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A GENERALIZATION OF BIHARI'S LEMMA FOR DISCONTINUOUS FUNCTIONS AND ITS APPLICATION TO THE STABILITY PROBLEM OF DIFFERENTIAL EQUATIONS WITH IMPULSE DISTURBANCE

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Abstract. This paper presents a generalization of nonlinear integral inequalities of the Gronwall–Bellman–Bihari type for discontinuous functions and its application to the investigation of the practical stability of solutions of systems of integro-differential equations with impulse perturbations at fixed moments of time.

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1. Introduction

Gronwall–Bellman–Bihari inequalities [1]–[3], [31] and their numerous linear and nonlinear generalizations [15]–[17], [19], [20], [22], [23] for continuous and discontinuous functions [4], [28] play an important role in investigating qualitative characteristics of solutions of differential equations such as existence, uniqueness, boundedness, stability with various kinds of perturbations [5]–[13], [21], [22], [24], [25]–[31].

In this paper we establish a new nonlinear integral inequality ("integro-sum" inequality) for discontinuous functions. This term was for the first time used in [28] and subsequently in [9], [10], [30].

In Section 2 we consider the mathematical model of some processes described by a system of integro-differential equations with impulse perturbations at fixed moments of time.

In Section 3 the result of [2], [3], [25] is generalized to the case of discontinuous functions.

In Section 4 the sufficient conditions of boundedness, stability, practical stability [14] of an undisturbed motion system are established for different kinds of nonlinearities on the right-hand side of a system of equations with impulse perturbations.

2. Mathematical Model

Let us consider a system of equations of the form

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$$\begin{gathered}\frac{dx}{dt} = f(t, x, K[x(t)]), \quad t \neq t_i, \\ \triangle x \mid_{t=t_i} = I_i(x),\end{gathered} \tag{1}$$

where $x \in R^n$, $f \in R^n$, $I_i \in R^n$, $i = 1, 2, \ldots$, $t \geq t_0 \geq 0$ are defined in the domain

$$W = \{(t, x) : \ t \in J = [t_0, T], \ \ T \leq \infty, \ \ \|x\| \leq h = const > 0\},$$
$$\lim_{i \to \infty} t_i = \infty, \quad t_{i-1} < t_i, \quad \forall\, i = 1, 2, \ldots,$$
$$K[x(t)] = \int\limits_{t_0}^{t} k(t, \tau, x(\tau))d\tau, \quad \Delta x|_{t=t_i} = x(t_i + 0) - x(t_i - 0) = I_i(x(t_i - 0)).$$

The solution $x(t)$ of system (1) is continuous from the left at the points $\{t_i\}$ and has first kind discontinuity at $\{t_i\}$.

Let us assume that $f(t, 0, K[0]) = I_i(0) = 0$, $\forall i \in N$ and the functions f, I_i, k satisfy the following conditions:

i) $\|f(t, x, y)\| \leq l(t)\big[\|x\| + \|y\|\big]$, $l(t) \geq 0$, $\forall t \geq t_0$,

ii) $\|k(t, s, x)\| \leq \xi(s)\|x\|^m$, $\forall s : \ t_0 \leq s \leq t$, $\xi(s) \geq 0$, $m > 0$,

iii) $\|I_i(x) - I_i(y)\| \leq \beta_i\|x - y\|$, $\forall x, y \in W$, $\beta_i = const > 0$ in domain W.

Consider the solution $x(t) = x(t, t_0, x_0)$ of the Cauchy problem for system (1) with the initial condition $x(t_0) = x_0$. It is obvious that

$$x(t, t_0, x_0) = x_0 + \int\limits_{t_0}^{t} f(\tau, x(\tau), \ K[x(\tau)])d\tau + \sum_{t_0 < t_i < t} I_i\Big(x(t_i - 0)\Big). \tag{2}$$

We call (2) the “integro-sum” representation of the solution $x(t, t_0, x_0)$.

