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EXISTENCE RESULTS FOR FIRST AND SECOND ORDER NONCONVEX SWEEPING PROCESSES WITH PERTURBATIONS AND WITH DELAY: FIXED POINT APPROACH

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Abstract. We are interested in existence results for nonconvex functional differential inclusions. First, we prove an existence result, in separable Hilbert spaces, for first order nonconvex sweeping processes with perturbation and with delay. Then, by using this result and a fixed point theorem we prove an existence result for second order nonconvex sweeping processes with perturbation and with delay of the form $\dot{u}(t) \in C(u(t))$, $\ddot{u}(t) \in -N^P(C(u(t)); \dot{u}(t)) + F(t, \dot{u}_t)$ when C is a nonconvex bounded Lipschitz set-valued mapping and F is a set-valued mapping with convex compact values taking their values in finite dimensional spaces.

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1. Introduction

This paper is concerned with some existence results of solutions for first and second order functional differential inclusions of the form

$$\begin{cases} \dot{u}(t) \in -N^P_{C(t)}(u(t)) + F(t, u_t) \qquad \text{a.e. on} \quad I, \\ u(t) \in C(t), \qquad \forall t \in I, \\ u(s) = T(0)u(s) = \varphi(s), \qquad \forall s \in [-\tau, 0], \end{cases} \tag{FNSPPD}$$

and

$$\begin{cases} \ddot{u}(t) \in -N^P_{C(u(t))}(\dot{u}(t)) + F(t, \dot{u}_t) \qquad \text{a.e. on} \quad I, \\ \dot{u}(t) \in C(u(t)), \qquad \forall t \in I, \\ T(0)\dot{u} = \varphi, \qquad \in [-\tau, 0], \end{cases} \tag{SNSPPD}$$

We will call them *First* (*resp. Second*) *order Nonconvex Sweeping Process with Perturbation and with Delay*. Such problems have been studied by many authors (see, for example, [3], [5], [6], [7], [8], [11], [12], [13], [14], [15], [16], [17] [18], and the references therein). In [13], some topological properties of solutions sets for the (FNSPPD) problem in the convex case are established, and in [14], the compactness of solution sets in $\mathbb{R}^n$ is obtained in the nonconvex case. Using some new properties and caracterizations of uniformly r-prox regular nonconvex sets, obtained in [9], [10], [14], [19], we establish in the present work the existence of Lipschitz solutions to the first problem. We use this result to

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prove the existence of solutions for the (SNSPPD) by applying the Kakutani Ky-Fan's fixed point theorem.

2. Preliminaries and Fundamental Results

Let H be a real separable Hilbert space and let S be a nonempty closed subset of H. We will denote by $d_S(\cdot)$ (or by $d(\cdot; S)$) the usual distance function to the subset S. We recall (see, e.g., [10]) that *the proximal normal cone* $N^P(S;x)$ to S at $x \in S$ (also denoted by $N_S^P(x)$) is defined by

$$N^P(S;x) = \{\xi \in H : \exists \alpha > 0 : x \in \text{Proj}(x + \alpha\xi, S)\}$$

where $\text{Proj}(u, S) := \{y \in S : d_S(u) = \|u - y\|\}$. Recall also (see for instance [10]) that *the proximal subdifferential* $\partial^P f(\bar{x})$ of a Lipschitz function $f : H \to \mathbb{R}$ at a point $\bar{x} \in H$ is the set of all $\xi \in H$ for which there exist $\delta, \sigma > 0$ such that for all $x' \in x + \delta\mathbb{B}$

$$\left\langle \xi, x' - x \right\rangle \leq f(x') - f(x) + \sigma\|x' - x\|^2.$$

Here $\mathbb{B}$ denotes the closed unit ball centered at the origin of H. Recall now (see [19]) that for a given $r \in]0, +\infty]$, a subset S is uniformly r-prox-regular if and only if every nonzero proximal to S can be realized by an r-ball, this means that for all $y \in S$ and all $\xi \in N^P(S;y), \xi \neq 0$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x - y \right\rangle \leq \frac{1}{2r}\|x - y\|^2,$$

for all $x \in S$. We make the convention $\frac{1}{r} = 0$ for $r = +\infty$ (in this case, the uniform r-prox-regularity is equivalent to the convexity of S).

