t, x, u) = \left(T(t, x, u, s), X(t, x, u, s), U(t, x, u, s)\right) \,,$$
$$s \in (-\varepsilon, \varepsilon) \,, \; \varepsilon > 0 \,,$$

with $h^0(t, x, u) = (t, x, u)$ for all $(t, x, u) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^r$, and satisfying

$$L\left(t, x(t), u(t)\right) = L \circ h^s\left(t, x(t), u(t)\right) \frac{d}{dt} T\left(t, x(t), u(t), s\right) \,, \tag{8}$$

$$\frac{d}{dt} X\left(t, x(t), u(t), s\right) = \varphi \circ h^s\left(t, x(t), u(t)\right) \frac{d}{dt} T\left(t, x(t), u(t), s\right) \,, \tag{9}$$

$$g\left(t, x(t), u(t)\right) = g \circ h^s\left(t, x(t), u(t)\right) \frac{d}{dt} T\left(t, x(t), u(t), s\right) \,, \tag{10}$$

$$U\left(t, x(t), u(t), s\right) \in \Omega \,, \forall s \in (-\varepsilon, \varepsilon) \,, \tag{11}$$

then,

$$\psi(t) \cdot \frac{\partial}{\partial s} X\left(t, x(t), u(t), s\right)\Big|_{s=0} - H(t, x(t), u(t), \psi_0, \psi(t), \lambda) \frac{\partial}{\partial s} T\left(t, x(t), u(t), s\right)\Big|_{s=0} = \text{const}$$

is a conservation law for problem (P_1), with H the Hamiltonian (5) *associated to problem (P_1).*

Proof. Using the fact that $h^0(t, x, u) = (t, x, u)$, from condition (8) one gets

$$\begin{aligned} 0 &= \frac{d}{ds}\left(L \circ h^s\left(t, x(t), u(t)\right) \frac{d}{dt} T\left(t, x(t), u(t), s\right)\right)\Bigg|_{s=0} \\ &= \frac{\partial L}{\partial t} \left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial x} \cdot \left.\frac{\partial X}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial u} \cdot \left.\frac{\partial U}{\partial s}\right|_{s=0} + L \frac{d}{dt} \left.\frac{\partial T}{\partial s}\right|_{s=0} \,, \end{aligned} \tag{12}$$

while conditions (9) and (10) yield

$$\frac{d}{dt}\left.\frac{\partial X}{\partial s}\right|_{s=0} = \frac{\partial \varphi}{\partial t}\left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial \varphi}{\partial x}\cdot\left.\frac{\partial X}{\partial s}\right|_{s=0} + \frac{\partial \varphi}{\partial u}\cdot\left.\frac{\partial U}{\partial s}\right|_{s=0} + \varphi\frac{d}{dt}\left.\frac{\partial T}{\partial s}\right|_{s=0}, \quad (13)$$

$$0 = \frac{\partial g}{\partial t}\left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial g}{\partial x}\cdot\left.\frac{\partial X}{\partial s}\right|_{s=0} + \frac{\partial g}{\partial u}\cdot\left.\frac{\partial U}{\partial s}\right|_{s=0} + g\frac{d}{dt}\left.\frac{\partial T}{\partial s}\right|_{s=0}. \quad (14)$$

Multiplying (12) by ψ_0, (13) by $\psi(t)$, and (14) by λ, we can write

$$\begin{aligned}\psi_0 &\left(\frac{\partial L}{\partial t}\left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial x}\cdot\left.\frac{\partial X}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial u}\cdot\left.\frac{\partial U}{\partial s}\right|_{s=0} + L\frac{d}{dt}\left.\frac{\partial T}{\partial s}\right|_{s=0}\right)\\ &+ \psi(t)\cdot\left(\frac{\partial \varphi}{\partial t}\left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial \varphi}{\partial x}\cdot\left.\frac{\partial X}{\partial s}\right|_{s=0} + \frac{\partial \varphi}{\partial u}\cdot\left.\frac{\partial U}{\partial s}\right|_{s=0}\right.\\ &\qquad\left. + \varphi\frac{d}{dt}\left.\frac{\partial T}{\partial s}\right|_{s=0} - \frac{d}{dt}\left.\frac{\partial X}{\partial s}\right|_{s=0}\right)\\ &+ \lambda\cdot\left(\frac{\partial g}{\partial t}\left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial g}{\partial x}\cdot\left.\frac{\partial X}{\partial s}\right|_{s=0} + \frac{\partial g}{\partial u}\cdot\left.\frac{\partial U}{\partial s}\right|_{s=0} + g\frac{d}{dt}\left.\frac{\partial T}{\partial s}\right|_{s=0}\right) = 0. \quad (15)\end{aligned}$$

