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A DIRICHLET PROBLEM FOR THE INHOMOGENEOUS POLYHARMONIC EQUATION IN THE UPPER HALF PLANE

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In memory of Professor Gaetano Fichera

Abstract. On the basis of a higher order integral representation formula related to the polyharmonic differential operator and obtained through a certain polyharmonic Green function, a Dirichlet problem is explicitly solved in the upper half plane.

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1. Introduction

There are many Dirichlet problems for the polyharmonic operator. For the biharmonic equation $w_{z\bar{z}z\bar{z}} = f$ there are four, namely on the boundary either

1. $w = \gamma_0$, $w_{z\bar{z}} = \gamma_1$,
2. $w = \gamma_0$, $w_{\bar{z}} = \gamma_1$ or $w = \gamma_0$, $w_z = \gamma_1$,
3. $w = \gamma_0$, $w_z + w_{\bar{z}} = \gamma_1$,
4. $w = \gamma_0$, $\partial_\nu w = \gamma$,

are prescribed where ∂_ν denotes the outward normal derivative. When higher order powers of the Laplace operator are considered many combinations of these four kinds are possible. But the boundary data have to satisfy different kinds of smoothness properties. For type 1, the continuity is sufficient. For the other three types certain differentiability conditions have to be satisfied in order that the problem be unconditionally solvable. This is demonstrated in [8] where the problem of type 2 is treated in the unit disc. As technical details are somehow more involved in the case of a disc, the same problem is here investigated for the upper half plane. The problem was first studied in the Ph.D. thesis [10] of the second author. As the boundary data were assumed there just to be continuous, solvability conditions became involved. Under proper smoothness assumptions, these conditions are avoided so that the problem is well posed.

A proper polyharmonic Green function for the upper half plane $H = \{z \in \mathbb{C} : 0 < \operatorname{Im} z\}$ is given by

$$(n-1)!^2 G_n(z,\zeta) = |\zeta - z|^{2(n-1)} \log \left| \frac{\bar{\zeta} - z}{\zeta - z} \right|^2 + \sum_{\nu=1}^{n-1} \frac{1}{\nu} |\zeta - z|^{2(n-1-\nu)} (\zeta - \bar{\zeta})^\nu (z - \bar{z})^\nu .$$

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$G_2(z,\zeta)$ is the proper biharmonic Green function in H for Dirichlet problems 2, 3, 4 but not for 1 where a different Green function is needed, see [4], [5], [6] for the disc.

The Green function can be used to create a proper Cauchy–Pompeiu representation, see [2], [8], [10]. The usual way of iterating representation formulas [1] to get higher order Cauchy–Pompeiu formulas is in the case of unbounded domains not appropriate as the convolution of the respective integrals may not exist.

Theorem 1. *Any $w \in C^{2n}(H;\mathbb{C}) \cap C^{2n-1}(\overline{H};\mathbb{C})$ for which $|z|^{2\nu+\delta}(\partial_z\partial_{\bar z})^\nu w$ for $0 \leqslant \nu \leqslant n$, $|z|^{2\nu+1+\delta}\partial_z^\nu \partial_{\bar z}^{\nu+1} w$, $|z|^{2\nu+1+\delta}\partial_z^{\nu+1}\partial_{\bar z}^\nu w$ for $0 \leqslant \nu \leqslant n-1$ are bounded in H can be represented as*

$$\begin{aligned} w(z) = &- \sum_{\nu=0}^{[\frac{n-1}{2}]} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1} G_n(z,t)(\partial_\zeta\partial_{\bar\zeta})^\nu w(t)dt \\ &- \sum_{\nu=0}^{[\frac{n}{2}]-1} \frac{1}{2\pi i} \int_{-\infty}^{\infty} (\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1} G_n(z,t)\partial_\zeta^\nu\partial_{\bar\zeta}^{\nu+1} w(t)dt \\ &- \frac{1}{\pi}\int_H G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta. \end{aligned} \tag{1.1}$$

Proof. For formula (1.1) observe

$$\begin{aligned} \frac{1}{\pi}\int_H G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta &= \frac{1}{\pi}\int_H \big\{\partial_\zeta[G_n(z,\zeta)\partial_\zeta^{n-1}\partial_{\bar\zeta}^n w(\zeta)] \\ &\quad - \partial_{\bar\zeta}[\partial_\zeta G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-1} w(\zeta)] + \partial_\zeta\partial_{\bar\zeta} G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-1} w(\zeta)\big\} d\xi d\eta \\ &= \frac{1}{\pi}\int_H \Bigg\{ \sum_{\nu=0}^{n-2} [\partial_\zeta[(\partial_\zeta\partial_{\bar\zeta})^\nu G_n(z,\zeta)\partial_\zeta^{n-\nu-1}\partial_{\bar\zeta}^{n-\nu} w(\zeta)] - \partial_{\bar\zeta}[\partial_\zeta^{\nu+1}\partial_{\bar\zeta}^\nu \\ &\quad \times G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1} w(\zeta)]] + (\partial_\zeta\partial_{\bar\zeta})^{n-1} G_n(z,\zeta)\partial_\zeta\partial_{\bar\zeta} w(\zeta)\Bigg\} d\xi d\eta, \end{aligned}$$

