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ON A DOUBLE SERIES OF CHAN AND ONG

KENNETH S. WILLIAMS

Abstract. An arithmetic identity is used to prove a relation satisfied by the double series $\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}$. As an application an explicit formula is given for the number of representations of the positive integer n by the form $x_1^2+x_1x_2+2x_2^2+x_3^2+x_3x_4+2x_4^2+x_5^2+x_5x_6+2x_6^2+x_7^2+x_7x_8+2x_8^2$.

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1. Introduction.

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, real numbers, complex numbers, respectively. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ we define

$$\sigma_m(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d^m, \tag{1.1}$$

where d runs through the positive integers dividing n. We also set $\sigma(n) = \sigma_1(n) = \sum_{d \mid n} d$ and $d(n) = \sigma_0(n) = \sum_{d \mid n} 1$. If $n \notin \mathbb{N}$, we set $\sigma_m(n) = 0$. The Bernoulli numbers $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}, \dots$ are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad x \in \mathbb{R}, \quad |x| < 2\pi. \tag{1.2}$$

The Eisenstein series $E_k(q)$ $(k \in \mathbb{N})$ is defined by

$$E_k(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \quad q \in \mathbb{C}, \quad |q| < 1. \tag{1.3}$$

We set

$$L(q) := E_1(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n. \tag{1.4}$$

In this paper we use a recent elementary arithmetic identity due to Huard, Ou, Spearman and Williams [3] to prove in Section 5 the following result, after some preliminary results are proved in Sections 2, 3 and 4.

Theorem 1.1. *Let $n \in \mathbb{N}$. Set $n = 7^\alpha N$, where $\alpha \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\gcd(N, 7) = 1$. Then the number of $(x, y, z, t) \in \mathbb{Z}^4$ such that*

$$n = x^2 + xy + 2y^2 + z^2 + zt + 2t^2$$

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is

$$4\sigma(n) - 28\sigma\left(\frac{n}{7}\right) = 4\sigma(N) = 4\sum_{\substack{d \mid n \\ 7 \nmid d}} d.$$

In 1999 H. H. Chan and Y. L. Ong [2] introduced the two-dimensional theta series

$$S(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}, \quad q \in \mathbb{C}, \quad |q| < 1. \tag{1.5}$$

They proved a result equivalent to the following identity [2, Remark 3, p. 1742].

Theorem 1.2. $S^2(q) = \frac{7}{6}L(q^7) - \frac{1}{6}L(q)$.

This identity is also equivalent to the one stated by Ramanujan as entry 5 of his second notebook [10] and first proved by Berndt [1, p. 467, entry 5(i)]. Both Berndt and Chan and Ong used modular equations of degree 7 in their proofs of Theorem 1.2. We show in Section 6 that Theorem 1.2 is a simple consequence of Theorem 1.1 and thus can be viewed as an elementary identity.

Klein and Fricke in their book [6, p. 400] gave an analytic proof of the following theorem.

Theorem 1.3. *Let $n \in \mathbb{N}$. Then the number of $(x, y, z, t) \in \mathbb{Z}^4$ such that*

$$4n = x^2 + y^2 + 7z^2 + 7t^2, \quad x \equiv z \pmod{2}$$

is

$$4\sum_{\substack{d \mid n \\ 7 \nmid d}} d.$$

We show in Section 7 that Theorem 1.3 is also an elementary consequence of Theorem 1.1, thus providing an elementary proof of Theorem 1.3. The elementary proof of Theorem 1.3 given by Humbert [4] is restricted to odd n.

Next, making use of a result, which was proved recently by Lemire and Williams [8, Lemma 4.6, p. 113] in order to evaluate the convolution sum

$$\sum_{\substack{m \in \mathbb{N} \\ 1 \leq m < \frac{n}{7}}} \sigma(m)\sigma(n-7m),$$

in conjunction with Theorem 1.2, we prove in Section 8 the following result.