By virtue of i)–iii), the “integro-sum’ inequality for $\|x(t)\|$ $\big(x(t) = x(t, t_0, x_0)\big)$ can be written in the form

$$\begin{aligned}\|x(t)\| &\leq \|x_0\| + \int\limits_{t_0}^{t} l(\tau)\left[\|x(\tau)\| + \|K[x(\tau)]\|\right] d\tau + \sum_{t_0 < t_i < t} \|I_i\Big(x(t_i - 0)\Big)\| \\ &\leq \|x_0\| + \int\limits_{t_0}^{t} l(\tau)\|x(\tau)\|d\tau + \int\limits_{t_0}^{t}\left[l(s)\left(\int\limits_{t_0}^{s}\xi(\sigma)\|x(\sigma)\|^m d\sigma\right)\right]ds \\ &\quad + \sum_{t_0 < t_i < t}\beta_i\|x(t_i - 0)\|.\end{aligned} \tag{3}$$

We use estimates for $\|x(t)\|$ to investigate different qualitative characteristics of solutions of system (1).

3. The Bellman–Bihari–Rakhmatullina Generalization Method for "Integro-Sum" Inequalities

Using the results of the monograph [30], let us consider the "integro-sum" inequality

$$u(t) \leq \varphi(t) + \int\limits_{t_0}^{t} \Phi(t, s, u(s))ds + \sum_{t_0<t_i<t} \Psi(t, t_k)\mu_k(u(t_k - 0)), \tag{4}$$

where $u(t)$, $\varphi(t)$, $\Psi(t, t_k)$ are continuous nonnegative functions for $t \geq t_0$, $u(t)$ having first kind discontinuities at the points t_k :

$$t_0 < t_1 < \cdots, \quad \lim_{i\to\infty} t_i = \infty;$$

$\Phi(t, s, u)$ is nonnegative at $t \geq s \geq t_0$, defined in domain $t \geq s \geq t_0$, $\mid u \mid \leq k = const < \infty$ and, for fixed t and s, it is nondecreasing with respect to u; the functions $\mu_k(u)$ are continuous nonnegative and nondecreasing with respect to u.

Then, for arbitrary $t \in [t_0, \infty[$ $u(t)$ satisfy the inequality $u(t) \leq \sigma_\varphi(t)$ [30, Theorem 3.1, p. 174], where $\sigma_\varphi(t)$ is some solution of the equation

$$\sigma(t) = \varphi(t) + \int\limits_{t_0}^{t} \Phi\left(t, s, \sigma(s)\right) ds + \sum_{t_0<t_k<t} \Psi(t, t_k)\mu_k\left(\sigma(t_k - 0)\right), \tag{5}$$

continuous on each intervals $[t_k, t_{k+1}[$, $k = 0, 1, \ldots$. Here $\sigma(t_i - 0) = \lim\limits_{t\to t_{i-0}} \sigma(t)$.

The following statement is true.

Lemma. *Let the nonnegative function $\varphi(t)$ with first kind discontinuities at the points $t_i : t_1 < t_2 < \cdots$, $\lim\limits_{i\to\infty} = \infty$ satisfy the "integro-sum" inequality*

$$\begin{aligned}\varphi(t) \leq C + \int\limits_{t_0}^{t} q(s)\varphi(s)ds + \int\limits_{t_0}^{t} q(s)\left(\int\limits_{t_0}^{s} g(\sigma)\varphi^m(\sigma)d\sigma\right)ds \\ + \sum_{t_0<t_i<t} \beta_i\varphi(t_i - 0), \ m > 0,\end{aligned} \tag{6}$$

where $C \geq 0$, $q(t) \geq 0$, $g(t) \geq 0$, $\beta_i = const \geq 0$.

Then the following estimates are valid:

$$\begin{aligned}\text{I) } \varphi(t) \leq \exp\left[\int\limits_{t_0}^{t} q(\tau)d\tau\right]\Bigg[\Big\{C \prod_{t_0<t_i<t}(1+\beta_i)\Big\}^{1-m} \\ + (1-m)\int\limits_{t_0}^{t} g(s)\Big[\exp(m-1)\int\limits_{t_0}^{s} q(\sigma)d\sigma\Big]ds\Bigg]^{\frac{1}{1-m}}, \quad 0<m<1;\end{aligned} \tag{7}$$

$$\text{II) } \varphi(t) \le C \prod_{t_0<t_i<t} (1+\beta_i) \exp\left[\int_{t_0}^{t} (q(\tau)+g(\tau))\,d\tau\right], \quad m=1; \tag{8}$$