where the last term is

$$\begin{aligned} &\frac{1}{\pi}\int_H (\partial_\zeta\partial_{\bar\zeta})^{n-1} G_n(z,\zeta)\partial_\zeta\partial_{\bar\zeta} w(\zeta)d\xi d\eta \\ &\qquad = \frac{1}{\pi}\int_H \big\{\partial_\zeta[(\partial_\zeta\partial_{\bar\zeta})^{n-1} G_n(z,\zeta)\partial_{\bar\zeta} w(\zeta)] - \partial_\zeta^n\partial_{\bar\zeta}^{n-1} G_n(z,\zeta)\partial_{\bar\zeta} w(\zeta)\big\} d\xi d\eta \end{aligned}$$

and

$$\frac{1}{\pi}\int_H \partial_\zeta^n\partial_{\bar\zeta}^{n-1} G_n(z,\zeta)\partial_{\bar\zeta} w(\zeta)d\xi d\eta = w(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\partial_\zeta^n\partial_{\bar\zeta}^{n-1} G_n(z,t)w(t)dt,$$

as we have

$$\partial_\zeta^n \partial_{\bar\zeta}^{n-1} G_n(z,\zeta) = (-1)^n \left(\frac{z-\bar z}{\bar z-\zeta}\right)^n \frac{1}{z-\zeta} = \left(\frac{z-\bar z}{\zeta-\bar z}\right)^n \frac{1}{z-\zeta}$$
$$= -\left(\frac{z-\bar z}{\zeta-\bar z}\right)^n \frac{1}{\zeta-z}.$$

Thus

$$w(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \partial_\zeta^n \partial_{\bar\zeta}^{n-1} G_n(z,t) w(t)dt - \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\partial_\zeta\partial_{\bar\zeta})^{n-1} G_n(z,t)\partial_{\bar\zeta} w(t)dt$$
$$-\sum_{\nu=0}^{n-2}\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^\nu G_n(z,t)\partial_\zeta^{n-\nu-1}\partial_{\bar\zeta}^{n-\nu} w(t)dt\right.$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_\zeta^{\nu+1}\partial_{\bar\zeta}^{\nu} G_n(z,t)(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1} w(t)dt$$
$$\left.-\frac{1}{\pi}\int_H G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)\right\}d\xi d\eta$$
$$= -\sum_{\nu=0}^{n-1}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^\nu G_n(z,t)\partial_\zeta^{n-\nu-1}\partial_{\bar\zeta}^{n-\nu} w(t)dt$$
$$-\sum_{\nu=0}^{n-1}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_\zeta^{\nu+1}\partial_{\bar\zeta}^{\nu} G_n(z,t)(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1} w(t)dt$$
$$-\frac{1}{\pi}\int_H G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta$$
$$= -\sum_{\nu=0}^{[\frac{n-1}{2}]}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1} G_n(z,t)(\partial_\zeta\partial_{\bar\zeta})^{\nu} w(t)dt$$
$$-\sum_{\nu=0}^{[\frac{n}{2}]-1}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1} G_n(z,t)\partial_\zeta^{\nu}\partial_{\bar\zeta}^{\nu+1} w(t)dt$$
$$-\frac{1}{\pi}\int_H G_n(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta.$$

This is representation (1.1). □

2. Some Properties of the Green Function

For expressing the kernels in the Cauchy–Pompeiu representation explicitly the Green function [9] has to be investigated. This is done by using the following rule.

Lemma 1. *For* $f, g \in C^{2\rho}(D; \mathbb{C})$, $1 \leqslant \rho$, $D \subset \mathbb{C}$ *open and* g *harmonic, i.e.* $\partial_z \partial_{\bar{z}} g = 0$ *in* D, *we have*

$$(\partial_z \partial_{\bar{z}})^\rho (fg) = g(\partial_z \partial_{\bar{z}})^\rho f + \sum_{\tau=0}^{\rho-1} \binom{\rho}{\tau} \big\{ \partial_z^\tau \partial_{\bar{z}}^\rho f \partial_z^{\rho-\tau} g + \partial_z^\rho \partial_{\bar{z}}^\tau f \partial_{\bar{z}}^{\rho-\tau} g \big\}.$$

The proof follows by induction.

The proof of the next lemma follows by direct calculations, see [10].