Theorem 1.4. *Let $n \in \mathbb{N}$. Then the number of $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8$ such that*

$$n = x_1^2 + x_1x_2 + 2x_2^2 + x_3^2 + x_3x_4 + 2x_4^2 + x_5^2 + x_5x_6 + 2x_6^2 + x_7^2 + x_7x_8 + 2x_8^2$$

is given by

$$\frac{24}{5}\sigma_3(n) + \frac{1176}{5}\sigma_3\left(\frac{n}{7}\right) + \frac{16}{5}c_7(n),$$

where the $c_7(n)$ $(n \in \mathbb{N})$ *are integers defined by*

$$\sum_{n=1}^{\infty} c_7(n)q^n = q\left(\prod_{n=1}^{\infty}(1-q^n)^{16}(1-q^{7n})^8 + 13q\prod_{n=1}^{\infty}(1-q^n)^{12}(1-q^{7n})^{12} + 49q^2\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{7n})^{16}\right)^{\frac{1}{3}}. \tag{1.6}$$

This result should be compared with that of Kachakhidze [5].

Finally, we make use of a classical identity of Jacobi, which is given for example in [7, Corollary 6, p. 37], to prove the following formula for $c_7(n)$ $(n \in \mathbb{N})$ in Section 9.

Theorem 1.5. *For* $n \in \mathbb{N}$ *we have*

$$c_7(n) = \sum_{\substack{(r,s)\in\mathbb{N}_0^2 \\ \frac{r(r+1)}{2}+7\frac{s(s+1)}{2}=\ n-1}} (-1)^{r+s}(2r+1)(2s+1) + 2\sum_{\substack{(r,s,t)\in\mathbb{N}_0^2\times\mathbb{N} \\ \frac{r(r+1)}{2}+7\frac{s(s+1)}{2}+t=\ n-1}} (-1)^{r+s}(2r+1)(2s+1)\sum_{\substack{d\in\mathbb{N}\\ d\mid t}}\left(\frac{-7}{d}\right).$$

Here

$$\left(\frac{-7}{d}\right) = \begin{cases} 1, & \text{if } d \equiv 1,2,4 \pmod 7, \\ -1, & \text{if } d \equiv 3,5,6 \pmod 7, \\ 0, & \text{if } d \equiv 0 \pmod 7, \end{cases}$$

is the Legendre–Jacobi–Kronecker symbol for discriminant -7.

2. Some properties of $F_k(n)$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ we define

$$F_k(n) := \begin{cases} 1, & \text{if } \ k \mid n, \\ 0, & \text{if } \ k \nmid n. \end{cases} \tag{2.1}$$

Let $a \in \mathbb{Z}$. Denote the gcd of k and a by (k,a). Clearly

$$F_k(an) = F_{k/(k,a)}(n). \tag{2.2}$$

For $x \in \mathbb{R}$ we denote the greatest integer less than or equal to x by $[x]$. The following results are easily proved:

$$\sum_{d\mid n} F_k(d) = d\left(\frac{n}{k}\right); \tag{2.3}$$

$$\sum_{d\mid n} dF_k(d) = k\sigma\left(\frac{n}{k}\right); \tag{2.4}$$

$$\sum_{d\mid n} \frac{n}{d}F_k(d) = \sigma\left(\frac{n}{k}\right); \tag{2.5}$$

$$\sum_{1\le l\le m} F_k(l) = \left[\frac{m}{k}\right]; \tag{2.6}$$

$$\sum_{\substack{1\le l\le m\\ 2|l}} F_k(l) = \begin{cases} \left[\dfrac{m}{k}\right], & \text{if } \ 2|k,\\ \left[\dfrac{m}{2k}\right], & \text{if } \ 2\nmid k; \end{cases} \tag{2.7}$$

$$\sum_{\substack{1\le l\le m\\ 2\nmid l}} F_k(l) = \begin{cases} 0, & \text{if } \ 2|k,\\ \left[\dfrac{m+k}{2k}\right], & \text{if } \ 2\nmid k; \end{cases} \tag{2.8}$$

$$\sum_{\substack{d|n\\ 2|d}} \sum_{\substack{1\le l\le d\\ 2|l}} F_k(l) = \begin{cases} \displaystyle\sum_{d|\frac{n}{2}} \left[\frac{2d}{k}\right], & \text{if } \ 2|k,\\ \displaystyle\sum_{d|\frac{n}{2}} \left[\frac{d}{k}\right], & \text{if } \ 2\nmid k; \end{cases} \tag{2.9}$$