Let Ω be an open subset of a normed vector space Z and let $C : \Omega \rightrightarrows H$ be a set-valued mapping with compact values. We will say that C is Lipschitz with ratio λ if for any $z, z' \in \Omega$ one has

$$\mathcal{H}(C(z), C(z')) \leq \lambda\|z - z'\|.$$

Here $\mathcal{H}$ stands for the Hausdorff distance relative to the norm associated with the Hilbert space H defined by

$$\mathcal{H}(A, B) := \max\big\{\sup_{a\in A} d_B(a), \sup_{b\in B} d_A(b)\big\}.$$

Let $\Phi : X \rightrightarrows Y$ be a set-valued mapping defined between two topological vector spaces X and Y, we will say that Φ is upper semi-continuous (in short u.s.c.) at $x \in \text{dom}\,(\Phi) := \{x' \in X : \Phi(x') \neq \varnothing\}$ if for any open O containing $\Phi(x)$ there exists a neighborhood V of x such that $\Phi(V) \subset O$.

Let $T > 0$. We will deal with a finite delay $\tau > 0$. If $u : [-\tau, T] \to H$, then for every $t \in [0, T]$, we define the function $u_t(s) = T(t)u(s) = u(t + s), s \in [-\tau, 0]$ and the Banach space $\mathcal{C}_T := \mathcal{C}_T([-\tau, T], H)$ (resp. $\mathcal{C}_0$) of all continuous mapping from $[-\tau, T]$ (resp. $[-\tau, 0]$) to H with the norm given by $\|\varphi\|_T = \max\{\|\varphi(s)\| : s \in [-\tau, T]\}$. Clearly, if $u \in \mathcal{C}_T$, then $u_t \in \mathcal{C}_0$, and the mapping $u \to u_t$ is continuous in the sense of uniform convergence.

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel of the paper (see [9], [19]).

Proposition 2.1. *Let S be a nonempty closed subset of H and $x \in S$. The following assertions hold:*

(1) $\partial^P d_S(x) = N^P(S,x) \bigcap \mathbb{B}$;
(2) *If S is uniformly r-prox-regular, then $\partial^P d_S(x)$ is a closed convex set in H and for any $x \in H$ with $d_S(x) < r$ one has* $\mathrm{Proj}(x,S) \neq \varnothing$.

The following closedness property of the proximal subdifferential due to Bounkhel and Thibault [10]. It is one of the powerful results used to prove our existence results in this paper.

Proposition 2.2. *Let $r \in]0,\infty], \Omega$ be an open subset in a normed vector space Z, and $C : \Omega \rightrightarrows H$ be a Lipschitz continuous set-valued mapping. Assume that $C(z)$ is uniformly r-prox-regular for all $z \in \Omega$, then the set-valued mapping $(z,x) \to \partial^P d_{C(z)}(x)$ from $Z \times H$ to H endowed with the weak topology is u.s.c., which is equivalent to the u.s.c. of the function $(z,x) \to \sigma(\partial^P d_{C(z)}(x), p)$ for any $p \in H$. Here $\sigma(S,p)$ denotes the support function associated with S, i.e., $\sigma(S,p) = \sup\limits_{s\in S}\langle s,p\rangle$.*

3. Main Results

The following theorem provides an existence result of solutions for the problem (FNSPPD) that will be used next.