Lemma 2. *For* $1 \leqslant \rho < n$, $z, \zeta \in \overline{H}$, $z \neq \zeta$,

$$(\partial_\zeta \partial_{\bar{\zeta}})^\rho G_n(z,\zeta) = G_{n-\rho}(z,\zeta) + \sum_{\mu=0}^{\rho-1} \frac{(z-\bar{z})^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!} (\partial_\zeta \partial_{\bar{\zeta}})^{\rho-\mu} \breve{G}_{n-\mu}(z,\zeta)$$

with

$$\partial_\zeta \partial_{\bar{\zeta}} \breve{G}_\tau(z,\zeta) = (\zeta - \bar{\zeta})^{\tau-2} g_1(z,\zeta)$$

for $2 \leqslant \tau$ *and*

$$g_1(z,\zeta) = \frac{1}{\bar{\zeta} - z} - \frac{1}{\zeta - \bar{z}}.$$

Lemma 3. *For* $t \in \mathbb{R}$, $z \in H$ *and* $0 \leqslant 2\nu \leqslant n-2$

$$\begin{aligned}(\partial_\zeta \partial_{\bar{\zeta}})^{n-\nu-1} G_n(z,t) = & \sum_{\mu=0}^{n-2\nu-2} (-1)^{n-\mu-\nu} \frac{(n-\mu-\nu-2)!}{\nu!(n-\mu-1)!} \\ & \times (z-\bar{z})^{n-\mu-1} g_{n-\mu-2\nu-1}(z,t), \end{aligned} \tag{2.1}$$

where for $1 \leqslant \alpha$,

$$g_\alpha(z,\zeta) = \frac{1}{(\bar{\zeta} - z)^\alpha} + \frac{(-1)^\alpha}{(\zeta - \bar{z})^\alpha}.$$

Proof. For $1 \leqslant k$,

$$\partial_\zeta^\rho \partial_{\bar{\zeta}}^\tau (\zeta - \bar{\zeta})^k = \begin{cases} (-1)^\tau \frac{k!}{(k-\rho-\tau)!} (\zeta - \bar{\zeta})^{k-\rho-\tau}, & \rho + \tau \leqslant k, \\ 0, & k < \rho + \tau. \end{cases}$$

Moreover, for $0 < \sigma$

$$\partial_\zeta^\sigma g_1(z,\zeta) = (-1)^{\sigma+1} \frac{\sigma!}{(\zeta - \bar{z})^{\sigma+1}},$$

$$\partial_{\bar{\zeta}}^\sigma g_1(z,\zeta) = (-1)^\sigma \frac{\sigma!}{(\bar{\zeta} - z)^{\sigma+1}}.$$

Thus applying Lemma 1,

$$(\partial_\zeta \partial_{\bar{\zeta}})^{n-\nu-\mu-1} \breve{G}_{n-\mu}(z,\zeta) = g_1(z,\zeta)(\partial_\zeta \partial_{\bar{\zeta}})^{n-\nu-\mu-2} (\zeta - \bar{\zeta})^{n-\mu-2}$$

$$+\sum_{\tau=0}^{\min\{\nu,n-\mu-\nu-3\}}(-1)^{n-\nu-\mu}\binom{n-\mu-\nu-2}{\tau}$$
$$\times\frac{(n-\mu-2)!(n-\mu-\nu-\tau-2)!}{(\nu-\tau)!}(\zeta-\bar\zeta)^{\nu-\tau}g_{n-\mu-\nu-\tau-1}(z,\zeta).$$

For $\zeta=\bar\zeta$ this is

$$(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-\mu-1}\breve G_n(z,\zeta)=\begin{cases}(-1)^{n-\nu-\mu}\frac{(n-\mu-2)!(n-\mu-2-\nu)!}{\nu!}\\ \quad\times g_{n-\mu-2\nu-1}(z,\zeta), & 2\nu+\mu+2\leqslant n,\\ 0, & n<2\nu+\mu+2,\end{cases}$$

so that for $\zeta=\bar\zeta$,

$$(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1}G_n(z,\zeta)=\sum_{\mu=0}^{n-2\nu-2}(-1)^{n-\nu-\mu}\frac{(n-\nu-\mu-2)!}{\nu!(n-\mu-1)!}$$
$$\times(z-\bar z)^{n-\mu-1}g_{n-\mu-2\nu-1}(z,\zeta). \qquad\square$$

Lemma 4. *For $t\in\mathbb{R}$, $z\in H$ and $2\leqslant 2\nu\leqslant n-2$,*

$$\partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1}G_n(z,t)$$
$$=\sum_{\mu=0}^{n-2\nu-1}(-1)^{n-\nu-\mu}\frac{(n-\nu-\mu-2)!}{(\nu-1)!(n-\mu-1)!}(z-\bar z)^{n-\mu-1}g_{n-\mu-2\nu}(z,t)$$
$$+\sum_{\mu=0}^{n-2\nu-2}(-1)^{n-\nu-\mu}\frac{(n-\nu-\mu-2)!}{\nu!(n-\mu-1)!}(z-\bar z)^{n-\mu-1}\partial_\zeta g_{n-\mu-2\nu-1}(z,t) \tag{2.2}$$

and

$$\partial_\zeta^{n}\partial_{\bar\zeta}^{n-1}G_n(z,t)=-\left(\frac{z-\bar z}{t-\bar z}\right)^n\frac{1}{t-z}. \tag{2.3}$$

Proof. For proper μ and ν,

$$\partial_\zeta^{n-\nu-\mu-1}\partial_{\bar\zeta}^{n-\nu-\mu-2}(\zeta-\bar\zeta)^{n-\mu-2}=\begin{cases}(-1)^{n-\nu-\mu}\frac{(n-\mu-2)!}{(2\nu+\mu+1-n)!}\\ \quad\times(\zeta-\bar\zeta)^{2\nu+\mu+1-n}, & n\leqslant 2\nu+\mu+1,\\ 0, & 2\nu+\mu+1<n.\end{cases}$$

