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EXACT DISTRIBUTIONS OF THE PRODUCT AND RATIO OF ABSOLUTE VALUES OF PEARSON TYPE VII AND BESSEL FUNCTION RANDOM VARIABLES

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Abstract. Exact distributions of $|XY|$ and $|X/Y|$ are derived when X and Y are Pearson type VII and Bessel function random variables distributed independently of each other.

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1. Introduction

Pearson type VII and Bessel function distributions have found applications in a variety of areas that range from image and speech recognition and ocean engineering to finance. They are rapidly becoming distributions of first choice whenever "something" with heavier than Gaussian tails is observed in the data (see the books by Kotz *et al* [8] and Kotz and Nadarajah [9] for applications). A random variable X is said to have Pearson type VII distribution if its probability density function (pdf) is given by

$$f_X(x) = \frac{\Gamma(M-1/2)}{\sqrt{N\pi}\,\Gamma(M-1)}\left(1+\frac{x^2}{N}\right)^{1/2-M} \tag{1}$$

for $-\infty < x < \infty$, $M > 1$ and $N > 0$. A random variable Y is said to have Bessel function distribution if its pdf is given by

$$f_Y(y) = \frac{|y|^m}{\sqrt{\pi}\,2^m b^{m+1}\Gamma(m+1/2)} K_m\left(\left|\frac{y}{b}\right|\right) \tag{2}$$

for $-\infty < y < \infty$, $b > 0$ and $m > 1$, where

$$K_m(x) = \frac{\sqrt{\pi}x^m}{2^m\Gamma(m+1/2)}\int_1^\infty \left(t^2-1\right)^{m-1/2}\exp(-xt)\,dt$$

is the modified Bessel function of the second kind. The aim of this note is to derive the exact distributions of $|XY|$ and $|X/Y|$.

The distributions of $|XY|$ and $|X/Y|$ have been studied by several authors especially when X and Y are independent random variables and come from the same family. With respect to products of random variables, see the papers by Sakamoto [19] for the uniform family, Harter [5] and Wallgren [24] for Student's t family, Springer and Thompson [20] for the normal family, Stuart [22] and

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Podolski [14] for the gamma family, Steece [21], Bhargava and Khatri [3] and Tang and Gupta [23] for the beta family, AbuSalih [1] for a power function family, and Malik and Trudel [10] for the exponential family (for a comprehensive review of the known results see also the paper [18] by Rathie and Rohrer). With respect to ratios of random variables, see the papers by Marsaglia [11] and Korhonen and Narula [7] for the normal family, Press [15] for Student's t family, Basu and Lochner [2] for the Weibull family, Hawkins and Han [6] for the non-central chi-squared family, Provost [16] for the gamma family, and Pham-Gia [13] for the beta family.

However, there are relatively a small number of works of the above kind when X and Y belong to different families. In this note, we derive explicit expressions for the cumulative distribution functions (cdfs) of $|XY|$ and $|X/Y|$ when X and Y are independent random variables with pdfs (1) and (2), respectively. The derivations are presented in Sections 3 and 4. Some mathematical preliminaries are noted in Section 2.

2. Some Preliminaries

Pearson type VII distribution given by (1) is related to Student's t distribution as follows: if $M = 1 + a/2$ and

$$U = \sqrt{\frac{a}{N}} X, \tag{3}$$

then U is Student's t random variable with a degrees of freedom. Note that the pdf of Student's t random variable with ν degrees of freedom is given by

$$f(x) = \frac{1}{\sqrt{\nu} B\left(\nu/2, 1/2\right)} \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \tag{4}$$

for $-\infty < x < \infty$. Nadarajah and Kotz in [12] have shown that the cdf corresponding to (4) can be expressed as

$$F(x) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{\pi} \arctan\left(\dfrac{x}{\sqrt{\nu}}\right) \\ \quad + \dfrac{1}{2\pi} \displaystyle\sum_{l=1}^{(\nu-1)/2} B\left(l, \frac{1}{2}\right) \frac{\nu^{l-1/2} x}{\left(\nu + x^2\right)^l} & \text{if } \nu \text{ is odd,} \\ \dfrac{1}{2} + \dfrac{1}{2\pi} \displaystyle\sum_{l=1}^{\nu/2} B\left(l - \frac{1}{2}, \frac{1}{2}\right) \frac{\nu^{l-1} x}{\left(\nu + x^2\right)^{l-1/2}} & \text{if } \nu \text{ is even.} \end{cases} \tag{5}$$