Theorem 3.1. *Let H be a separable Hilbert space, $T > 0$, $I := [0,T]$, and $r \in]0,+\infty]$. Assume that C is Lipschitz with ratio $\lambda > 0$ and $C(t)$ is r-prox-regular for every $t \in I$. Let $F : I \times \mathcal{C}_0 \rightrightarrows H$ be a set-valued mapping with convex compact values in H such that $F(t,\cdot)$ is u.s.c. on $\mathcal{C}_0$ for any fixed $t \in I$ and $F(\cdot,\varphi)$ admits a measurable selection on I for any fixed $\varphi \in \mathcal{C}_0$. Assume that $F(t,\varphi) \subset l\mathbb{B}$ for all $(t,\varphi) \in I \times \mathcal{C}_0$, for some $l > 0$. Assume that $C(t)$ is strongly compact for every $t \in I$. Then for every $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$, there exists a continuous mapping $u : [-\tau,T] \to H$ such that u is Lipschitz continuous on I and satisfies :*

$$\begin{cases} \dot{u}(t) \in -N^P_{C(t)}(u(t)) + F(t,u_t) & a.e. \text{ on } \quad I, \\ u(t) \in C(t), \qquad \forall t \in I, \\ u(s) = T(0)u(s) = \varphi(s), \qquad \forall s \in [-\tau,0], \end{cases} \tag{FNSPPD}$$

and

$$\|\dot{u}(t)\| \leq (2l+\lambda) \qquad a.e. \text{ on } \quad I.$$

Proof. We prove the conclusion of our theorem when F is globally u.s.c. on $I \times \mathcal{C}_0$ and then, as in [13] (see also [4]), we can proceed by approximation to prove it when $F(t,\cdot)$ is u.s.c. on $\mathcal{C}_0$ for any fixed $t \in I$ and $F(\cdot,\varphi)$ admits a measurable selection on I for any fixed $\varphi \in \mathcal{C}_0$.

We construct via discretization a sequence of continuous mappings $\{u_n\}_n$ in $\mathcal{C}_T$. For every $n \in \mathbb{N}$, we consider the following partition of I:

$$t_{n,i} := \frac{iT}{2^n} \quad (0 \le i \le 2^n) \tag{3.1}$$

and

$$I_{n,i} :=]t_{n,i}, t_{n,i+1}] \quad \text{if} \quad 0 \le i \le 2^n - 1.$$

Put

$$\mu_n := \frac{T}{2^n}\,.$$

Fix $n_0 \ge 1$ satisfying for every $n \ge n_0$

$$(2l + 3\lambda)\mu_n < r. \tag{3.2}$$

First, we put $u_n(s) := \varphi(s)$, for all $s \in [-\tau, 0]$ and for all $n \ge n_0$.

For every $n \ge n_0$ and for every $t \in I_{n,i}$, we define by induction

$$u_{n,i+1} = \mathrm{Proj}(u_{n,i} - \mu_n f_0(t_{n,i}, T(t_{n,i})u_n), C(t_{n,i+1})), \tag{3.3}$$

$$f_n(t) := f_0(t_{n,i}, T(t_{n,i})u_n), \tag{3.4}$$

$$u_n(t) := u_{n,i} + \frac{t - t_{n,i}}{\mu_n}\,(u_{n,i+1} - u_{n,i}), \tag{3.5}$$

where $f_0(t_{n,i}, T(t_{n,i})u_n)$ is a minimal norm element of $F(t_{n,i}, T(t_{n,i})u_n)$, i.e.,

$$\|f_0(t_{n,i}, T(t_{n,i})u_n)\| = \min\{\|y\| :\ y \in F(t_{n,i}, T(t_{n,i})u_n)\} \le l$$

and

$$T(t_{n,i})u_n := (u_n)_{t_{n,i}}.$$

The above construction is possible despite the nonconvexity of the images of C. Indeed, we can show that for every $n \ge n_0$ we have

$$d_{C(t_{n,i+1})}(u_{n,i} - \mu_n f_0(t_{n,i}, T(t_{n,i})u_n)) \le l\mu_n + \lambda|t_{n,i+1} - t_{n,i}| \le (l + \lambda)\mu_n \le \frac{r}{2}$$

and hence as C has uniformly r-prox-regular values, by Proposition 2.1 one can choose a point $u_{n,i+1} = \mathrm{Proj}(u_{n,i} - \mu_n f_0(t_{n,i}, T(t_{n,i})u_n), C(t_{n,i+1}))$, for all $n \ge n_0$. Note that from (3.3) one deduces for every $0 \le i < 2^n$

$$\|u_{n,i+1} - (u_{n,i} - \mu_n f_0(t_{n,i}, T(t_{n,i})u_n))\| \le (l + \lambda)\mu_n. \tag{3.6}$$