Lemma 1 applied shows

$$\partial_\zeta^{n-\nu-\mu}\partial_{\bar\zeta}^{n-\nu-\mu-1}\breve G_{n-\mu}(z,\zeta)$$
$$=g_1(z,\zeta)\partial_\zeta(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-\mu-2}+\partial_\zeta g_1(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-\mu-2}(\zeta-\bar\zeta)^{n-\mu-2}$$
$$+\sum_{\tau=0}^{\min\{\nu,n-\nu-\mu-3\}}(-1)^{n-\mu-1}\binom{n-\mu-\nu-2}{\tau}\frac{(n-\mu-2)!(n-\mu-\nu-\tau-2)!}{(\nu-\tau)!}$$
$$\times\left\{(\nu-\tau)(\zeta-\bar\zeta)^{\nu-\tau-1}g_{n-\mu-\nu-\tau-1}(z,\zeta)+(\zeta-\bar\zeta)^{\nu-\tau}\partial_\zeta g_{n-\mu-\nu-\tau-1}(z,\zeta)\right\}.$$

Arguing as in the preceding proof, this is, for $\zeta=\bar\zeta$ and $1\leqslant\nu$,

$$\partial_\zeta^{n-\nu-\mu}\partial_{\bar\zeta}^{n-\nu-\mu-1}\breve{G}_{n-\mu}(z,\zeta) = \begin{cases} (-1)^{n-\mu-\nu}(n-\mu-2)!g_1(z,\zeta), & n=2\nu+\mu+1,\\ (-1)^{n-\mu-\nu}\frac{(n-\mu-2)!(n-\mu-\nu-2)!}{\nu!}\{\nu g_{n-\mu-2\nu}(z,\zeta) & \\ \quad +\partial_\zeta g_{n-\mu-2\nu-1}(z,\zeta)\}, & 2\nu+\mu+2\leqslant n,\\ 0, & n<2\nu+\mu+1,\end{cases}$$

and for $\nu = 0$

$$\partial_\zeta^{n-\mu}\partial_{\bar\zeta}^{n-\mu-1}\breve{G}_{n-\mu}(z,\zeta) = \begin{cases} (-1)^{n-\mu}(n-\mu-2)!g_1(z,\zeta), & n=\mu+1,\\ (-1)^{n-\mu}(n-\mu-2)!^2\partial_\zeta g_{n-\mu-1}(z,\zeta), & \mu+2\leqslant n,\\ 0, & n<\mu+1.\end{cases}$$

Hence for $1 \leqslant \nu$ and $\zeta = \bar\zeta$

$$\partial_\zeta^{n-\nu}\partial_{\bar\zeta}^{n-\nu-1}G_n(z,\zeta) = (-1)^{\nu-1}\frac{(z-\bar z)^{2\nu}}{(2\nu)!}g_1(z,\zeta) + \sum_{\mu=0}^{n-2\nu-2}(-1)^{n-\mu-\nu}\frac{(n-\mu-\nu-2)!}{\nu!(n-\mu-1)!}\left\{\nu g_{n-\mu-2\nu}(z,\zeta)+\partial_\zeta g_{n-\mu-2\nu-1}(z,\zeta)\right\}.$$

Observing for $\zeta = \bar\zeta$

$$\partial_\zeta G_1(z,\zeta) = -g_1(z,\zeta),$$
$$\partial_\zeta G_{\nu+1}(z,\zeta) = 0$$

for $1 \leqslant \nu$, then for $\zeta = \bar\zeta$

$$\partial_\zeta^{n}\partial_{\bar\zeta}^{n-1}G_n(z,\zeta) = \frac{1}{\zeta-\bar z} - \frac{1}{\zeta - z} + \sum_{\mu=0}^{n-2}\frac{(z-\bar z)^{n-\mu-1}}{(\zeta-\bar z)^{n-\mu}} = -\left(\frac{z-\bar z}{\zeta-\bar z}\right)^n\frac{1}{\zeta-z}. \qquad \square$$

3. Dirichlet Problem

Basic for Dirichlet problems is the existence of a Poisson kernel [4], [7]. For the upper half plane it is [10]

$$\frac{1}{2i}\frac{z-\bar z}{|t-z|^2},$$

where $z \in H$, $t \in \mathbb{R}$.

