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WEIGHTED INEQUALITIES FOR INTEGRAL OPERATORS WITH ALMOST HOMOGENEOUS KERNELS

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Abstract. Let $m \in N$ and $a_1, \dots, a_m$ be real numbers such that for each i, $a_i \neq 0$ and $a_i \neq a_j$ if $i \neq j$. In this paper we study integral operators of the form

$$Tf(x) = \int k_1(x - a_1 y) \cdots k_m(x - a_m y) f(y)\, dy,$$

with $f, \varphi_{i,j} : R^n \to R$, $k_i(y) = \sum_{j \in Z} 2^{\frac{jn}{q_i}} \varphi_{i,j}(2^j y)$, $1 \leq q_i < \infty$, $i = 1, \dots, m$, $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = 1$.

If $\varphi_{i,j}$ satisfy certain uniform regularity conditions out of the origin, we obtain the boundedness of $T : L^p(w) \to L^p(w)$ for all power weights w in adequate Muckenhoupt classes.

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1. Introduction

In [7] the authors obtain the L^p boundedness, $p > 1$, for a class of maximal operators on the three-dimensional Heisenberg group. The operators they consider have relevance in the analysis on $Sl(R^3)$. Some of them actually arise in the study of the boundary behavior of Poisson integrals on the symmetric space $Sl(R^3)/SO(3)$. To obtain the principal results, they analyze the $L^2(R)$ boundedness of singular integral operators of the form

$$Tf(x) = \int |x - y|^{-\alpha} |x + y|^{\alpha - 1} f(y) dy,$$

$0 < \alpha < 1$.

A natural question was if these operators were also bounded on $L^p(R)$ for $p > 1$, $p \neq 2$ and if this result still holds for larger dimensions or for more general kernels. In [4] we study integral operators of the form

$$Tf(x) = \int_{R^n} |x - y|^{-\alpha} |x + y|^{-n+\alpha} f(y) dy,$$

$0 < \alpha < n$. We obtain the $L^p(R^n, dx)$ boundedness, $1 < p < \infty$, and the weak type $(1, 1)$ of them. We observe that the kernel is homogeneous of degree $-n$.

We take the Hardy–Littlewood maximal function as

$$Mf(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(x)|\, dx$$

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where the supremum is taken along all the cubes Q such that x belongs to Q. We recall that a weight w is a measurable, nonnegative and locally integrable function. It is well known that, for $p > 1$, M is bounded on $L^p(w)$ if and only if there exists $c > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} \leq c.$$

The class of functions that satisfy this inequality is called A_p. For $p = 1$, the class A_1 is defined by

$$Mw(x) \leq cw(x)$$

a.e. $x \in R^n$ and for some positive constant $c > 0$. The weak type $1 - 1$ of a maximal function is equivalent to $w \in A_1$. These classes A_p have been defined by Muckenhoupt (see [6]) in the one-dimensional case and for larger dimensions by Coifmann and Fefferman (see [1]).

In [9] the author proves very general weighted norm inequalities for maximal operators of the form

$$M_\mu f(x) = \sup_j |f * \mu_j(x)|,$$

where $\{\mu_j\}_{j \in Z}$ is a family of finite Borel measures on R^n, each one supported in $\|x\| \leq 2^j$, satisfying a certain decay of its Fourier transform. He also considers the singular integral operator $T_\sigma f = \sum_{j=-\infty}^{\infty} \sigma_j * f$, for a sequence $\{\sigma_j\}_{j \in Z}$ of signed measures on R^n with $\int d\sigma_j = 0$ that fall under the general theorems of J. Duoandikoetxea and J. L. Rubio de Francia stated in [3], which is an excellent reference for the unweighted theory. In particular the author studies the weight theory for

$$T_\Omega f(x) = p.v \int f(x - y) \frac{\Omega(y)}{|y|^n} dy,$$

where Ω is homogeneous of degree 0 and has mean value zero on S^{n-1}.

Let $m \in N$, let $a_1, \ldots, a_m$ be real numbers such that for each i, $a_i \neq 0$ and $a_i \neq a_j$ if $i \neq j$. Let $q_1, \ldots, q_m$ be real numbers, $1 \leq q_i < \infty$, such that

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = 1.$$

For each $1 \leq i \leq m$ let $\{\varphi_{i,j}\}_{j \in Z}$ be a family of nonnegative real functions defined on R^n, satisfying

H1) $\operatorname{supp} \varphi_{i,j} \subset \{y \in R^n : 2^{-1} \leq |y| \leq 2\}$.