By (3.3) and (3.5) we have for all $0 \le i < 2^n$, $u_{n,i} \in C(t_{n,i})$, and for every t and t' in $I_{n,i}$ $(0 \le i \le 2^n)$

$$u_n(t') - u_n(t) = \frac{t' - t}{\mu_n}\,(u_{n,i+1} - u_{n,i}).$$

Thus, in view of (3.6), if $t, t' \in I_{n,i}$ $(0 \le i < 2^n)$ with $t \le t'$, one obtains

$$\|u_n(t') - u_n(t)\| \le (2l + \lambda)(t' - t),$$

and, by addition, this also holds for all $t, t' \in I$ with $t \le t'$. This inequality implies that u_n is Lipschitz continuous with a constant $2l + \lambda$.

Coming back to the definition of u_n in (3.5), one observes that for $0 \le i < 2^n$

$$\dot{u}_n(t) = \frac{u_{n,i+1} - u_{n,i}}{\mu_n} \quad \text{for a.e.} \quad t \in I_{n,i}.$$

Then one obtains, in view of (3.6), for a.e. $t \in I$

$$\|\dot{u}_n(t) + f_n(t)\| \le (3l + \lambda). \tag{3.7}$$

Now, let θ_n, ρ_n be defined from I to I by $\theta_n(0) = 0$, $\rho_n(0) = 0$, and

$$\theta_n(t) = t_{n,i+1}, \quad \rho_n(t) = t_{n,i} \quad \text{if} \quad t \in I_{n,i} \quad (0 \le i < 2^n).$$

Then, by the relations (3.3), (3.4), and (3.5), and the properties of proximal normal cones to subsets, we have for a.e. $t \in I$

$$f_n(t) \in F(\rho_n(t), T(\rho_n(t))u_n)$$

and

$$\dot{u}_n(t) + f_n(t) \in -N^P(C(\theta_n(t)); u_n(\theta_n(t))). \tag{3.8}$$

By our construction and the Lipschitz continuity of C we have for any $n \ge n_0$ and any $t \in I$

$$\begin{aligned} d(u_n(t), C(t)) &\le \|u_n(t) - u_n(t_{n,i})\| + \mathcal{H}(C(t), C(t_{n,i})) \\ &\le (2l + \lambda)|t - t_{n,i}| + \lambda|t - t_{n,i}| \\ &\le 2(l + \lambda)|t - t_{n,i}| \le 2(l + \lambda)|\mu_n. \end{aligned} \tag{3.9}$$

Since $C(t)$ is strongly compact and $\mu_n \to 0$, (3.9) implies that the set $\{u_n(t) : n \ge n_0\}$ is relatively strongly compact in H for all $t \in I$. Thus, by Arzela–Ascoli's theorem we can extract a subsequence of the sequence $\{u_n\}_n$ still denoted $\{u_n\}_n$, which converges uniformly on $[-\tau, T]$ to a Lipschitz continuous function u which clearly satisfies $u \equiv \varphi$ on $[-\tau, 0]$. Now by letting $n \to +\infty$ we get for all $t \in I$

$$u(t) \in C(t).$$

On the one hand, this follows from our construction and by (3.2) and the uniform convergence of $\{u_n\}_n$ to u over I we get

$$\|u_n(\theta_n(t)) - u(t)\| \le \|u_n(\theta_n(t)) - u(\theta_n(t))\| + \|u(\theta_n(t)) - u(t)\| \to 0.$$

Now, using the same technique as in [13] and the relation (3.2) we obtain

$$\lim_{n\to\infty} \|T(\rho_n(t))u_n - T(t)u_n\| = 0 \quad \text{in} \quad \mathcal{C}_0.$$