Lemma 5. *Let $\gamma \in C(\mathbb{R};\mathbb{C})$ be bounded. Then*

$$u(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(z-\bar z)\gamma(t)}{|t-z|^2}dt \tag{3.1}$$

is harmonic in H satisfying for $\tau \in \mathbb{R}$

$$\lim_{z\to\tau} u(z) = \gamma(\tau).$$

Remark 1. The substitution $t = x + y\tan\tau$, where $z = x + iy$, $0 < y$, transforms (3.1) into

$$u(z) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \gamma(x + y\tan\tau)d\tau.$$

The Poisson kernel is the imaginary part of the Cauchy kernel

$$2i\,\mathrm{Im}\,\frac{1}{t-z} = \frac{z-\bar{z}}{|t-z|^2}.$$

If γ is real-valued and c a real constant then, see [10],

$$w(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\gamma(t)}{t-z}\,dt + ic$$

solves the Schwarz problem for analytic functions in the upper half plane: w is analytic and satisfies

$$\mathrm{Re}\,w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(z-\bar{z})\gamma(t)}{|t-z|^2}\,dt$$

so that $\lim_{z\to\tau}\mathrm{Re}\,w(z) = \gamma(\tau)$ for $\tau \in \mathbb{R}$. The constant c is related to $\mathrm{Im}\,w(i)$ by

$$\mathrm{Im}\,w(i) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\gamma(t)}{1+t^2} + c.$$

Dirichlet Problem. For given f satisfying $|z|^{2(n-1)}f(z) \in L_1(H;\mathbb{C})$, $\gamma_\nu \in C^{n-2\nu}(\mathbb{R};\mathbb{C})$ for $0 \leqslant 2\nu \leqslant n-1$, $\hat{\gamma}_\nu \in C^{n-1-2\nu}(\mathbb{R};\mathbb{C})$ for $0 \leqslant 2\nu \leqslant n-2$ with the respective derivatives bounded, find a solution to

$$\begin{gathered}(\partial_z\partial_{\bar{z}})^n w = f \quad \text{in} \quad H,\\ (\partial_z\partial_{\bar{z}})^\nu w = \gamma_\nu \quad \text{for} \quad 0 \leqslant 2\nu \leqslant n-1,\\ \partial_z^\nu\partial_{\bar{z}}^{\nu+1} w = \hat{\gamma}_\nu \quad \text{for} \quad 0 \leqslant 2\nu \leqslant n-2 \quad \text{on} \quad \mathbb{R}.\end{gathered} \tag{3.2}$$

Theorem 2. *The Dirichlet Problem* (3.2) *is uniquely solvable in a weak sense by*

$$w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \Big\{ \Big(\frac{z-\bar{z}}{t-\bar{z}}\Big)^n \frac{\gamma_0(t)}{t-z} + \sum_{\mu=1}^{n-1}(-1)^\mu \frac{(z-\bar{z})^\mu}{\mu} g_\mu(z,t)\hat{\gamma}_0(t)$$

$$+ \sum_{\nu=1}^{[\frac{n-1}{2}]} \Big\{ \sum_{\mu=2\nu}^{n-1} (-1)^{\mu-\nu} \frac{(\mu-\nu-1)!}{\mu!(\nu-1)!} \frac{(z-\bar{z})^\mu}{(t-z)^{\mu-2\nu+1}}$$

$$+ \sum_{\mu=2\nu}^{n-1} (-1)^{\nu-1} \frac{(\mu-\nu)!}{\mu!\nu!} \frac{(z-\bar{z})^\mu}{(t-\bar{z})^{\mu-2\nu+1}} \Big\} \gamma_\nu(t)$$

$$+ \sum_{\nu=1}^{[\frac{n}{2}]-1} \sum_{\mu=2\nu+1}^{n-1} (-1)^{\mu-\nu} \frac{(\mu-\nu-1)!}{\mu!\nu!} (z-\bar{z})^\mu g_{\mu-2\nu}(z,t) \hat{\gamma_\nu}(t) \Big\} dt$$

$$- \frac{1}{\pi} \int_H G_n(z,\zeta) f(\zeta) d\xi d\eta. \tag{3.3}$$

Remark 2. The smoothness conditions on the boundary data cannot be weakened as then the problem will need solvability conditions to be satisfied.

Proof. If the solution exists it can be represented according to Theorem 1. Modifying the kernel functions via Lemmas 3 and 4 proves (3.3). From this representation formula the homogeneous problem is also seen to be only trivially solvable. Thus the problem is uniquely solvable.

To prove existence, (3.3) is shown to be a solution. From the fact that $g_\nu(z,\zeta)$ are harmonic functions it is seen that the above boundary integral is a polyharmonic function, a solution to the related homogeneous equation $(\partial_z\partial_{\bar{z}})^n w = 0$ in H. By the properties of the Green function $G_n(z,\zeta)$ the above area integral satisfies the related homogeneous boundary conditions. Moreover, its lower order derivatives turn out to be continuous functions in H. Finally,

$$\partial_z^n \partial_{\bar{z}}^{n-1} \Big(-\frac{1}{\pi} \int_H G_n(z,\zeta) f(\zeta) d\xi d\eta \Big) = -\frac{1}{\pi} \int_H \Big(\frac{\bar{\zeta}-\zeta}{\bar{\zeta}-z} \Big)^n \frac{f(\zeta)}{\zeta - z} d\xi\, d\eta$$

as a Pompeiu type integral has a weak $\bar{z}$ derivative being $f(z)$. Hence the area integral is a weak solution to the equation $(\partial_z\partial_{\bar{z}})^n w = f$.