Representations (3) and (5) will be crucial for the derivations in Sections 3 and 4. The derivations involve the Euler psi function (the logarithmic derivative of $\Gamma(x)$), the hypergeometric functions defined by

$${}_0F_1\left(a; x\right) = \sum_{k=0}^{\infty} \frac{1}{(a)_k} \frac{x^k}{k!} \quad \text{and} \quad {}_1F_2\left(a; b, c; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k (c)_k} \frac{x^k}{k!}$$

(where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial), and the Lommel functions defined by

$$s_{\mu,\nu}(x) = \frac{x^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} \, {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{x^2}{4}\right)$$

and

$$\begin{aligned} S_{\mu,\nu}(x) = s_{\mu,\nu}(x) &+ \frac{2^{\mu+\nu-1}\Gamma(\nu)\Gamma((\mu+\nu+1)/2)}{x^{\nu}\Gamma((1+\nu-\mu)/2)} \, {}_0F_1\left(1-\nu; -\frac{x^2}{4}\right) \\ &+ \frac{2^{\mu-\nu-1}x^{\nu}\Gamma(-\nu)\Gamma((\mu-\nu+1)/2)}{\Gamma((1-\nu-\mu)/2)} \, {}_0F_1\left(1+\nu; -\frac{x^2}{4}\right). \end{aligned}$$

We also need the notation: $a = 2(M-1)$, $r = r(z) = \sqrt{a/N}z$ and C denoting Euler's constant. The properties of the above special functions can be found in the books by Prudnikov *et al.* [17] and Gradshteyn and Ryzhik [4].

3. Distribution of the Product

Theorem 1 derives an explicit expression for the cdf of $|XY|$ in terms of the hypergeometric function.

Theorem 1. *Suppose X and Y are distributed according to* (1) *and* (2), *respectively. If a is an odd integer, then the cdf of $Z = |XY|$ can be expressed as*

$$F_Z(z) = I(a) + \frac{r}{\pi^{3/2}\sqrt{a}2^{m-1}b^{m+1}\Gamma(m+1/2)} \sum_{k=1}^{(a-1)/2} B\left(k, \frac{1}{2}\right) A(k), \quad (6)$$

where $I(\cdot)$ denotes the integral

$$I(a) = \frac{1}{\pi^{3/2}2^{m-2}b^{m+1}\Gamma(m+1/2)} \int_0^{\infty} \arctan\left(\frac{r}{\sqrt{a}y}\right) y^m K_m\left(\frac{y}{b}\right) dy,$$

and

$$\begin{aligned} A(k) &= 2^{-(m+2)}(ab)^{-m}r^{2m}\Gamma(-m)B(-m, m+k) \\ &\quad \times \, {}_1F_2\left(m+k; 1+m, 1+m; -\frac{r^2}{4ab^2}\right) \\ &\quad - \{2C + \Psi(k)\} 2^{m-2}b^m\Gamma(m) \, {}_1F_2\left(k; 1-m, 1; -\frac{r^2}{4ab^2}\right). \end{aligned} \quad (7)$$

Proof. Using relation (3), one can write the cdf as $\Pr(|XY| \le z) = \Pr(|UY| \le r)$, which can be expressed as

$$F_Z(r) = \frac{1}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+1/2)} \int_{-\infty}^{\infty} \left\{ F\left(\frac{r}{|y|}\right) - F\left(-\frac{r}{|y|}\right) \right\} |y|^m K_m\left(\left|\frac{y}{b}\right|\right) dy$$

$$= \frac{1}{\sqrt{\pi} 2^{m-1} b^{m+1} \Gamma(m+1/2)} \int_0^\infty \left\{ F\left(\frac{r}{y}\right) - F\left(-\frac{r}{y}\right) \right\} y^m K_m\left(\frac{y}{b}\right) dy, \quad (8)$$

where $F(\cdot)$ inside the integrals denotes the cdf of Student's t random variable with a degrees of freedom. Substituting the form for $F(\cdot)$ given by (5) for odd degrees of freedom, (8) can be reduced to

$$F_Z(r) = I(a) + \frac{r}{\pi^{3/2} \sqrt{a} 2^{m-1} b^{m+1} \Gamma(m+1/2)} \sum_{k=1}^{(a-1)/2} B\left(k, \frac{1}{2}\right) J(k), \quad (9)$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{m+2k-1} K_m(y/b)}{\left(y^2 + r^2/a\right)^k} dy.$$