Therefore, as the uniform convergence of u_n to u on $[-\tau, T]$ implies the uniform convergence of $T(t)u_n$ to $T(t)u$ on $[-\tau, 0]$, we conclude that

$$T(\rho_n(t))u_n \longrightarrow T(t)u = u_t \quad \text{in} \quad \mathcal{C}_0. \tag{3.10}$$

On the other hand, by (3.7) the sequences (f_n) and $(\dot{u}_n)$ are bounded sequences in $L^\infty(I, H, dt)$, then by extracting subsequences (because it is the dual space of the separable space $L^1(I, H, dt)$) we may suppose without loss of generality that that f_n and $\dot{u}_n$ weakly-$\star$ converge in $L^\infty(I, H, dt)$ to some mappings f and ω respectively. Then, for all $t \in I$ one has

$$\varphi(0)+\int_0^t \dot{u}(s)ds = u(t) = \lim_{n\to\infty} u_n(t) = \varphi(0)+\lim_{n\to\infty}\int_0^t \dot{u}_n(s)ds = \varphi(0)+\int_0^t \omega(s)ds,$$

which proves that $\dot{u}(t) = \omega(t)$ for a.e. $t \in I$.

Using now Mazur's lemma, we obtain

$$\dot{u}(t) + f(t) \in \bigcap_n \overline{\text{co}}\{\dot{u}_k(t) - f_k(t) : k \geq n\} \qquad \text{a.e.} \quad t \in I.$$

Fix such t in I and any ξ in H, the last relation yields

$$\langle \dot{u}(t) + f(t), \xi\rangle \leq \inf_n \sup_{k\geq n} \langle \dot{u}_k(t) + f_k(t), \xi\rangle.$$

By (3.7), (3.8), and Proposition 2.1 we have for a.e. $t \in I$

$$\dot{u}_n(t) + f_n(t) \in N^P(C(\theta_n(t)); u_n(\theta_n(t))) \cap \delta\mathbb{B} = \partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t))),$$

where $\delta := (3l + \lambda)$. Hence, according to this last inclusion and Proposition 2.1 we get

$$\begin{aligned}\langle \dot{u}(t) + f(t), \xi\rangle &\leq \delta \limsup_n \sigma(-\partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t)); \xi)\\ &\leq \delta\sigma(-\partial^P d_{C(t)}(u(t)); \mu).\end{aligned}$$

Since $\partial^P d_{C(t)}(u(t))$ is closed convex, we obtain

$$\dot{u}(t) + f(t) \in -\delta\partial^P d_{C(t)}(u(t)) \subset -N^P_{C(t)}(u(t))$$

and then

$$-\dot{u}(t) \in N^P_{C(t)}(u(t)) + f(t)$$

because $u(t) \in C(t)$. Finally, from (3.10) and the global upper semicontinuity of F and the convexity of its values and with the same techniques used above we can prove that

$$f(t) \in F(t, T(t)u) = F(t, u_t) \qquad \text{a.e.} \quad t \in I.$$

Thus, the existence is proved. □

Now, we use Theorem 3.1 and standard methods for the fixed point theorem to prove an existence result for the second order nonconvex sweeping processes under consideration. In the next theorem we assume that H is a finite dimensional space.