$$\sum_{\substack{d|n\\ 2|d}} \sum_{\substack{1\le l\le d\\ 2\nmid l}} F_k(l) = \begin{cases} 0, & \text{if } \ 2|k,\\ \displaystyle\sum_{d|\frac{n}{2}} \left[\frac{2d+k}{2k}\right], & \text{if } \ 2\nmid k; \end{cases} \tag{2.10}$$

$$\sum_{\substack{d|n\\ 2\nmid d}} \sum_{\substack{1\le l\le d\\ 2|l}} F_k(l) = \begin{cases} \displaystyle\sum_{d|n} \left[\frac{d}{k}\right] - \sum_{d|\frac{n}{2}} \left[\frac{2d}{k}\right], & \text{if } \ 2|k,\\ \displaystyle\sum_{d|n} \left[\frac{d}{2k}\right] - \sum_{d|\frac{n}{2}} \left[\frac{d}{k}\right], & \text{if } \ 2\nmid k; \end{cases} \tag{2.11}$$

$$\sum_{\substack{d|n\\ 2\nmid d}} \sum_{\substack{1\le l\le d\\ 2\nmid l}} F_k(l) = \begin{cases} 0, & \text{if } \ 2|k,\\ \displaystyle\sum_{d|n} \left[\frac{d+k}{2k}\right] - \sum_{d|\frac{n}{2}} \left[\frac{2d+k}{2k}\right], & \text{if } \ 2\nmid k. \end{cases} \tag{2.12}$$

Adding (2.9) and (2.12) we obtain

$$\sum_{d|n} \sum_{\substack{1\le l\le d\\ l\equiv d \,(\mathrm{mod}\ 2)}} F_k(l)$$

$$= \begin{cases} \displaystyle\sum_{d|n/2} \left[\frac{2d}{k}\right], & \text{if } \ 2|k,\\ \displaystyle\sum_{d|n/2} \left[\frac{d}{k}\right] + \sum_{d|n} \left[\frac{d+k}{2k}\right] - \sum_{d|n/2} \left[\frac{2d+k}{2k}\right], & \text{if } \ 2\nmid k. \end{cases} \tag{2.13}$$

3. An identity of Huard, Ou, Spearman and Williams. Using nothing more than the rearrangement of terms in finite sums, Huard, Ou, Spearman and Williams [3] proved the following elementary arithmetic formula.

Theorem 3.1. *Let $F : \mathbb{Z}^4 \to \mathbb{C}$ be such that*

$$F(a,b,x,y) - F(x,y,a,b) = F(-a,-b,x,y) - F(x,y,-a,-b)$$

for all $(a,b,x,y) \in \mathbb{Z}^4$. Then, for $n \in \mathbb{N}$, we have

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} \Big(F(a,b,x,-y) - F(a,-b,x,y) + F(a,a-b,x+y,y)$$
$$- F(a,a+b,y-x,y) + F(b-a,b,x,x+y) - F(a+b,b,x,x-y) \Big)$$
$$= \sum_{\substack{d\in\mathbb{N} \\ d\mid n}} \sum_{x=1}^{d-1} \Big(F(0,n/d,x,d) + F(n/d,0,d,x) + F(n/d,n/d,d-x,-x)$$
$$- F(x,x-d,n/d,n/d) - F(x,d,0,n/d) - F(d,x,n/d,0) \Big).$$

Taking $F(a,b,x,y) = f(b)$ in Theorem 3.1, where $f : \mathbb{Z} \to \mathbb{C}$ is an even function, we obtain

Corollary 3.1. *Let $f : \mathbb{Z} \to \mathbb{C}$ be an even function. Then for $n \in \mathbb{N}$ we have*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} \Big(f(a-b) - f(a+b) \Big)$$
$$= f(0)(\sigma(n) - d(n)) + \sum_{d\mid n} f(d) - \sum_{d\mid n} df(d) + 2\sum_{d\mid n} \frac{n}{d} f(d) - 2\sum_{d\mid n} \sum_{1\le l\le d} f(l).$$

Corollary 3.1 was stated but not proved by Liouville in [9]. Replacing n by $2n$ in Theorem 3.1, and choosing $F(a,b,x,y) = F_2(a)f(b)F_2(y)$, where $f : \mathbb{Z}^4 \to \mathbb{C}$ is even, we obtain