The boundary integral in (3.3) remains to be shown to satisfy the boundary conditions.

For checking the boundary behavior of the function given in (3.3) besides the kernel functions (2.1), (2.2), (2.3), also their proper derivatives have to be calculated. On the basis of Lemmas 3 and 4 denote for $1 \leqslant n$, $t \in \mathbb{R}$, $z \in H$,

$$A = \partial_\zeta^n \partial_{\bar{\zeta}}^{n-1} G_n(z,t) = -\Big(\frac{z-\bar{z}}{t-\bar{z}} \Big)^n \frac{1}{t-z}, \tag{3.4}$$

$$B = (\partial_\zeta \partial_{\bar{\zeta}})^{n-1} G_n(z,t) = \sum_{\mu=1}^{n-1} (-1)^{\mu+1} \frac{1}{\mu} (z-\bar{z})^\mu g_\mu(z,t), \tag{3.5}$$

$$C_\nu = \partial_\zeta^{n-\nu} \partial_{\bar{\zeta}}^{n-\nu-1} G_n(z,t) = \sum_{\mu=2\nu}^{n-1} (-1)^{\mu-\nu-1} \frac{(\mu-\nu-1)!}{\mu!(\nu-1)!} \frac{(z-\bar{z})^\mu}{(t-z)^{\mu-2\nu+1}}$$

$$+ \sum_{\mu=2\nu}^{n-1} (-1)^\nu \frac{(\mu-\nu)!}{\mu!\nu!} \frac{(z-\bar{z})^\mu}{(t-\bar{z})^{\mu-2\nu+1}} \tag{3.6}$$

for $2 \leqslant 2\nu \leqslant n-1$,

$$D_\nu = (\partial_\zeta \partial_{\bar\zeta})^{n-\nu-1} G_n(z,t) = \sum_{\mu=2\nu+1}^{n-1} (-1)^{\mu-\nu-1} \frac{(\mu-\nu-1)!}{\mu!\nu!} (z-\bar z)^\mu g_{\mu-2\nu}(z,t) \tag{3.7}$$

for $2 \leqslant 2\nu \leqslant n-2$.

Lemma 6. *For $1 \leqslant n$, $t \in \mathbb{R}$ and $z \in H$*

$$\partial_{\bar z} A = -n \frac{(z-\bar z)^{n-1}}{(t-\bar z)^{n+1}},$$

$$\partial_z \partial_{\bar z} A = -n(n-1) \frac{(z-\bar z)^{n-2}}{(t-\bar z)^{n+1}},$$

$$\partial_z^\nu \partial_{\bar z}^{\nu+1} A = \sum_{\tau=0}^{\nu} (-1)^{\tau+1} \binom{\nu}{\tau} \frac{(n+\nu-\tau)!}{(n-\nu-\tau-1)!} \frac{(z-\bar z)^{n-\nu-\tau-1}}{(t-\bar z)^{n+\nu-\tau+1}}$$

for $2 \leqslant 2\nu \leqslant n-2$,

$$\partial_z^\nu \partial_{\bar z}^\nu A = \sum_{\tau=0}^{\nu-1} (-1)^{\tau+1} \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-1)!}{(n-\nu-\tau-1)!} \frac{(z-\bar z)^{n-\nu-\tau-1}}{(t-\bar z)^{n+\nu-\tau}}$$

for $2 \leqslant 2\nu \leqslant n-1$.

Proof. Rewrite (3.4) as

$$A = \frac{z-\bar z}{|t-z|^2} - \sum_{\nu=1}^{n-1} \frac{(z-\bar z)^\nu}{(t-\bar z)^{\nu+1}}.$$

Differentiating shows the first two formulas.

Applying $(\partial_z \partial_{\bar z})^\nu$ to the first formula and applying Lemma 1 show

$$\partial_z^\nu \partial_{\bar z}^{\nu+1} A = -\sum_{\tau=0}^{\nu} \binom{\nu}{\tau} \frac{n!(-1)^\tau}{(n-1-\nu-\tau)!} \frac{(z-\bar z)^{n-1-\nu-\tau}}{(t-\bar z)^{n+1+\nu-\tau}} \frac{(n+\nu-\tau)!}{n!}.$$

Similarly, from the second formula

$$\partial_z^\nu \partial_{\bar z}^\nu A = -\sum_{\tau=0}^{\nu-1} \binom{\nu-1}{\tau} \frac{n!(-1)^\tau}{(n-1-\nu-\tau)!} \frac{(z-\bar z)^{n-1-\nu-\tau}}{(t-\bar z)^{n+\nu-\tau}} \frac{(n+\nu-\tau-1)!}{n!}$$

follows. □

Lemma 7. *For $1 \leqslant n$, $t \in \mathbb{R}$ and $z \in H$*

$$\partial_{\bar z} B = -g_1(z,t) - \frac{(z-\bar z)^{n-1}}{(t-\bar z)^n} - \sum_{\mu=1}^{n-2} (-1)^\mu \frac{(z-\bar z)^\mu}{(t-z)^{\mu+1}} = -\frac{(z-\bar z)^{n-1}}{(t-\bar z)^n} + \left(\frac{\bar z - z}{t-z}\right)^{n-1} \frac{1}{t-\bar z},$$