By direct application of equation (2.16.3.13) in [17, vol. 2], one can easily see that $J(k) = A(k)$, where $A(k)$ is given by (7). The result of the theorem follows by substituting this form for $J(k)$ into (9). □

Theorem 2 is an analogue of Theorem 1 for the case where degrees of freedom a is an even integer.

Theorem 2. *Suppose X and Y are distributed according to* (1) *and* (2), *respectively. If a is an even integer, then the cdf of $Z = |XY|$ can be expressed as*

$$F_Z(z) = \frac{r}{\pi^{3/2} \sqrt{a} 2^{m-1} b^{m+1} \Gamma(m+1/2)} \sum_{k=1}^{a/2} B\left(k - \frac{1}{2}, \frac{1}{2}\right) A(k), \quad (10)$$

where

$$\begin{aligned} A(k) = {} & 2^{-(m+2)} (ab)^{-m} r^{2m} \Gamma(-m) B\left(-m, m+k-\frac{1}{2}\right) \\ & \times {}_1F_2\left(m+k-\frac{1}{2}; 1+m, 1+m; -\frac{r^2}{4ab^2}\right) \\ & - \left\{2C + \Psi\left(k - \frac{1}{2}\right)\right\} 2^{m-2} b^m \Gamma(m) \\ & \times {}_1F_2\left(k - \frac{1}{2}; 1-m, 1; -\frac{r^2}{4ab^2}\right). \end{aligned} \quad (11)$$

Proof. Substituting the form for $F(\cdot)$ given by (5) for even degrees of freedom, (8) can be reduced to

$$F_Z(r) = \frac{r}{\pi^{3/2} \sqrt{a} 2^{m-1} b^{m+1} \Gamma(m+1/2)} \sum_{k=1}^{a/2} B\left(k - \frac{1}{2}, \frac{1}{2}\right) J(k), \quad (12)$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{m+2k-2} K_m\left(y/b\right)}{\left(y^2 + r^2/a\right)^{k-1/2}}\, dy.$$

By direct application of equation (2.16.3.13) in [17, Vol. 2], one can easily see that $J(k) = A(k)$, where $A(k)$ is given by (11). The result of the theorem follows by substituting this form for $J(k)$ into (12). □