Corollary 3.2. *Let $f : \mathbb{Z} \to \mathbb{C}$ be an even function. Then for $n \in \mathbb{N}$ we have*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} \Big(f(2a-b) - f(2a+b) \Big)$$
$$= f(0)\left(\frac{1}{2}\sigma(n) - \frac{1}{2}d(n) - \frac{1}{2}d\left(\frac{n}{2}\right)\right)$$
$$+ \frac{1}{2}\sum_{d\mid n} f(d) - \frac{1}{2}\sum_{d\mid n} df(d) + 2\sum_{d\mid n} \frac{n}{d} f(d)$$
$$+ \frac{1}{2}\sum_{d\mid n} f(2d) + \frac{1}{2}\sum_{d\mid n} \frac{n}{d} f(2d) - \sum_{d\mid n} \sum_{1\le l\le 2d} f(l)$$

$$-\sum_{d \mid n} \sum_{\substack{1 \le l \le d \\ l \equiv d \pmod 2}} f(l).$$

Let $k \in \mathbb{N}$. Taking $f(x) = F_k(x)$ $(x \in \mathbb{Z})$ in Corollary 3.1 and appealing to (2.3), (2.4), (2.5) and (2.6), we obtain

Theorem 3.2. *Let $k, n \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} \Big(F_k(a-b) - F_k(a+b) \Big)$$

$$= \sigma(n) - (k-2)\sigma\left(\frac{n}{k}\right) - d(n) + d\left(\frac{n}{k}\right) - 2\sum_{d \mid n} \left[\frac{d}{k}\right].$$

Finally, taking $f(x) = F_k(x)$ $(x \in \mathbb{Z})$ in Corollary 3.2, and appealing to (2.2), (2.3), (2.4), (2.5), (2.6) and (2.13), we obtain

Theorem 3.3. *Let $k, n \in \mathbb{N}$. Then if k is odd we have*

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} (F_k(2a-b) - F_k(2a+b))$$

$$= \frac{1}{2}\sigma(n) + \frac{(5-k)}{2}\sigma\left(\frac{n}{k}\right) - \frac{1}{2}d(n) - \frac{1}{2}d\left(\frac{n}{2}\right) + d\left(\frac{n}{k}\right)$$

$$-\sum_{d \mid n} \left[\frac{2d}{k}\right] - \sum_{d \mid n/2} \left[\frac{d}{k}\right] - \sum_{d \mid n} \left[\frac{d+k}{2k}\right] + \sum_{d \mid n/2} \left[\frac{2d+k}{2k}\right]$$

and if k is even

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} (F_k(2a-b) - F_k(2a+b))$$

$$= \frac{1}{2}\sigma(n) + \frac{(4-k)}{2}\sigma\left(\frac{n}{k}\right) + \frac{1}{2}\sigma\left(\frac{n}{k/2}\right) - \frac{1}{2}d(n) - \frac{1}{2}d\left(\frac{n}{2}\right) + \frac{1}{2}d\left(\frac{n}{k}\right)$$

$$+ \frac{1}{2}d\left(\frac{n}{k/2}\right) - \sum_{d \mid n} \left[\frac{2d}{k}\right] - \sum_{d \mid n/2} \left[\frac{2d}{k}\right].$$

4. Evaluation of some finite sums. Our task in this section is to give the values of the sums $\sum_{d \mid n} \left[\frac{d}{k}\right]$, $\sum_{d \mid n} \left[\frac{2d}{k}\right]$, $\sum_{d \mid n/2} \left[\frac{d}{k}\right]$, $\sum_{d \mid n} \left[\frac{d+k}{2k}\right]$ and $\sum_{d \mid n/2} \left[\frac{2d+k}{2k}\right]$ occurring in Theorems 3.2 and 3.3 in the special case where $k = 7$.

For $a \in \mathbb{Z}$ and $m, n \in \mathbb{N}$ we define

$$d_{a,m}(n) := \sum_{\substack{d|n \\ d \equiv a \pmod{m}}} 1, \tag{4.1}$$

so that

$$\sum_{a=0}^{m-1} d_{a,m}(n) = d(n). \tag{4.2}$$

In particular we set

$$d_i := d_{i,7}(n), \quad i = 0, 1, 2, 3, 4, 5, 6, \tag{4.3}$$

and

$$e_i := d_{i,14}(n), \quad i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. \tag{4.4}$$