$$\text{III) } \varphi(t) \le C \prod_{t_0<t_i<t} (1+\beta_i) \exp\left[\int_{t_0}^{t} q(\tau)d\tau\right]\left[1-(m-1)\prod_{t_0<t_i<t}(1+\beta_i)^{m-1}\right.$$

$$\left.\times C^{m-1}\int_{t_0}^{t} g(s)\left(\exp(m-1)\int_{t_0}^{s} q(\sigma)d\sigma\right)ds\right]^{-\frac{1}{m-1}}, \quad m>1, \tag{9}$$

$$\forall t \ge t_0: \int_{t_0}^{t} g(s)\left(\exp[(m-1)\int_{t_0}^{s} q(\sigma)d\sigma]\right)ds$$

$$< \left[(m-1)\prod_{t_0<t_i<t}(1+\beta_i)^{m-1}C^{m-1}\right]^{-1}.$$

Proof. Suppose that $t \in [t_0, t_1[$. Then

$$\varphi(t) \le C + \int_{t_0}^{t} q(s)\varphi(s)ds + \int_{t_0}^{t} q(s)\left(\int_{t_0}^{s} g(\sigma)\varphi^m(\sigma)d\sigma\right)ds. \tag{10}$$

Denote

$$V(t) = C + \int_{t_0}^{t} q(s)\varphi(s)ds + \int_{t_0}^{t} q(s)\left(\int_{t_0}^{s} g(\sigma)\varphi^m(\sigma)d\sigma\right)ds.$$

It is obvious that $\varphi(t_0) = V(t_0) = C$, $\varphi(t) \le V(t)$, $\forall t \ge t_0$. Then

$$\frac{dV}{dt} = q(t)\varphi(t) + q(t)\int_{t_0}^{t} g(\sigma)\varphi^m(\sigma)d\sigma \le q(t)\left[V(t) + \int_{t_0}^{t} g(\sigma)V^m(\sigma)d\sigma\right].$$

Let $W(t) = V(t) + \int_{t_0}^{t} g(\sigma)V^m(\sigma)d\sigma$. Then $W(t_0) = V(t_0) = C$, $V(t) \le W(t)$, $\forall t \ge t_0$.

It easy to see that

$$\frac{dW}{dt} \le q(t)W(t) + g(t)W^m(t).$$

From the latter differential inequality we have the following estimates for $\varphi(t)$:

$$\varphi(t) \le \exp\left[\int_{t_0}^{t} q(\tau)d\tau\right]\left[C^{1-m} + (1-m)\int_{t_0}^{t} g(\tau)\right.$$

$$\times \exp\left[(m-1)\int_{t_0}^{t} q(\sigma)d\sigma\right]d\tau\Bigg]^{\frac{1}{1-m}} \quad \text{for } 0<m<1, \; t\ge t_0,$$

$$\varphi(t)\le C\exp\left[\int_{t_0}^{t}\left(q(\tau)+g(\tau)\right)d\tau\right], \quad \text{for } m=1, \; t\ge t_0, \tag{11}$$

$$\varphi(t)\le C\exp\left[\int_{t_0}^{t} q(\tau)d\tau\right]\Bigg[1-(m-1)\,C^{m-1}\int_{t_0}^{t} g(\tau)$$

$$\times \exp\left[(m-1)\int_{t_0}^{t} q(\sigma)d\sigma\right]d\tau\Bigg]^{-\frac{1}{m-1}}, \quad \text{for } m>1 \text{ and}$$

$$\forall t\ge t_0: \int_{t_0}^{t} g(\tau)\exp\left[(m-1)\int_{t_0}^{t} q(\sigma)d\sigma\right]d\tau<\left[(m-1)\,C^{m-1}\right]^{-1}.$$

From (11) it follows that inequalities (7)–(9) are fulfilled $\forall t\in[t_0,t_1[$.

Using the scheme described in [4] on the interval $[t_k,t_{k+1}[$, $k=1,2,\dots$, and the estimates for the function $\varphi(t)$ on the interval $[t_{k-1},t_k[$, we obtain (by induction) estimates (7)–(9) on the entire interval J. □

4. Practical Stability by Chetaev

Now we investigate the problem of practical stability of a trivial solution $x\equiv 0$ of system (1) with different kinds of nonlinearity f on right-hand side of (1).