H2) *For each* $1 \leq i \leq m$ *there exists* $p_i > q_i$ *such that* $\|\varphi_{i,j}\|_{p_i} \leq c$, *with* c *independent of* j.

Let $k_i(x) = \sum_{j \in Z} 2^{\frac{jn}{q_i}} \varphi_{i,j}(2^j x)$ and let T be the integral operator with kernel

$$k(x, y) = k_1(x - a_1 y) \cdots k_m(x - a_m y)$$

so that for a measurable and nonnegative f,

$$Tf(x) = \int k(x,y)\, f(y)\, dy. \tag{1}$$

In [5] we prove that T extends to a bounded operator on $L^p(R^n)$. We observe that if $\varphi_{i,j} \equiv \varphi_{i,j'}$ for all j, then $j' \in Z$, $k_i\left(2^l x\right) = 2^{-l\frac{n}{q_i}} k_i(x)$ for $l \in Z$, so it is "homogeneous" of degree $-\frac{n}{q_i}$ and then k is "homogeneous" of degree $-n$. Many authors have studied singular integral operators with kernels of the form $\sum_{j=-\infty}^{\infty} 2^{-jn} k\left(2^{-jx}\right)$, where $k \in L^1(R^n)$, having with some cancellation property and whose Fourier transform has a reasonable decay at infinity (see Theorem 8.23 in [2]). Such kernels have just one singularity at the origin. Now, our kernels have m different singularities.

In [8] we study the case $k_i(x) = |x|^{-\alpha_i}$, $\alpha_1 + \cdots + \alpha_m = n$ and obtain weighted inequalities for a wide class of weights w in A_p.

We recall that a power weight of the form $|x|^a$ belongs to A_p if and only if $-n < a < n(p-1)$ (see [2]). In this paper we prove

Theorem. *Let the operator T be defined by* (1)*. Let p_{m+1} be defined by $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{p_{m+1}} = 1$ and let $w(x) = |x|^a$ with $-n < a < n\left(\frac{p}{p_{m+1}} - 1\right)$. If $\max_{1\le i\le m}\left\{\frac{p_i}{p_i - q_i}\right\} < p < \infty$, then T is a bounded operator on $L^p(w)$.*

We also give some example of kernels of the above described type, for which the condition $-n < a < n\left(\frac{p}{p_{m+1}} - 1\right)$ becomes necessary for the boundedness, on $L^p\left(|x|^a dx\right)$, of the corresponding operator.

Throughout this paper, c will denote a positive constant not necessarily the same at each occurrence.

2. Proof of the Theorem

We will prove that for all $\max_{1\le i\le m}\left\{\frac{p_i}{p_i - q_i}\right\} < p < \infty$, we obtain the weak type condition

$$w\{x : |Tf(x)| > \lambda\} \le \frac{c}{\lambda^p} \|f\|_{p,w}^p$$

and the theorem will follow from the Marcinkiewicz interpolation theorem.

For $x \in R^n - \{0\}$ we define $l = l(x) \in Z$ such that $2^l < |x| \le 2^{l+1}$. We take positive numbers $d < 1$, $D > 1$ such that

$$d < \min\left(\min_{1\le i\le m}\left\{\frac{|a_i|}{2},\right\}, \min_{i\neq s}\left\{\frac{|a_i - a_s|}{2},\right\}\right)$$

and $D > 2\max_{1\le i\le n}\{|a_i|\}$. We define $r < 0$ and $R > 0$ such that $2^r < d \le 2^{r+1}$ and $2^R < D \le 2^{R+1}$, and we set

$$A_i = A_i(x) = \left\{y \in R^n : |y - a_i x| \le 2^l d\right\}, \quad 1 \le i \le m,$$

$$A_{m+1}=\left\{y\in R^{n}:|y|\leq 2^{l}D\right\}\bigcap\left(\bigcup_{1\leq i\leq m}A_{i}\right)^{c},$$

and

$$A_{m+2}=\left\{y\in R^{n}:|y|>2^{l}D\right\}\bigcap\left(\bigcup_{1\leq i\leq m+1}A_{i}\right)^{c}.$$

We also define, for $1\leq i\leq m+2$, operators T_i by

$$T_{i}f\left(x\right)=\int\limits_{A_{i}(x)}k\left(x,y\right)\left|f\left(y\right)\right|dy.$$

In order to obtain (1), we decompose

$$\left\{x:|Tf(x)|>\lambda\right\}=\bigcup_{i=1}^{m+2}\left\{x:T_{i}f\left(x\right)>\lambda/\left(m+2\right)\right\},$$

and measure each one of these sets separately.