Theorem 3.2. *Let Ω be an open subset of H, $r \in]0, +\infty]$, and $C : \Omega \rightrightarrows H$ be a set-valued mapping with nonempty closed uniformly r-prox-regular values. Assume that C is Lipschitz with ratio $\lambda > 0$, and $m = \sup_{x\in\Omega} |C(x)| < +\infty$. Let $F : [0, +\infty) \times \mathcal{C}_0 \rightrightarrows H$ be a set-valued mapping with nonempty closed convex compact values in H such that $F(t, .)$ is upper semicontinuous on $\mathcal{C}_0$ for any fixed $t \in [0, +\infty)$, and F is $\mathcal{L}([0, +\infty)) \otimes \mathcal{B}(\mathcal{C}_0)$-measurable on $[0, +\infty) \times \mathcal{C}_0$ and there exists $l > 0$ satisfying $|F(t, \varphi)| \leq l$ for all $(t, \varphi) \in [0, +\infty) \times \mathcal{C}_0$. Then, for* *every* $x^0 \in \Omega, u^0 \in C(x^0)$ *and* $\varphi \in \mathcal{C}_0$ *with* $u^0 = \varphi(0)$ *there exist* $T > 0$ *and two continuous mappings* $x : [-\tau, T] \to \Omega$ *and* $u : [-\tau, T] \to \Omega$ *Lipschitz continuous on* $[0, T]$ *with ratio* m *and* $\lambda m + 2l$ *respectively such that*

$$\begin{cases} x(t) = x^0 + \int_0^t u(s)ds, \quad \forall t \in I, \\ T(0)u = \varphi \quad in \quad [-\tau, 0], \\ u(t) \in C(x(t)), \quad \forall t \in I, \\ -\dot{u}(t) \in N^P_{C(x(t))}(u(t)) + F(t, T(t)u) \qquad a.e. \quad t \in I. \end{cases} \tag{SNSPPD}$$

Proof. Let $x^0 \in \Omega$ and $T > 0$ such that $x^0 + lT\mathbb{B} \subset \Omega$. Let us define the differentiable mapping $\phi(t) := \int_0^t \varphi(s)ds$ for all $t \in I_\tau := [-\tau, 0]$ and put

$$\mathcal{X} := \Big\{ x \in \mathcal{C}_T : x \equiv \phi \ \text{ on } \ I_\tau, \ \ x(t) = x^0 + \int_0^t \dot{x}(s)ds \ \text{ on } \ I, \\ \|\dot{x}(t)\| \le m \ \text{ a.e. } \ I \Big\},$$

$$\mathcal{U} := \Big\{ u \in \mathcal{C}_T : u \equiv \varphi \ \text{ on } \ I_\tau, \ \ u(t) = u^0 + \int_0^t \dot{u}(s)ds \ \text{ on } \ I, \\ \|\dot{u}(t)\| \le \lambda m + 2l \ \text{ a.e. } \ I \Big\}.$$

Then Arzela–Ascoli's theorem ensures that $\mathcal{X}$ and $\mathcal{U}$ are convex compact sets in $\mathcal{C}_T$ and we also have by the choice of x^0 and T that $x(t) \in \Omega$, for all $x \in \mathcal{X}$, and all $t \in I$. Therefore for any $x \in \mathcal{X}$ the set-valued mapping $C \circ x : I \to H$ is λm-Lipschitz with nonempty closed bounded uniformly r-prox-regular values. Then, using Theorem 3.1, for any $x \in \mathcal{X}$ there exist an integrable mapping $f \in L^1(I, H)$ and a continuous mapping $u_x : [-\tau, T] \to H$ Lipschitz continuous on I with ratio $\lambda m + 2l$ satisfying

$$\begin{cases} \dot{u}_x(t) \in -N^P_{C(x(t))}(u_x(t)) + f(t), \qquad \text{a.e. on} \quad I, \\ f(t) \in F(t, T(t)u_x), \qquad \text{a.e. on} \quad I, \\ u_x(t) \in C(x(t)), \qquad \forall t \in I, \\ T(0)u_x = \varphi \qquad \text{in} \quad [-\tau, 0]. \end{cases} \tag{3.11}$$

Let us consider the set-valued mapping $\Phi : \mathcal{X} \to \mathcal{U}$ such that $\Phi(x) = \{u_x \in \mathcal{U} : u_x$ *is a solution of* $(3.11)\}$, and let us show that its graph $gph(\Phi) = \{(x, u_x) \in \mathcal{X} \times \mathcal{U} : u_x \in \Phi(x)\}$ is sequentially closed in $\mathcal{X} \times \mathcal{U}$.