A trivial solution of system (1) is called practically stable with respect to (λ,Λ,J) if there exists exist a continuous and monotonously increasing function $\varphi(t_0,\|x_0\|)$ with respect to the second argument, such that for an arbitrary solution $x(t,t_0,x_0)\ne 0$ of system (1) the estimate

$$\|x(t,t_0,x_0)\|\le\varphi(t_0,\|x_0\|), \quad \forall t\in J,$$

is valid, where $\varphi(t_0,\|x_0\|)<\Lambda$, if only $\|x_0\|<\lambda$, $\forall t\ge t_0$, $t\in J$. Here $\lambda<\Lambda$, $J=[t_0,T]$, $T\le\infty$.

A trivial solution of system (1) is called uniformly practically stable with respect to t_0 relative to present values (λ,Λ,J), if for an arbitrary nontrivial solution $x(t,t_0,x_0)$ of system (1) the estimate $\|x(t,t_0,x_0)\|\le\varphi(\|x_0\|)$, $\forall t\in J$ holds, where $\varphi(\|x_0\|)<\Lambda$, if and only if $\|x_0\|<\lambda$. Here $\varphi(u)$ is a monotonous increasing function with respect to u.

Consider the case where the parameter $0<m<1$.

The following result is valid.

Proposition 1. *Let for system* (1) *the following conditions be fulfilled:*

a) *inequalities* i)–iii) *are fulfilled for* $m\in\,]0,1[$*;*

b) $\exists \pi(t_0) = const > 0:$

$$\prod_{t_0 < t_i < t} (1 + \beta_i) \leq \pi(t_0) < \infty, \quad \forall t \in J;$$

c) $\exists\ I_i(t_0) = const > 0 \quad (i = 1, 2):$

$$\int_{t_0}^{t} l(s) ds \leq I_1(t_0), \quad \forall t \in J,$$

$$\int_{t_0}^{s} \xi(\tau) \exp\left[(m-1) \int_{t_0}^{\tau} l(\sigma) d\sigma\right] d\tau \leq I_2(t_0) < \infty.$$

Then all solutions of system (1) *are bounded. If, in addition to to* a)–c)*, the inequality*

$$\exp\left[(1-m) I_1(t_0)\right] \left\{\left[\lambda \pi(t_0)\right]^{1-m} + (1-m) I_2(t_0)\right\} < \Lambda^{1-m} \tag{12}$$

holds, then a trivial solution of system (1) *is practically stable with respect to* (λ, Λ, J).

Proof. Using (2) and the "integro-sum" inequality (3), it is obvious that an arbitrary solution $x(t, t_0, x_0) \neq 0$ of system (1) satisfies the inequality

$$\|x(t, t_0, x_0)\| \leq \exp\left[\int_{t_0}^{t} l(\tau) d\tau\right] \left[\|x_0\|^{1-m} \prod_{t_0 < t_i < t} (1 + \beta_i)^{1-m} + (1-m) \right.$$
$$\left. \times \int_{t_0}^{t} \xi(\tau) \exp\left[(m-1) \int_{t_0}^{s} l(\sigma) d\sigma\right] d\tau\right]^{\frac{1}{1-m}}. \tag{13}$$

Using (13) and the conditions a)–c), it is easy to verify that solutions of system (1) are bounded. □

Let

$$\varphi(t_0, \|x_0\|) \overset{\text{def}}{=} \left[\|x_0\|^{1-m} \pi^{1-m}(t_0) + (1-m) I_2(t_0)\right]^{\frac{1}{1-m}} \exp\left[I_1\left(t_0\right)\right].$$

It is obvious that $\varphi(t_0, \lambda) < \Lambda$ if and only if $\|x_0\| < \lambda$ and (12) holds.

Remark 1. If in Proposition 1 $\pi(t_0)$, $I_i(t_0)$ are independent of t_0, then a trivial solution is practically stable uniformly with respect to t_0.

Consider the case where $m = 1$. The next result is valid.

Proposition 2.