If $2^{l}\leq|x|<2^{l+1}$, $y\in A_{i}(x)$ and $2^{j}\left(y-a_{i}x\right)\in\operatorname{supp}\varphi_{i,j}$, then

$$2^{-1}\leq 2^{j}\left|y-a_{i}x\right|<2^{j+l+r+1}$$

and so $j\geq -l-r-2$. Also, for $s\neq i$,

$$\left|y-a_{s}x\right|\leq\left|y-a_{i}x\right|+\left|\left(a_{i}-a_{s}\right)x\right|\leq\left(d+D\right)2^{l}$$

and

$$\left|y-a_{s}x\right|\geq\left|\left(a_{i}-a_{s}\right)x\right|-\left|y-a_{i}x\right|\geq d2^{l},$$

and so, if $2^{k}\left|y-a_{s}x\right|\in\operatorname{supp}\varphi_{s,k}$, then $2^{-1}\leq 2^{k}\left|y-a_{s}x\right|\leq\left(d+D\right)2^{k+l}$ and $2\geq 2^{k}\left|y-a_{s}x\right|\geq d2^{k+l}$, then $-l-R-3\leq k\leq -l-r+1$. So

$$\begin{aligned}\int\limits_{A_{i}(x)}\left|k\left(x,y\right)f(y)\right|dy&\leq c\int\limits_{A_{i}(x)}\sum_{j\geq -l-r-2}2^{\frac{nj}{q_{i}}}\left|\varphi_{i,j}\left(2^{j}\left(y-a_{i}x\right)\right)\right|\\&\quad\times\prod_{s\neq i}2^{\frac{-nl}{q_{s}}}\left|\varphi_{s,-l}\left(2^{-l}\left(y-a_{s}x\right)\right)f(y)\right|dy\\&\leq c\sum_{\substack{h\geq 0,\\ j\geq -l-r-2}}\ \int\limits_{d2^{l-h-1}\leq|y-a_{i}x|\leq d2^{l-h}}2^{\frac{nj}{q_{i}}}\left|\varphi_{i,j}\left(2^{j}\left(y-a_{i}x\right)\right)\right|\\&\quad\times\prod_{s\neq i}2^{\frac{-nl}{q_{s}}}\left|\varphi_{s,-l}\left(2^{-l}\left(y-a_{s}x\right)\right)f(y)\right|dy,\end{aligned}$$

but if $2^{j}\left(y-a_{i}x\right)\in\operatorname{supp}\varphi_{i,j}$ and $2^{l-h-1}\leq\left|y-a_{i}x\right|\leq 2^{l-h}$, then $2^{-1}\leq 2^{j}\left|y-a_{i}x\right|\leq 2^{l-h+j}$ and $2^{l-h-1+j}\leq 2^{j}\left|y-a_{i}x\right|\leq 2$, so $h-l-1\leq j\leq h-l+2$. Again, for the sake of simplicity, we study only the case $j=h-l$, since the other cases are similar.

Then

$$\int\limits_{A_{i}(x)}\left|k\left(x,y\right)f(y)\right|dy$$

$$\leq c\sum_{h\geq 0}\int_{2^{l-h-1}\leq|y-a_ix|\leq 2^{l-h}} 2^{n\left(\frac{h}{q_i}-l\right)}\left|\varphi_{i,h-l}\left(2^{h-l}\left(y-a_ix\right)\right)\right|$$
$$\times\prod_{s\neq i}\left|\varphi_{s,-l}\left(2^{-l}\left(y-a_sx\right)\right)f(y)\right|dy$$
$$\leq c\sum_{h\geq 0}2^{n\left(\frac{h}{q_i}-l+\frac{l-h}{p_i}+\sum_{s\neq i}\frac{l}{p_s}\right)}\left\|\varphi_{i,h-l}\right\|_{p_i}$$
$$\times\prod_{s\neq i}\left\|\varphi_{s,-l}\right\|_{p_s}\left\|f\chi_{2^{l-h-1}\leq|y-a_ix|\leq 2^{l-h}}\right\|_{p_{m+1}}$$
$$\leq c\sum_{h\geq 0}2^{n\left(\frac{h}{q_i}-l+\frac{l-h}{p_i}+\sum_{s\neq i}\frac{l}{p_s}+\frac{l-h}{p_{m+1}}\right)}\left(\frac{1}{2^{(l-h)n}}\int_{|y-a_ix|\leq 2^{l-h}}|f(y)|^{p_{m+1}}dy\right)^{\frac{1}{p_{m+1}}}$$
$$\leq c\sum_{i\geq 1}2^{nh\left(\frac{1}{q_i}-\frac{1}{p_i}-\frac{1}{p_{m+1}}\right)}\left(\frac{1}{2^{(l-h)n}}\int_{|y-a_ix|\leq 2^{l-h}}|f(y)|^{p_{m+1}}dy\right)^{\frac{1}{p_{m+1}}}$$
$$\leq c\left(M\left(f^{p_{m+1}}\left(a_ix\right)\right)\right)^{\frac{1}{p_{m+1}}}.$$