According to the maximality condition of Pontryagin's maximum principle, and given (11), the function

$$\psi_0 L\left(t, x(t), U\left(t, x(t), u(t), s\right)\right) + \psi(t)\cdot\varphi\left(t, x(t), U\left(t, x(t), u(t), s\right)\right) + \lambda\cdot g\left(t, x(t), U\left(t, x(t), u(t), s\right)\right)$$

attains an extremum for $s = 0$. Therefore

$$\psi_0\frac{\partial L}{\partial u}\cdot\left.\frac{\partial U}{\partial s}\right|_{s=0} + \psi(t)\cdot\frac{\partial \varphi}{\partial u}\cdot\left.\frac{\partial U}{\partial s}\right|_{s=0} + \lambda\cdot\frac{\partial g}{\partial u}\cdot\left.\frac{\partial U}{\partial s}\right|_{s=0} = 0$$

and (15) simplifies to

$$\begin{aligned}\psi_0 &\left(\frac{\partial L}{\partial t}\left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial x}\cdot\left.\frac{\partial X}{\partial s}\right|_{s=0} + L\frac{d}{dt}\left.\frac{\partial T}{\partial s}\right|_{s=0}\right)\\ &+ \psi(t)\cdot\left(\frac{\partial \varphi}{\partial t}\left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial \varphi}{\partial x}\cdot\left.\frac{\partial X}{\partial s}\right|_{s=0} + \varphi\frac{d}{dt}\left.\frac{\partial T}{\partial s}\right|_{s=0} - \frac{d}{dt}\left.\frac{\partial X}{\partial s}\right|_{s=0}\right)\\ &+ \lambda\cdot\left(\frac{\partial g}{\partial t}\left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial g}{\partial x}\cdot\left.\frac{\partial X}{\partial s}\right|_{s=0} + g\frac{d}{dt}\left.\frac{\partial T}{\partial s}\right|_{s=0}\right) = 0. \quad (16)\end{aligned}$$

From the adjoint system $\dot{\psi} = -\frac{\partial H}{\partial x}$ and equality (6), we know that

$$\dot{\psi} = -\psi_0\frac{\partial L}{\partial x} - \psi\cdot\frac{\partial \varphi}{\partial x} - \lambda\cdot\frac{\partial g}{\partial x},$$

$$\frac{d}{dt}H = \psi_0\frac{\partial L}{\partial t} + \psi\cdot\frac{\partial \varphi}{\partial t} + \lambda\cdot\frac{\partial g}{\partial t},$$

and one concludes that (16) is equivalent to

$$\frac{d}{dt}\left(\psi(t)\cdot\left.\frac{\partial X}{\partial s}\right|_{s=0} - H\left.\frac{\partial T}{\partial s}\right|_{s=0}\right) = 0.$$

The proof is complete. □

We now introduce the notion of unimprovable or Pareto conservation law.

Definition 4.2. An equation $C(t, x(t), u(t), \psi(t), \lambda) = \text{constant}$, valid in $t \in [a, b]$ for any quadruple $(x(\cdot), u(\cdot), \psi(\cdot), \lambda)$ satisfying "the general theorem of optimal control" (Theorem 3.2), is called an *unimprovable conservation law* or a *Pareto conservation law* for problem (P).

Given the relation between problems (P_1) and (P) (cf. Section 3), we obtain from Theorem 4.1 the following corollary.

Theorem 4.2 (Noether's theorem for vector-valued optimal control systems)**.** *If* $(x(\cdot), u(\cdot), \psi(\cdot), \lambda)$ *satisfy the conditions of Theorem* 3.2 *and there exists a* C^2*-smooth one-parameter group of transformations*

$$h^s : [a, b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r ,$$
$$h^s(t, x, u) = (T(t, x, u, s), X(t, x, u, s), U(t, x, u, s)) ,$$
$$s \in (-\varepsilon, \varepsilon) , \varepsilon > 0 ,$$

with $h^0(t, x, u) = (t, x, u)$ *for all* $(t, x, u) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^r$*, and satisfying*

$$U(t, x(t), u(t), s) \in \Omega , \forall s \in (-\varepsilon, \varepsilon) ,$$
$$\frac{d}{dt} X(t, x(t), u(t), s) = \varphi \circ h^s(t, x(t), u(t)) \frac{d}{dt} T(t, x(t), u(t), s) , \tag{17}$$
$$L(t, x(t), u(t)) = L \circ h^s(t, x(t), u(t)) \frac{d}{dt} T(t, x(t), u(t), s) \tag{18}$$

($L = (L_1, \dots, L_N)$*), then*

$$\psi(t) \cdot \frac{\partial}{\partial s} X(t, x(t), u(t), s)\big|_{s=0} - \mathcal{H}(t, x(t), u(t), \psi(t), \lambda) \frac{\partial}{\partial s} T(t, x(t), u(t), s)\big|_{s=0} = \text{const} \tag{19}$$

is an unimprovable conservation law for problem (P)*, with* $\mathcal{H}$ *the Hamiltonian* (7) *associated to problem* (P)*.*

Remark 4.1. Theorems 4.1 and 4.2 remain still valid in the situation where the boundary values of the state variables and/or the initial-terminal moments of time (a, b) are not fixed. We have considered conditions (4) and fixed both initial and terminal times only to simplify the presentation of Pontryagin's maximum principle: initial and terminal transversality conditions are not relevant to the proof of our Noether type theorems.