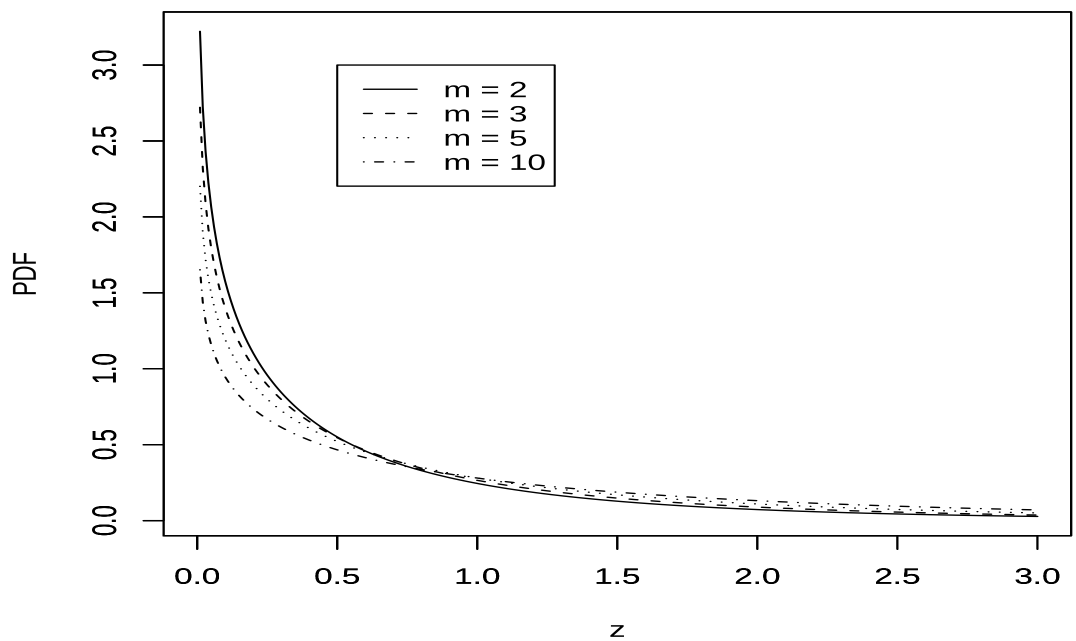

Fig. 1. Plots of the pdf of (6) and (10) for $M = 3.5$, $N = 1$, $b = 1$ and $m = 2, 3, 5, 10$

Figure 1 illustrates possible shapes of the pdf of $|XY|$ for $M = 3.5$, $N = 1$, $b = 1$ and a range of values of m. Note that the shapes are unimodal and that the value of m largely dictates the behavior of the pdf near $z = 0$.

4. Distribution of the Ratio

Theorem 3 derives an explicit expression for the cdf of $|X/Y|$ in terms of the Lommel function.

Theorem 3. *Suppose X and Y are distributed according to* (1) *and* (2), *respectively. If a is an odd integer then the cdf of $Z = |X/Y|$ can be expressed as*

$$F_Z(z) = I(a) + \frac{2a^{m/2}\Gamma(m+1)}{\pi^{3/2} b^{m+1} \Gamma\left(m + 1/2\right) r^m}$$

$$\times \sum_{k=1}^{(a-1)/2} \frac{a^{k/2} B\left(k, 1/2\right)}{b^{1-k} r^k} S_{-(m+k),1+m-k} \left(\frac{\sqrt{a}}{br} \right), \tag{13}$$

where $I(\cdot)$ denotes the integral

$$I(a) = \frac{1}{\pi^{3/2} 2^{m-2} b^{m+1} \Gamma\left(m+1/2\right)} \int_0^\infty \arctan\left(\frac{ry}{\sqrt{a}} \right) y^m K_m \left(\frac{y}{b} \right) dy.$$

Proof. Using relation (3), one can write the cdf as $\Pr(|X/Y| \le z) = \Pr(|U/Y| \le r)$, which can be expressed as

$$\begin{aligned} F_Z(r) &= \frac{1}{\sqrt{\pi} 2^m b^{m+1} \Gamma\left(m+1/2\right)} \int_{-\infty}^\infty \left\{ F\left(r|y|\right) - F\left(-r|y|\right) \right\} |y|^m K_m \left(\left| \frac{y}{b} \right| \right) dy \\ &= \frac{1}{\sqrt{\pi} 2^{m-1} b^{m+1} \Gamma\left(m+1/2\right)} \int_0^\infty \left\{ F\left(ry\right) - F\left(-ry\right) \right\} y^m K_m \left(\frac{y}{b} \right) dy, \end{aligned} \tag{14}$$

where $F(\cdot)$ inside the integrals denotes the cdf of Student's t random variable with a degrees of freedom. Substituting the form for $F(\cdot)$ given by (5) for odd degrees of freedom, (14) can be reduced to

$$F_Z(r) = I(a) + \frac{2r}{\pi^{3/2} \sqrt{a} 2^m b^{m+1} \Gamma\left(m+1/2\right)} \sum_{k=1}^{(a-1)/2} a^k r^{-2k} B\left(k, \frac{1}{2}\right) J(k), \tag{15}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{m+1} K_m\left(y/b\right)}{\left(y^2 + a/r^2\right)^k} dy. \tag{16}$$

By direct application of equation (2.16.3.14) in [17, Vol. 2], one can calculate (16) as

$$J(k) = 2^m m! a^{(m-k+1)/2} b^{k-1} r^{k-m-1} S_{-(m+k),1+m-k} \left(\frac{\sqrt{a}}{br} \right). \tag{17}$$

The result of the theorem follows by substituting (17) into (15). □

Theorem 4 is an analogue of Theorem 3 for the case where a degrees of freedom is an even integer.