The second inequality follows by Hölder's inequality and the last one follows since $p_s > q_s$, $1\leq s\leq m$ and so $\frac{1}{q_i}-\frac{1}{p_i}-\frac{1}{p_{m+1}}=\frac{1}{q_i}-1+\sum_{s\neq i}\frac{1}{p_s}<\frac{1}{q_i}-1+\sum_{s\neq i}\frac{1}{q_s}<0$.

Thus

$$w\left\{x:\int_{A_1(x)}|k\left(x,y\right)f(y)|\,dy>\lambda\right\}\leq w\left\{x:M\left(f^{p_{m+1}}\left(a_ix\right)\right)>c\lambda^{p_{m+1}}\right\}$$
$$\leq c\frac{1}{\left(\lambda^{p_{m+1}}\right)^{\frac{p}{p_{m+1}}}}\left\|f^{p_{m+1}}\right\|_{\frac{p}{p_{m+1}},w}^{\frac{p}{p_{m+1}}}=\frac{c}{\lambda^p}\left\|f\right\|_{p,w}^{p}.$$

Now, it is easy to check that $p>\max_{1\leq i\leq m}\left\{\frac{p_i}{p_i-q_i}\right\}$ implies $p>p_{m+1}$. Indeed,

$$\frac{1}{p_{m+1}}=1-\left(\frac{1}{p_1}+\cdots+\frac{1}{p_m}\right)=\left(\frac{1}{q_1}+\cdots+\frac{1}{q_m}\right)-\left(\frac{1}{p_1}+\cdots+\frac{1}{p_m}\right)$$
$$=\frac{p_1-q_1}{q_1p_1}+\cdots+\frac{p_m-q_m}{q_mp_m}=\frac{1}{q_1}\left(\frac{p_1-q_1}{p_1}\right)+\cdots+\frac{1}{q_m}\left(\frac{p_m-q_m}{p_m}\right)$$
$$\geq\min_{1\leq i\leq m}\left\{\frac{p_i-q_i}{p_i}\right\}>\frac{1}{p},$$

then the last inequality follows since $w\in A_{\frac{p}{p_{m+1}}}$ and so M is of weak type $\frac{p}{p_{m+1}}$ with respect to the weight w.

Now, for $y \in \Big(\bigcup\limits_{1 \le i \le m} A_i \Big)^c$,

$$|y - a_i x| \le |y - a_s x| + |(a_s - a_i)\, x| \le \left(1 + \frac{D}{d}\right) |y - a_s x| .$$

If $2^j (y - a_i x) \in \operatorname{supp} \varphi_{i,j}$ and $2^k (y - a_s x) \in \operatorname{supp} \varphi_{s,k}$, we obtain as before that $j, k \le -l - r + 1$. We also have

$$\frac{1}{2} \le 2^j |y - a_i x| \le 2^j \left(1 + \frac{D}{d}\right) |y - a_s x| \le 2^{j-k} 2 \left(1 + \frac{D}{d}\right) \le 2^{j-k+R-r+2},$$

so $j > k - R + r - 3$ and, analogously,

$$2^{j-k} \frac{1}{2\left(1 + \frac{D}{d}\right)} \le 2^{j-k} \frac{1}{\left(1 + \frac{D}{d}\right)} 2^k |y - a_s x| \le 2^j |y - a_i x| \le 2,$$

so $j < k + R - r + 3$. Again we estimate only the case $j = k$.