Let (x_n, u_n) be a sequence of elements of $gph(\Phi)$ converging uniformly to $(x, \omega) \in \mathcal{X} \times \mathcal{U}$. We have to show that $\omega \in \Phi(x)$. By (3.11) there exists for each

$n \in \mathbb{N}$ an integrable mapping $f_n \in L^1(I,H)$ satisfying

$$\begin{cases} \dot{u}_n \in -N^P_{C(x_n(t))}(u_n(t)) + f_n(t), & \text{a.e. on} \quad I, \\ f_n(t) \in F(t, T(t)u_n), & \text{a.e. on} \quad I, \\ u_n(t) \in C(x_n(t)), \quad \forall t \in I, \\ T(0)u_n = \varphi \quad \text{in} \quad I_\tau \end{cases} \tag{3.12}$$

with

$$\|\dot{u}_n(t)\| \le \lambda m + 2l.$$

Thus, since $u_n(t) \in C(x_n(t)) \subset m\mathbb{B}$, for all $t \in I$ and all $n \in \mathbb{N}$, there exists (by Theorem 0.3.4. in [2]) a subsequence of $(\dot{u}_n)$ again denoted $(\dot{u}_n)$ such that $(\dot{u}_n)$ weakly converges in $L^1(I,H)$ to $\dot{\omega}$ and $\omega(t) = u_0 + \int_0^t \dot{\omega}(s)ds$. Furthermore, it is clear that if $\omega \in \mathcal{C}_T$, then $\omega_t = T(t)\omega \in \mathcal{C}_0$ and the mapping $\omega \longmapsto \omega_t$ is continuous from $\mathcal{C}_T$ to $\mathcal{C}_0$. Thus the uniform convergence of u_n to ω implies the uniform convergence of $T(t)u_n$ to $T(t)\omega$ on $[-\tau, 0]$.

Now, we wish to prove that ω satisfies (3.11), i.e., $\omega = u_x$. By (3.12) one has for almost every t in I

$$-\dot{u}_n(t) + f_n(t) \in N^P_{C(x_n(t))}(u_n(t)). \tag{3.13}$$

Since f_n is uniformly bounded in $L^1(I,H)$, by extracting a subsequence, if necessary, we may assume that f_n weakly converges in $L^1(I,H)$ to some f and then by our assumptions on F and by the classical semi-continuity results (see for instance Corollary 2.4.1 in [1]), we get $f(t) \in F(t, T(t)\omega)$. By (3.12) once again one has $u_n(t) \in C(x_n(t))$, for all $t \in I$. Since C is Lipschitz, we have $d_{C(x(t))}(u_n(t)) \le \lambda \|x(t) - x_n(t)\| \to 0$, and hence one obtains $\omega(t) \in C(x(t))$ because the set $C(x(t))$ is closed.

On the other hand, we have

$$\| - \dot{u}_n(t) + f_n(t)\| \le \lambda m + 2l =: \alpha,$$

i.e., $-\dot{u}_n(t) + f_n(t) \in \alpha\mathbb{B}$. Then by Proposition 2.1 we obtain

$$-\dot{u}_n(t) + f_n(t) \in \alpha \partial^P d_{C(x_n(t))}(u_n(t)) \quad \text{a.e. on} \quad I.$$

Now, as $(\dot{u}_n - f_n)$ weakly converges to $\dot{\omega} - f$ in $L^1(I,H)$, Mazur's lemma ensures that for a.e., t in I

$$-\dot{\omega}(t) + f(t) \in \bigcap_n \overline{\text{co}}\{-\dot{u}_k(t) + f_k(t) : k \ge n\}.$$

Fix such t in I and any μ in H, then the last relation gives

$$\langle -\dot{\omega}(t) + f(t), \mu \rangle \le \inf_n \sup_{k \ge n} \langle -\dot{u}_n(t) + f_n(t), \mu \rangle,$$

and hence according to (3.13) we get

$$\begin{aligned} \langle -\dot{\omega}(t) + f(t), \mu \rangle &\le \limsup_n \sigma(\alpha \partial^P d_{C(x_n(t))n}(u_n(t)), \mu) \\ &\le \sigma(\alpha \partial^P d_{C(x(t))}(\omega(t)), \mu). \end{aligned}$$