In the next section we illustrate Theorem 4.2 with an example of five state variables $(n = 5)$, two controls $(r = 2)$, and two functionals to minimize $(N = 2)$.

5. Example for the Flight of a Pilotless Aircraft

We borrow from [6, §3.4] the problem of optimizing a vector-valued functional with two components representing fuel consumption (I_1) and flight time (I_2),

$$I_1 = \int_0^T u_1(t)\mathrm{d}t\,, \quad I_2 = \int_0^T 1\mathrm{d}t\,,$$

subject to a dynamical control system describing the motion of a pilotless aircraft

$$\begin{cases} \dot{x}_1(t) = x_3(t)\,, \\ \dot{x}_2(t) = x_4(t)\,, \\ \dot{x}_3(t) = c_1\frac{u_1(t)}{x_5(t)}\cos(u_2(t))\,, \\ \dot{x}_4(t) = c_1\frac{u_1(t)}{x_5(t)}\sin(u_2(t)) - c_2\,, \\ \dot{x}_5(t) = -u_1(t)\,. \end{cases}$$

Here x_1 is the range of the aircraft, x_2 the altitude, x_3 the horizontal component of the velocity, x_4 the vertical component of the velocity, x_5 the mass of the aircraft (which depends on its fuel quantity), u_1 the rate of fuel consumption, u_2 the thrust angle relative to the horizontal, c_1 and c_2 given constants. A full description of the model and a complete analysis of its solution, can be found in [6, §3.4]. Our objective here is to obtain a nontrivial unimprovable conservation law for the problem with the help of Theorem 4.2. As to the model, it is enough for our purposes to say that there are physical constraints on the control values, under which it makes sense to consider $\tan(u_2)$ (cf. [6, (3.42)]). Two trivial unimprovable conservation laws are $\psi_1(t) = \text{const}$ (obtained from Theorem 4.2 by setting $T = t$, $X_1 = x_1 + s$, $X_i = x_i$, $i = 2, \ldots, 5$, $U_j = u_j$, $j = 1, 2$), and $\psi_2(t) = \text{const}$ (obtained from Theorem 4.2 by setting $T = t$, $X_2 = x_2 + s$, $X_i = x_i$, $i = 1, 3, 4, 5$, $U_j = u_j$, $j = 1, 2$). We claim that

$$\psi_1 x_1(t) + 2\psi_2 x_2(t) + \psi_3(t)x_3(t) + 2\psi_4(t)x_4(t) = \text{const} \tag{20}$$

is also an unimprovable conservation law for the problem. We remark that (20) is nontrivial, and difficult to obtain without Theorem 4.2. To prove it with the help of Theorem 4.2, one just needs to show that the problem is invariant (satisfies conditions (17) and (18)) with $T = t$, $X_1 = \mathrm{e}^s x_1$, $X_2 = \mathrm{e}^{2s} x_2$, $X_3 = \mathrm{e}^s x_3$, $X_4 = \mathrm{e}^{2s} x_4$, $X_5 = x_5$, $U_1 = u_1$, and $U_2 = \arctan(\mathrm{e}^s \tan u_2)$ ($\sin U_2 = \mathrm{e}^{2s} \sin u_2$, $\cos U_2 = \mathrm{e}^s \cos u_2$). This is done by direct calculations ($\frac{d}{dt}T = 1$):

$$\begin{aligned} &\frac{d}{dt}X_1 = \mathrm{e}^s \dot{x}_1 = \mathrm{e}^s x_3 = X_3 \frac{d}{dt}T\,, \\ &\frac{d}{dt}X_2 = \mathrm{e}^{2s} \dot{x}_2 = \mathrm{e}^{2s} x_4 = X_4 \frac{d}{dt}T\,, \\ &\frac{d}{dt}X_3 = \mathrm{e}^s \dot{x}_3 = c_1 \frac{u_1}{x_5} \mathrm{e}^s \cos u_2 = c_1 \frac{U_1}{X_5} \cos U_2 \frac{d}{dt}T\,, \end{aligned}$$

$$\frac{d}{dt}X_4 = \mathrm{e}^{2s}\dot{x}_4 = c_1\frac{u_1}{x_5}\mathrm{e}^{2s}\sin u_2 - c_2 = \left(c_1\frac{U_1}{X_5}\sin U_2 - c_2\right)\frac{d}{dt}T\,,$$
$$\frac{d}{dt}X_5 = \dot{x}_5 = -u_1 = -U_1\frac{d}{dt}T\,,$$

and therefore equations (17) are verified,

$$L_1 = u_1 = U_1\frac{d}{dt}T\,,$$
$$L_2 = 1 = \frac{d}{dt}T\,,$$

and equations (18) are also satisfied. Equality (19) takes then form (20).

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