$$\partial_z \partial_{\bar z} B = -(n-1)(z-\bar z)^{n-2} g_n(z,t),$$

$$\partial_z^\nu \partial_{\bar z}^{\nu+1} B = \sum_{\tau=0}^{\nu-1} (-1)^{\nu-1} \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-2)!}{(n-\nu-\tau-2)!} \frac{(z-\bar z)^{n-\nu-\tau-2}}{(t-z)^{n+\nu-\tau-1}} + \sum_{\tau=0}^{\nu} (-1)^{n+1-\tau} \binom{\nu}{\tau} \frac{(n+\nu-\tau-1)!}{(n-\nu-\tau-1)!} \frac{(z-\bar z)^{n-\nu-\tau-1}}{(t-\bar z)^{n+\nu-\tau}}$$

for $2 \leqslant 2\nu \leqslant n-2$,

$$\partial_z^\nu \partial_{\bar z}^{\nu} B = \sum_{\tau=0}^{\nu-1} (-1)^{\nu} \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-2)!}{(n-\nu-\tau-1)!} (z-\bar z)^{n-\nu-\tau-1} g_{n+\nu-\tau-1}(z,t)$$

for $2 \leqslant 2\nu \leqslant n-1$.

Proof. Differentiating (1.5) gives

$$\begin{aligned}\partial_{\bar z} B &= \sum_{\mu=1}^{n-1} \left\{ (-1)^\mu (z-\bar z)^{\mu-1} g_\mu(z,t) - \frac{(z-\bar z)^\mu}{(t-\bar z)^{\mu+1}} \right\} \\ &= -g_1(z,t) + \sum_{\mu=2}^{n-1} \left\{ (-1)^\mu \frac{(z-\bar z)^{\mu-1}}{(t-z)^\mu} + \frac{(z-\bar z)^{\mu-1}}{(t-\bar z)^\mu} \right\} - \sum_{\mu=2}^{n} \frac{(z-\bar z)^{\mu-1}}{(t-\bar z)^\mu}\end{aligned}$$

which is the first expression. Differentiating again shows

$$\begin{aligned}\partial_z \partial_{\bar z} B &= -(n-1) \frac{(z-\bar z)^{n-2}}{(t-\bar z)^n} - (n-1) \left(\frac{\bar z - z}{t-z} \right)^{n-2} \frac{1}{(t-z)^2} \\ &= -(n-1)(z-\bar z)^{n-2} g_n(z,t).\end{aligned}$$

Proceeding as in the preceding proof leads to

$$\begin{aligned}\partial_z^\nu \partial_{\bar z}^\nu B = &- \sum_{\tau=0}^{\nu-1} \binom{\nu-1}{\tau} \frac{(n-1)!(-1)^\nu}{(n-1-\tau-\nu)!} \frac{(z-\bar z)^{n-1-\tau-\nu}}{(t-z)^{n+\nu-1-\tau}} \frac{(n-1+\nu-1-\tau)!}{(n-1)!} \\ &- \sum_{\tau=0}^{\nu-1} \binom{\nu-1}{\tau} \frac{(n-1)!(-1)^{\tau+n}}{(n-1-\tau-\nu)!} \frac{(z-\bar z)^{n-1-\tau-\nu}}{(t-\bar z)^{n+\nu-1-\tau}} \frac{(n+\nu-\tau-2)!}{(n-1)!} \\ = &\sum_{\tau=0}^{\nu-1} (-1)^\nu \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-2)!}{(n-\nu-\tau-1)!} (z-\bar z)^{n-\nu-\tau-1} g_{n+\nu-\tau-1}(z,t)\end{aligned}$$

and

$$\begin{aligned}\partial_z^\nu \partial_{\bar z}^{\nu+1} B = &\sum_{\tau=0}^{\nu-1} (-1)^{\nu+1} \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-2)!}{(n-\nu-\tau-2)!} (z-\bar z)^{n-\nu-\tau-2} g_{n+\nu-\tau-1}(z,t) \\ &+ \sum_{\tau=0}^{\nu-1} (-1)^{n-\tau-1} \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-1)!}{(n-\nu-\tau-1)!} \frac{(z-\bar z)^{n-\nu-\tau-1}}{(t-\bar z)^{n+\nu-\tau}} \\ = &\sum_{\tau=1}^{\nu} (-1)^{\nu+1} \binom{\nu-1}{\tau-1} \frac{(n+\nu-\tau-1)!}{(n-\nu-\tau-1)!} (z-\bar z)^{n-\nu-\tau-1} \frac{(-1)^{n+\nu-\tau}}{(t-\bar z)^{n+\nu-\tau}}\end{aligned}$$

$$+\sum_{\tau=0}^{\nu-1}(-1)^{\nu+1}\binom{\nu-1}{\tau}\frac{(n+\nu-\tau-1)!}{(n-\nu-\tau-1)!}(z-\bar{z})^{n-\nu-\tau-1}\frac{(-1)^{n+\nu-\tau}}{(t-\bar{z})^{n+\nu-\tau}}$$
$$+\sum_{\tau=0}^{\nu-1}(-1)^{\nu+1}\binom{\nu-1}{\tau}\frac{(n+\nu-\tau-2)!}{(n-\nu-\tau-2)!}\frac{(z-\bar{z})^{n-\nu-\tau-2}}{(t-z)^{n+\nu-\tau-1}}.$$