For $y \in A_{m+1},$ we also have $|y - a_i x| \le 2^{l-1} \frac{D}{3}$ and so if $2^j |y - a_i x| \in \operatorname{supp} \varphi_{i,j}$, then $-l - R - 1 \le j \le -l - r + 1$. Thus we obtain

$$\int\limits_{A_{m+1}} |k(x,y)\, f(y)|\, dy \le c \int\limits_{\left\{y : |y| \le 2^l D\right\}} 2^{-nl} \prod_{i=1}^{m} \left| \varphi_{i,-l} \left(2^{-l} (y - a_i x)\right) \right| |f(y)|\, dy$$

$$\le c 2^{-nl + \sum\limits_{i=1}^{m} 2^{\frac{nl}{p_i}}} \Bigg(\int\limits_{\{y : |y| \le D|x|\}} |f(y)|^{p_{m+1}} \Bigg)^{\frac{1}{p_{m+1}}} = \Bigg(2^{-nl} \int\limits_{\{y : |y| \le D|x|\}} |f(y)|^{p_{m+1}} \Bigg)^{\frac{1}{p_{m+1}}}$$

$$\le c \left(M f^{p_{m+1}} (x) \right)^{\frac{1}{p_{m+1}}},$$

and proceed as in the first case to obtain the desired result.

Finally, by Hölder's inequality

$$\int\limits_{A_{m+2}} |k(x,y)\, f(y)|\, dy \le \left\| k(x,.)\, \chi_{A_{m+2}} w^{-\frac{1}{p}} \right\|_p \left\| f \chi_{A_{m+2}} w^{\frac{1}{p}} \right\|_p ,$$

now, Minkowsky's integral inequality implies that

$$\left\| k(x,.)\, \chi_{A_{m+2}} w^{-\frac{1}{p}} \right\|_p \le c \sum_{j \le -l-r-1} 2^{jn} \Bigg(\int\limits_{A_{m+2}} \prod_{i=1}^{m} \varphi_{i,j}^{\acute{p}} \left(2^j (y - a_i x)\right) |y|^{-\frac{a\acute{p}}{p}}\, dy \Bigg)^{\frac{1}{p'}}$$

$$\le c \sum_{j \le -l} 2^{jn + j\frac{a}{p} - \sum\limits_{i=1}^{m} \frac{jn}{\acute{p} q_i}} \prod_{i=1}^{m} \|\varphi_{i,j}\|_{\acute{p} q_i} \le c\, |x|^{\frac{-a}{p} + \frac{n}{r_2}} .$$

The second inequality follows since the hypothesis $p > \max\limits_{1 \le i \le m} \left\{ \frac{p_i}{p_i - q_i} \right\}$ implies $\acute{p} q_i < p_i$, $1 \le i \le m$, and since, for the involved $i, \acute{j}s$, $|y| \le |y - a_i x| + |a_i x| \le 2^{-j+1} + D 2^{l+1} \le c 2^{-j}$.

From this last inequality we obtain

$$\int\limits_{A_{m+2}} |k(x,y) f(y)| \, dy \le c\,|x|^{\frac{-a}{p}-\frac{n}{p}} \|f\|_{p,w},$$

and so

$$w\left\{x : \int\limits_{A_{m+2}} |k(x,y) f(y)| \, dy > \lambda\right\} \le w\left\{x : c\,|x|^{\frac{-a}{p}-\frac{n}{p}} \|f\|_{p,w} > \lambda\right\}$$

$$\le w\left\{x : |x| < \left(\frac{\|f\|_{p,w}}{c\lambda}\right)^{\frac{p}{n+a}}\right\} = \int\limits_{|x|<\left(\frac{\|f\|_{p,w}}{c\lambda}\right)^{\frac{p}{n+a}}} |y|^a \, dy = c\left(\frac{\|f\|_{p,w}}{\lambda}\right)^p.$$

3. Necessary Conditions

Next, we give some examples of operators T defined by (1) for which the condition $-n < a < n\left(\frac{p}{p_{m+1}} - 1\right)$ becomes necessary for the boundedness of T on $L^p(|x|^a \, dx)$, $1 \le p < \infty$.

Example 1. $k_i(x) = |x|^{-\alpha_i}$, $\alpha_1 + \cdots + \alpha_m = n$. It can be seen as $\sum_{j\in Z} 2^{j\alpha_i} \frac{\chi_{B_1(x)}}{|x|^{\alpha_i}}$ with $B_1 = \left\{x \in R^n : \frac{1}{2} < |x| \le 1\right\}$. So $\varphi_{i,j}(x) = \varphi_i(x) = \frac{\chi_{B_1}(x)}{|x|^{\alpha_i}}$, $q_i = \frac{n}{\alpha_i}$, $1 \le i \le m$. Since any φ_i belongs to $L^\infty(B_1)$, it belongs to L^{p_i} for all $1 \le p_i \le \infty$, too. So we will show that if the operator T defined by (1) is bounded on $L^p(|x|^a \, dx)$ then $-n < a < n(p-1)$. It is a reciprocal result of the theorem proved in [8], at least for power weights in A_p.