Clearly,

$$d_i = e_i + e_{i+7}, \quad i = 0, 1, 2, 3, 4, 5, 6. \tag{4.5}$$

Also,

$$d_0 = d_{0,7}(n) = \sum_{\substack{d|n \\ d \equiv 0 \pmod{7}}} 1 = \sum_{d|n/7} 1 = d\left(\frac{n}{7}\right) \tag{4.6}$$

and, similarly,

$$e_0 = d\left(\frac{n}{14}\right). \tag{4.7}$$

Thus

$$e_7 = d_0 - e_0 = d\left(\frac{n}{7}\right) - d\left(\frac{n}{14}\right). \tag{4.8}$$

We need the following results, all of which are simple to prove.

$$d(n) = e_0 + e_1 + e_2 + \cdots + e_{13}. \tag{4.9}$$

$$d\left(\frac{n}{2}\right) = e_0 + e_2 + e_4 + \cdots + e_{12}. \tag{4.10}$$

$$d\left(\frac{n}{7}\right) = e_0 + e_7. \tag{4.11}$$

$$d_{i,7}\left(\frac{n}{2}\right) = e_{2i}, \quad i = 0, 1, 2, \ldots, 6. \tag{4.12}$$

$$\begin{aligned}\sum_{d|n} \left[\frac{d}{7}\right] = \frac{1}{7}\sigma(n) - \frac{1}{7}(&e_1 + 2e_2 + 3e_3 + 4e_4 + 5e_5 + 6e_6 \\ &+ e_8 + 2e_9 + 3e_{10} + 4e_{11} + 5e_{12} + 6e_{13}).\end{aligned} \tag{4.13}$$

$$\begin{aligned}\sum_{d|n} \left[\frac{2d}{7}\right] = \frac{2}{7}\sigma(n) - \frac{1}{7}(&2e_1 + 4e_2 + 6e_3 + e_4 + 3e_5 + 5e_6 \\ &+ 2e_8 + 4e_9 + 6e_{10} + e_{11} + 3e_{12} + 5e_{13}).\end{aligned} \tag{4.14}$$

$$\sum_{d|n}\left[\frac{d+7}{14}\right]=\frac{1}{14}\sigma(n)-\frac{1}{14}(e_1+2e_2+3e_3+4e_4+5e_5+6e_6 - 7e_7-6e_8-5e_9-4e_{10}-3e_{11}-2e_{12}-e_{13}). \tag{4.15}$$

$$\sum_{d|n/2}\left[\frac{d}{7}\right]=\frac{1}{7}\sigma\left(\frac{n}{2}\right)-\frac{1}{7}(e_2+2e_4+3e_6+4e_8+5e_{10}+6e_{12}). \tag{4.16}$$

$$\sum_{d|n/2}\left[\frac{2d+7}{14}\right]=\frac{1}{7}\sigma\left(\frac{n}{2}\right)-\frac{1}{7}(e_2+2e_4+3e_6-3e_8-2e_{10}-e_{12}). \tag{4.17}$$

We are now in a position to prove the three theorems that we will need in the proof of Theorem 1.1 in Section 5.

Theorem 4.1. *Let $n \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}}\Big(F_7(a-b)-F_7(a+b)\Big) = \frac{5}{7}\sigma(n)-5\sigma\left(\frac{n}{7}\right)-\frac{5}{7}d_1-\frac{3}{7}d_2-\frac{1}{7}d_3+\frac{1}{7}d_4+\frac{3}{7}d_5+\frac{5}{7}d_6.$$

Proof. This result follows by taking $k=7$ in Theorem 3.2 and appealing to (4.5), (4.9), (4.11) and (4.13). □

Theorem 4.2. *Let $n \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}}\Big(F_7(2a-b)-F_7(2a+b)\Big) = \frac{1}{7}\sigma(n)-\sigma\left(\frac{n}{7}\right)-\frac{1}{7}d_1-\frac{2}{7}d_2+\frac{4}{7}d_3-\frac{4}{7}d_4+\frac{2}{7}d_5+\frac{1}{7}d_6.$$

Proof. This result follows by taking $k=7$ in Theorem 3.3 and appealing to (4.5), (4.9), (4.10), (4.11), (4.14), (4.15), (4.16) and (4.17). □