Theorem 4. *Suppose X and Y are distributed according to (1) and (2), respectively. If a is an even integer, then the cdf of $Z = |X/Y|$ can be expressed as*

$$\begin{aligned} F_Z(z) = & \frac{2a^{m/2}\Gamma(m+1)}{\pi^{3/2} b^{m+1} \Gamma\left(m+1/2\right) r^m} \\ & \times \sum_{k=1}^{a/2} \frac{a^{k/2} B\left(k-1/2, 1/2\right)}{b^{3/2-k} r^k} S_{1/2-m-k,3/2+m-k} \left(\frac{\sqrt{a}}{br} \right). \end{aligned} \tag{18}$$

Proof. Substituting the form for $F(\cdot)$ given by (5) for odd degrees of freedom, (14) can be reduced to

$$F_Z(r) = \frac{2r}{\pi^{3/2}\sqrt{a}2^m b^{m+1}\Gamma\left(m+1/2\right)} \sum_{k=1}^{a/2} a^{k-1/2} r^{1-2k} B\left(k-\frac{1}{2}, \frac{1}{2}\right) J(k), \quad (19)$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{m+1} K_m\left(y/b\right)}{\left(y^2 + a/r^2\right)^{k-1/2}} \, dy. \quad (20)$$

By direct application of equation (2.16.3.14) in [17, Vol. 2], one can calculate (20) as

$$J(k) = 2^m m! a^{(m-k+3/2)/2} b^{k-3/2} r^{k-m-3/2} S_{1/2-m-k,3/2+m-k}\left(\frac{\sqrt{a}}{br}\right). \quad (21)$$

The result of the theorem follows by substituting (21) into (19). □

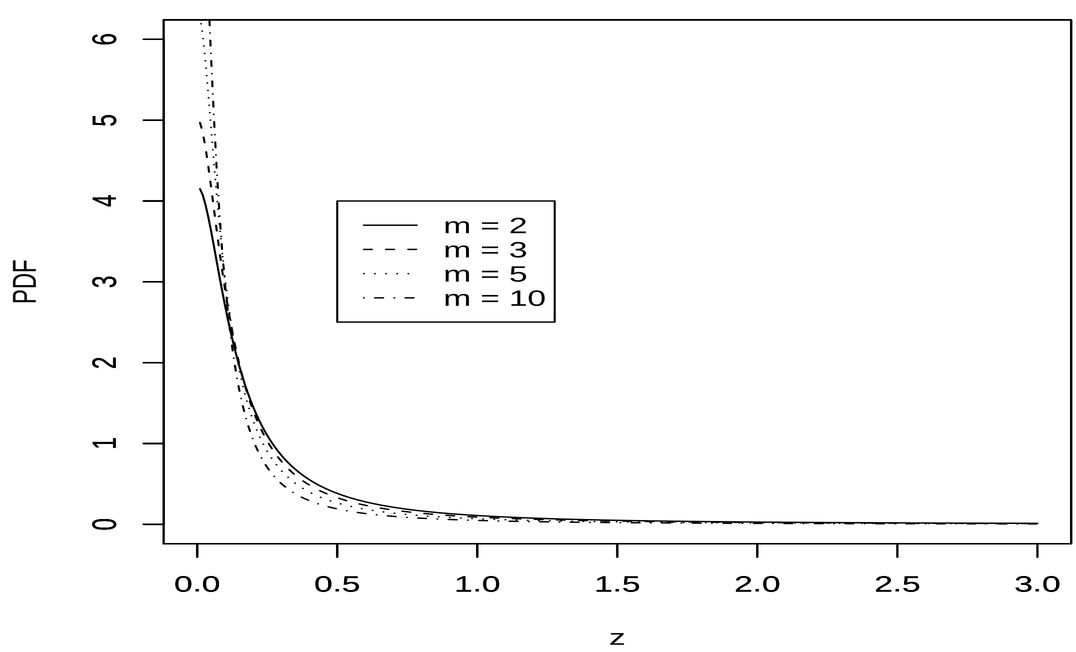

FIG. 2. Plots of the pdf of (13) and (18) for $M = 6$, $N = 1$, $b = 1$ and $m = 2, 3, 5, 10$

Figure 2 illustrates possible shapes of the pdf of $|X/Y|$ for $M = 6$, $N = 1$, $b = 1$ and a range of values of m. Note that the shapes are unimodal and that the value of m largely dictates the behavior of the pdf near $z = 0$.

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