Let us prove this assertion. $a > -n$ is necessary for the local integrability of $|x|^a$. To check the other inequality, we take $B_l = \left\{x \in R^n : 2^{-l} < |x| \le 2^{-l+1}\right\}$, $f = \chi_{B_1}$ and $l \gg 1$.

$$\|Tf\|^p_{L^p(|x|^a dx)} \ge \int \left| \int\limits_{B_1} \prod_{1\le i\le m} |y - a_i x|^{-\alpha_i} \, dy \right|^p |x|^a \, dx$$

$$\ge \int\limits_{B_l} \left| \int\limits_{B_1} |y - a_i x|^{-\alpha_i} \, dy \right|^p |x|^a \, dx \ge \int\limits_{B_l} |x|^{-n\,p+a} dx \ge 2^{l(a+n(1-p))}.$$

Since $2^{l(a+n(1-p))}$ must be bounded for $l \gg 1$, we obtain $a + n(1-p) < 0$.

Example 2. We take $n = 1$, $a_1 = 1$, $a_2 = -1$, $q_1 = q > q' = q_2$. $\varphi_1(x) = \frac{\chi_{[1/2,1]}(x)}{|x-1/2|^{1/2q}}$, $\varphi_2(x) = \frac{\chi_{[1/2,1]}(x)}{|x|^{1/q'}}$ and $\varphi_{1,j}(x) = \varphi_1(x)$, $\varphi_{2,j}(x) = \varphi_2(x)$, $j \in Z$. We observe that $k_2(x) = \frac{1}{|x|^{1/q'}}$ a.e. It is easy o check that $\varphi_{i,j}$, $i = 1, 2$, satisfy the hypotheses H1) and H2) stated in the introduction for any p_1 and p_2 such that $q \le p_1 < 2q$, $q' \le p_2 < \infty$. Let p_3 be defined by $\frac{1}{p_3} + \frac{1}{2q} = 1$. We suppose that the operator T defined by (1) is bounded on $L^p\left(|x|^a\,dx\right)$, for some p with $1 \leq p < \infty$. We will show that $-1 < a < \frac{p}{p_3} - 1$.

Indeed, $a > -1$ is necessary for the local integrability of $|x|^a$. To obtain the other inequality, we take $j \ll 0$ and set $I_j = (2^{j-1,}2^j]$ and $f = \chi_{I_1}$.

$$Tf(x) \geq \int\limits_{I_1} 2^{j/q}\varphi_1(2^j\,(y-x))k_2((y+x))dy,$$

$$\begin{aligned}\|Tf\|^p_{L^p(|x|^a dx)} &\geq \int\limits_{I_{-j-2}} 2^{jp/q}\left|\int\limits_{I_1}\varphi_1(2^j\,(y-x))\,(y+x)^{-1/q\acute{}}dy\right|^p |x|^a\,dx\\
&\geq c\int\limits_{I_{-j-2}} 2^{\frac{jp}{q}+\frac{jp}{q\acute{}}-a}\left|\int\limits_{I_1}\varphi_1(2^j\,(y-x))dy\right|^p dx\\
&\geq c\int\limits_{I_{-j-2}} 2^{\frac{jp}{q}+\frac{jp}{q\acute{}}-a}\left|\int\limits_{\left\{y:0<2^j(x-y)-\frac{1}{2}2^j\right\}}\varphi_1(2^j\,(y-x))dy\right|^p dx\\
&\geq c\int\limits_{I_{-j-2}} 2^{\frac{jp}{q}+\frac{jp}{q\acute{}}-a-\frac{jp}{2q}}dx = c2^{j\left(p-a-\frac{p}{2q}-1\right)} = c2^{j\ \frac{p}{p_3}-a-1}\ .\end{aligned}$$

The fourth inequality follows since it is easy to check that, for $x \in I_{-j-2}$, $\left\{y : 0 < 2^j\,(x-y) - \frac{1}{2} < 2^j\right\} \subset I_1$. Now, for $2^{j\ \frac{p}{p_3}-a-1}$ to be bounded on $j < 0$, it must happen that $\frac{p}{p_3} - a - 1 > 0$.

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