Theorem 4.3. *For all $a,b \in \mathbb{N}$*

$$\left(\frac{-7}{ab}\right)=\Big(F_7(a-b)-F_7(a+b)\Big)+\Big(F_7(a-2b)-F_7(a+2b)\Big) + \Big(F_7(2a-b)-F_7(2a+b)\Big).$$

Proof. If $a\equiv 0 \pmod 7$ or $b \equiv 0 \pmod 7$ both the left-hand side and right-hand side of the asserted formula are zero. Thus we may suppose that $a \not\equiv 0 \pmod 7$ and $b \not\equiv 0 \pmod 7$. Define $c \not\equiv 0 \pmod 7$ by $a \equiv bc \pmod 7$. Then the assertion of the theorem becomes

$$\left(\frac{-7}{c}\right)=\Big(F_7(c-1)-F_7(c+1)\Big)+\Big(F_7(c-2)-F_7(c+2)\Big)$$

$$+\Big(F_7(2c-1)-F_7(2c+1)\Big).$$

This is easily checked for the six cases $c \equiv 1,2,3,4,5,6 \pmod 7$. □

5. Proof of Theorem 1.1. For $m \in \mathbb{N}_0$ we let

$$r(m) = \text{number of } (x,y) \in \mathbb{Z}^2 \text{ such that } x^2+xy+2y^2 = m. \tag{5.1}$$

Clearly,

$$r(0) = 1. \tag{5.2}$$

For $n \in \mathbb{N}$ it is a classical result that

$$r(n) = 2\sum_{d \mid n} \left(\frac{-7}{d}\right). \tag{5.3}$$

Thus

$$r(n) = 2d_1 + 2d_2 - 2d_3 + 2d_4 - 2d_5 - 2d_6. \tag{5.4}$$

The number of $(x,y,z,t) \in \mathbb{Z}^4$ such that

$$n = x^2+xy+2y^2+z^2+zt+2t^2$$

is (appealing to (5.2), (5.3), Theorem 4.3, Theorem 4.1, Theorem 4.2 and (5.4))

$$\begin{aligned}
\sum_{\substack{(k,l)\in\mathbb{N}_0^2\\ k+l=n}} r(k)r(l) &= 2r(n) + \sum_{k=1}^{n-1} r(k)r(n-k)\\
&= 2r(n) + \sum_{k=1}^{n-1}\Bigg(2\sum_{a\mid k}\left(\frac{-7}{a}\right)\Bigg)\Bigg(2\sum_{b\mid n-k}\left(\frac{-7}{b}\right)\Bigg)\\
&= 2r(n) + 4\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}}\left(\frac{-7}{ab}\right)\\
&= 2r(n) + 4\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}}\Big(F_7(a-b)-F_7(a+b)\Big)\\
&\qquad + 4\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}}\Big(F_7(a-2b)-F_7(a+2b)\Big)\\
&\qquad + 4\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}}\Big(F_7(2a-b)-F_7(2a+b)\Big)\\
&= 2r(n) + 4\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}}\Big(F_7(a-b)-F_7(a+b)\Big)
\end{aligned}$$

$$+ 8 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} \Big(F_7(2a-b) - F_7(2a+b) \Big)$$

$$= 2r(n) + 4\left(\frac{5}{7}\sigma(n) - 5\sigma\left(\frac{n}{7}\right) - \frac{5}{7}d_1 - \frac{3}{7}d_2 - \frac{1}{7}d_3 + \frac{1}{7}d_4 + \frac{3}{7}d_5 + \frac{5}{7}d_6 \right)$$

$$+ 8\left(\frac{1}{7}\sigma(n) - \sigma\left(\frac{n}{7}\right) - \frac{1}{7}d_1 - \frac{2}{7}d_2 + \frac{4}{7}d_3 - \frac{4}{7}d_4 + \frac{2}{7}d_5 + \frac{1}{7}d_6 \right)$$

$$= 2r(n) + 4\sigma(n) - 28\sigma\left(\frac{n}{7}\right) - 4d_1 - 4d_2 + 4d_3 - 4d_4 + 4d_5 + 4d_6$$

$$= 4\sigma(n) - 28\sigma\left(\frac{n}{7}\right).$$

This completes the proof of Theorem 1.1. □

6. Proof of Theorem 1.2. We have by (1.5), Theorem 1.1 and (1.4)

$$\begin{aligned} S^2(q) &= \sum_{x,y,z,t\in\mathbb{Z}} q^{x^2+xy+2y^2+z^2+zt+2t^2} \\ &= 1 + \sum_{n=1}^{\infty} \left(4\sigma(n) - 28\sigma\left(\frac{n}{7}\right) \right) q^n \\ &= \frac{7}{6}L(q^7) - \frac{1}{6}L(q). \end{aligned}$$