The last inequality follows from the upper semicontinuity of the proximal subdifferential in Proposition 2.2. As the set $\partial^P d_{C(x(t))}(\omega(t))$ is closed convex (see Proposition 2.1), we obtain

$$-\dot{\omega}(t) + f(t) \in \alpha \partial^P d_{C(x(t))}(\omega(t))$$

and then

$$-\dot{\omega}(t) + f(t) \in N^P_{C(x(t))}(\omega(t)),$$

because $\omega(t) \in C(x(t))$.

This can be rewritten as

$$\begin{cases} \dot{\omega}(t) \in -N^P_{C(x(t))}(\omega(t)) + f(t), \qquad \text{a.e. on} \quad I, \\ f(t) \in F(t, T(t)\omega), \qquad \text{a.e. on} \quad I, \\ \omega(t) \in C(x(t)), \quad \forall t \in I, \\ T(0)\omega = \varphi, \quad \text{in} \quad I_\tau. \end{cases}$$

In other words, ω is of the form u_x with

$$\begin{cases} \dot{u}_x \in -N^P_{C(x(t))}(u_x(t)) + f(t), \qquad \text{a.e. on} \quad I, \\ f(t) \in F(t, T(t)u_x), \qquad \text{a.e. on} \quad I, \\ u_x(t) \in C(x(t)), \quad \forall t \in I, \\ T(0)u_x = \varphi, \quad \text{in} \quad I_\tau. \end{cases}$$

Then, $gph(\Phi)$ is sequentially closed in $\mathcal{X} \times \mathcal{U}$. Now, let us consider the set-valued mapping $A : \mathcal{X} \rightrightarrows \mathcal{C}_T$ defined by

$$A(x) = \left\{ v_x \in \mathcal{C}_T : v_x \equiv \varphi \ \text{ on } \ I_\tau, \ \ v_x(t) = x^0 + \int_0^t u_x(s)ds \ \text{ on } \ I, \ \ u_x \in \Phi(x) \right\}.$$

It is clear that A has compact convex values in $\mathcal{C}_T$ and for any $v_x \in A(x)$, and for almost every $t \in I$, one has $\dot{v}_x(t) = u_x(t) \in C(x(t)) \subset m\mathbb{B}$. Then $v_x \in \mathcal{X}$, and so $A(x) \subset \mathcal{X}$. Moreover, the sequential closedness of $gph(\Phi)$ in $\mathcal{X} \times \mathcal{U}$ ensures the sequential closedness of $gph(A)$ in $\mathcal{X} \times \mathcal{X}$. Consequently, we get the upper semicontinuity of A and so by Kakutani–Ky-Fan's theorem, the set-valued mapping A admits a fixed point, i.e., there exists $x \in \mathcal{X}$ such that $x \in A(x)$ and hence

$$\begin{cases} \dot{u}_x \in -N^P_{C(x(t))}(u_x(t)) + f(t), \qquad \text{a.e. on} \quad I, \\ f(t) \in F(t, T(t)u_x), \qquad \text{a.e. on} \quad I, \\ u_x(t) \in C(x(t)), \quad \forall t \in I, \\ T(0)u_x = \varphi, \quad \text{in} \quad I_\tau, \\ x(t) = x^0 + \displaystyle\int_0^t u_x(s)ds, \quad \forall t \in I. \end{cases}$$

This completes the proof. □

Remark 3.1. In the case where the perturbation depends only on the first and the second variable, i.e., $F(t, T(t)x)$, we can combine our ideas and our techniques of Nonsmooth Analysis used here and the proof in [16] to prove an existence result for the (SNSPPD) problem with F of the form $F(t, T(t)x)$. We point out that in [16] the authors gave an existence result in the convex case and with a perturbation F of the above form.

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