I) *Let the condition* b) *of Proposition* 1 *be fulfilled and inequalities* i)–iii) *hold for* $m = 1$*. If* $\exists I_3(t_0) = const > 0$*:* $\int_{t_0}^{t} (l(\tau) + \xi(\tau))\, d\tau \leq I_3(t_0) < \infty$, $\forall t \geq t_0$*, then all solutions of system* (1) *are bounded.*

II) *Assume that part* I) *of the proposition is fulfilled and the values of the initial and next perturbations of system* (1) *satisfy the inequality*

$$\frac{\Lambda}{\lambda} > \pi(t_0) \exp\left[I_3(t_0)\right]. \tag{14}$$

Then a trivial solution of system (1) *is practically stable* (*uniformly with respect to* t_0 *if* I_3, π *are independent of* t_0).

The proof of Proposition 2 follows from the "integro-sum" inequality

$$\|x(t,t_0,x_0)\| \le \|x_0\| + \int_{t_0}^{t} l(\tau)\|x(\tau,t_0,x_0)\|d\tau + \int_{t_0}^{t} l(s)\left(\int_{t_0}^{s} \xi(\sigma)\|x(s,t_0,x_0)\|d\sigma\right)ds$$

$$\overset{\text{lemma}}{\Longrightarrow} \|x(t,t_0,x_0)\| \le \|x_0\| \prod_{t_0<t_i<t}(1+\beta_i)\exp\left[\int_{t_0}^{t}[l(s)+\xi(s)]ds\right] \le \|x_0\|\pi(t_0)\exp[I_3(t_0)] \overset{\text{def}}{=} \varphi(t_0,\|x_0\|).$$

Remark 2. It can be easily verified that when the conditions of Proposition 2 are satisfied, the estimate

$$\|x(t,t_0,x_0)\| \le \|x_0\| \prod_{t_0<t_i<t}(1+\beta_i)\exp\left[\int_{t_0}^{t}[l(s)+\xi(s)]ds\right] \tag{15}$$

implies that a trivial solution of system (1) is stable by Lyapunov.

Now consider the case where $m > 1$. The next result is valid.

Proposition 3. *Let system* (1) *satisfy conditions* i)–iii) *and the inequalities* b), c) *of Proposition* 1 *be fulfilled. Then:*

A) *all solutions of system* (1) *are bounded;*

B) *a trivial solution is*

1) *practically stable if and only if the initial and next perturbations satisfy the inequalities*

1) (*p.s.*), *if only values of initial and next perturbations satisfy inequalities*

$$\begin{gathered} \lambda(t_0)\pi(t_0) < \left[(m-1)I_2(t_0)\right]^{-\frac{1}{m-1}}, \\ \exp\left[I_1(t_0)\right]\pi(t_0)\left[1-(m-1)\pi^{m-1}(t_0)\lambda^{m-1}(t_0)I_2(t_0)\right]^{-\frac{1}{m-1}} < \Lambda; \end{gathered} \tag{16}$$

2) *uniformly practically stable if* (16) *holds and* $I_i(t_0) = I_i$, $\pi(t_0) = \pi$, $\forall t_0 \ge 0$ (*independent of* t_0)*;*

C) *a trivial solution is stable by Lyapunov if and only if*

$$\int_{t_0}^{t} \xi(s) \exp\left((m-1)\int_{t_0}^{s} l(\sigma)d\sigma\right)ds < \frac{1}{(m-1)\prod\limits_{t_0<t_i<t}(1+\beta_i)^{m-1}\|x_0\|^{m-1}} \quad \forall t \geq t_0 \geq 0 \qquad (17)$$

Statements A)–C) follow from the estimate of the norm of a solution of system (1)

$$\|x(t,t_0x_0)\| \leq \|x_0\| \prod_{t_0<t_i<t}(1+\beta_i)\left[1-(m-1)\prod_{t_0<t_i<t}(1+\beta_i)^{m-1}\|x_0\|^{m-1} \times \int_{t_0}^{t}\xi(s)\exp\left((m-1)\int_{t_0}^{s}l(\sigma)d\sigma\right)ds\right]^{-\frac{1}{m-1}} \exp\left(\int_{t_0}^{t}l(\tau)d\tau\right),$$

where $t \geq t_0$, and therefore inequality (17) is satisfied.

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