□

7. Proof of Theorem 1.3. Let $n \in \mathbb{N}$. Set

$$A(n) := \{(x,y,z,t) \in \mathbb{Z}^4 \mid 4n = x^2 + y^2 + 7z^2 + 7t^2, x \equiv z \pmod 2\} \tag{7.1}$$

and

$$B(n) := \{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + xy + 2y^2 + z^2 + zt + 2t^2\}. \tag{7.2}$$

Let $(x,y,z,t) \in A(n)$. Then $4n = x^2 + y^2 + 7z^2 + 7t^2$ and $x \equiv z \pmod 2$ so $\frac{x-z}{2} \in \mathbb{Z}$ and

$$y - t \equiv y^2 - t^2 \equiv y^2 + 7t^2 = 4n - x^2 - 7z^2 \equiv x - z \equiv 0 \pmod 2$$

so $\frac{y-t}{2} \in \mathbb{Z}$. Further

$$\begin{aligned} \left(\frac{x-z}{2}\right)^2 + \left(\frac{x-z}{2}\right)z + 2z^2 + \left(\frac{y-t}{2}\right)^2 + \left(\frac{y-t}{2}\right)t + 2t^2 \\ = \frac{1}{4}(x^2 + 7z^2 + y^2 + 7t^2) = n \end{aligned}$$

so $\left(\frac{x-z}{2}, z, \frac{y-t}{2}, t\right) \in B(n)$. Thus we can define $\lambda : A(n) \to B(n)$ by

$$\lambda((x,y,z,t)) = \left(\frac{x-z}{2}, z, \frac{y-t}{2}, t\right).$$

Clearly, λ is injective. Let $(x_1, y_1, z_1, t_1) \in B(n)$. Set $x = 2x_1 + y_1 \in \mathbb{Z}$, $y = 2z_1 + t_1 \in \mathbb{Z}$, $z = y_1 \in \mathbb{Z}$, $t = t_1 \in \mathbb{Z}$. Clearly, $x \equiv y_1 \equiv z \pmod 2$. Also

$$\begin{aligned} x^2 + y^2 + 7z^2 + 7t^2 &= (2x_1 + y_1)^2 + (2z_1 + t_1)^2 + 7y_1^2 + 7t_1^2 \\ &= 4(x_1^2 + x_1y_1 + 2y_1^2 + z_1^2 + z_1t_1 + 2t_1^2) = 4n. \end{aligned}$$

Hence $(x, y, z, t) \in A(n)$. Moreover

$$\lambda((x, y, z, t)) = \left(\frac{x - z}{2}, z, \frac{y - t}{2}, t\right) = (x_1, y_1, z_1, t_1)$$

so λ is surjective. Thus λ is a bijection and we have by Theorem 1.1

$$\text{card } A(n) = \text{card } B(n) = 4\sigma(n) - 28\sigma\left(\frac{n}{7}\right) = 4\sum_{\substack{d|n \\ 7\nmid d}} d$$

as asserted. □

8. Proof of Theorem 1.4. Let $n \in \mathbb{N}$. Let $N(n)$ denote the number of $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8$ such that

$$n = x_1^2 + x_1x_2 + 2x_2^2 + x_3^2 + x_3x_4 + 2x_4^2 + x_5^2 + x_5x_6 + 2x_6^2 + x_7^2 + x_7x_8 + 2x_8^2.$$

Then by Theorem 1.2 we have

$$\sum_{n=0}^{\infty} N(n)q^n = \left(\sum_{(x,y)\in\mathbb{Z}^2} q^{x^2+xy+2y^2}\right)^4 = S^4(q) = \frac{1}{36}\Big(L(q) - 7L(q^7)\Big)^2.$$

Appealing to [8, Lemma 4.6, p. 113] we obtain

$$\sum_{n=0}^{\infty} N(n)q^n = 1 + \sum_{n=1}^{\infty}\left(\frac{24}{5}\sigma_3(n) + \frac{1176}{5}\sigma_3\left(\frac{n}{7}\right) + \frac{16}{5}c_7(n)\right)q^n.$$

Equating coefficients of q^n $(n \in \mathbb{N})$, we obtain the asserted result. □

9. Proof of Theorem 1.5. As in [8, equation (4.1), p. 112] we define

$$H = \left(\frac{A^7}{C} + 13qA^3C^3 + 49q^2\frac{C^7}{A}\right)^{\frac{1}{3}}, \tag{9.1}$$

where

$$A := \prod_{n=1}^{\infty}(1 - q^n), \quad C := \prod_{n=1}^{\infty}(1 - q^{7n}). \tag{9.2}$$

From the proof of Theorem 1.4 and [8, Lemma 4.2, p. 112] we have

$$S^4(q) = \frac{1}{36}\Big(L(q) - 7L(q^7)\Big)^2 = H^4,$$

so that $S(q) = \omega(q)H(q)$, where $\omega(q)^4 = 1$. From (1.5) and (9.1) we find for $|q| < 1$ that $S(q) = 1 + 2q + 4q^2 + O(q^3)$ and $H = 1 + 2q + 4q^2 + O(q^3)$ so that $\omega(q) = 1$ and

$$H = S(q). \tag{9.3}$$

This also follows from [2, Lemma 2.2, p. 1737] (with a typo corrected). Next, by [8, Lemma 4.4, p. 112] (with a typo corrected) and (9.3) we have

$$\sum_{n=1}^{\infty} c_7(n)q^n = qA^3C^3H = qA^3C^3S(q). \tag{9.4}$$

Now, by (1.5), (5.1), (5.2) and (5.3), we have

$$S(q) = \sum_{x,y=-\infty}^{\infty} q^{x^2+xy+2y^2} = \sum_{n=0}^{\infty} r(n)q^n = 1 + 2\sum_{n=1}^{\infty}\sum_{d|n}\left(\frac{-7}{d}\right)q^n. \tag{9.5}$$

Hence, from (9.2), (9.4) and (9.5), we deduce

$$\sum_{n=1}^{\infty} c_7(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^3\prod_{n=1}^{\infty}(1-q^{7n})^3 \times\left(1+2\sum_{n=1}^{\infty}\sum_{d|n}\left(\frac{-7}{d}\right)q^n\right). \tag{9.6}$$

By Jacobi's identity [7, Corollary 6, p. 37]

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{r=0}^{\infty}(-1)^r(2r+1)q^{\frac{r(r+1)}{2}},$$

equation (9.6) becomes

$$\begin{aligned}
\sum_{n=1}^{\infty} c_7(n)q^{n-1} &= \left(\sum_{r=0}^{\infty}(-1)^r(2r+1)q^{\frac{r(r+1)}{2}}\right)\left(\sum_{s=0}^{\infty}(-1)^s(2s+1)q^{\frac{7s(s+1)}{2}}\right)\\
&+ 2\left(\sum_{r=0}^{\infty}(-1)^r(2r+1)q^{\frac{r(r+1)}{2}}\right)\left(\sum_{s=0}^{\infty}(-1)^s(2s+1)q^{\frac{7s(s+1)}{2}}\right)\left(\sum_{t=1}^{\infty}\sum_{d|t}\left(\frac{-7}{d}\right)q^t\right)\\
&= \sum_{n=1}^{\infty}\left(\sum_{\substack{r,s=0\\ \frac{r(r+1)}{2}+\frac{7s(s+1)}{2}=n-1}}^{\infty}(-1)^{r+s}(2r+1)(2s+1)\right)q^{n-1}\\
&\quad + 2\sum_{n=1}^{\infty}\left(\sum_{\substack{r,s=0\\ \frac{r(r+1)}{2}+\frac{7s(s+1)}{2}+t=n-1}}^{\infty}\sum_{t=1}^{\infty}(-1)^{r+s}(2r+1)(2s+1)\sum_{d|t}\left(\frac{-7}{d}\right)\right)q^{n-1}.
\end{aligned}$$

Equating coefficients of q^{n-1} ($n \in \mathbb{N}$) we obtain the asserted formula for $c_7(n)$. □

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Author's address:

Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics
Carleton University
Ottawa, Ontario K1S 5B6
Canada
E-mail: kwilliam@connect.carleton.ca