These are the formulas in the lemma. □

Lemma 8. *For $2\leqslant 2\nu\leqslant n-1$, $t\in\mathbb{R}$, $z\in H$*

$$(\partial_z\partial_{\bar{z}})^\rho C_\nu=\sum_{\mu=2\nu}^{n-1}\sum_{\tau=0}^{\rho}(-1)^{\rho+\mu-\nu-1}\binom{\rho}{\tau}\frac{(\mu-\nu-1)!(\mu-2\nu+\rho-\tau)!}{(\nu-1)!(\mu-\rho-\tau)!(\mu-2\nu)!}$$
$$\times\frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-z)^{\mu-2\nu+\rho-\tau+1}}$$
$$+\sum_{\mu=2\nu}^{n-1}\sum_{\tau=0}^{\rho}(-1)^{\nu+\tau}\binom{\rho}{\tau}\frac{(\mu-\nu)!(\mu-2\nu+\rho-\tau)!}{\nu!(\mu-\rho-\tau)!(\mu-2\nu)!}\frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-\bar{z})^{\mu-2\nu+\rho-\tau+1}}$$

for $0\leqslant\rho\leqslant\nu$,

$$(\partial_z\partial_{\bar{z}})^\nu C_\nu=\sum_{\mu=2\nu}^{n-1}\sum_{\tau=0}^{\nu}\Big\{(-1)^{\mu-1}\binom{\nu}{\tau}\binom{\mu-\nu-1}{\nu-1}\frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-z)^{\mu-\nu-\tau+1}}$$
$$+(-1)^{\nu+\tau}\binom{\nu}{\tau}\binom{\mu-\nu}{\nu}\frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-\bar{z})^{\mu-\nu-\tau+1}}\Big\}$$
$$=-\sum_{\tau=0}^{\nu}\binom{\nu}{\tau}(z-\bar{z})^{\nu-\tau}g_{\nu-\tau+1}(z,t)$$
$$+\sum_{\mu=2\nu+1}^{n-1}\sum_{\tau=0}^{\nu}(-1)^{\mu-1}\binom{\nu}{\tau}\Big\{\binom{\mu-\nu-1}{\nu-1}\frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-z)^{\mu-\nu-\tau+1}}$$
$$+(-1)^{\mu-\nu-\tau+1}\binom{\mu-\nu}{\nu}\frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-\bar{z})^{\mu-\nu-\tau+1}}\Big\}$$
$$=\sum_{\mu=2\nu}^{n-1}\Big\{(-1)^{\mu-1}\binom{\mu-\nu-1}{\nu-1}\frac{(z-\bar{z})^{\mu-2\nu}(t-\bar{z})^\nu}{(t-z)^{\mu-\nu+1}}$$
$$+\binom{\mu-\nu}{\nu}\frac{(z-\bar{z})^{\mu-2\nu}(t-z)^\nu}{(t-\bar{z})^{\mu-\nu+1}}\Big\},$$

$$\partial_z^\rho\partial_{\bar{z}}^{\rho+1}C_\nu=\sum_{\mu=2\nu}^{n-1}\sum_{\tau=0}^{\rho}(-1)^{\rho+\mu-\nu}\binom{\rho}{\tau}\frac{(\mu-\nu-1)!(\mu-2\nu+\rho-\tau)!}{(\nu-1)!(\mu-\rho-\tau-1)!(\mu-2\nu)!}$$
$$\times\frac{(z-\bar{z})^{\mu-\rho-\tau-1}}{(t-z)^{\mu-2\nu+\rho-\tau+1}}$$

$$+\sum_{\mu=2\nu}^{n-1}\sum_{\tau=0}^{\rho+1}(-1)^{\tau+\nu}\binom{\rho+1}{\tau}\frac{(\mu-\nu)!(\mu-2\nu+\rho-\tau+1)!}{\nu!(\mu-\rho-\tau)!(\mu-2\nu)!}\times\frac{(z-\bar z)^{\mu-\rho-\tau}}{(t-\bar z)^{\mu-2\nu+\rho-\tau+2}}$$

for $0\leqslant\rho<\nu$,

$$\partial_z^{\nu}\partial_{\bar z}^{\nu+1}C_\nu=(-1)^{n-1}\frac{(n-\nu-1)!}{(\nu-1)!(n-2\nu-1)!}\frac{(z-\bar z)^{n-2\nu-1}(t-\bar z)^{\nu-1}}{(t-z)^{n-\nu}}+\frac{(\mu-\nu)!}{\nu!(\mu-2\nu-1)!}\frac{(z-\bar z)^{n-2\nu-1}(t-z)^{\nu}}{(t-\bar z)^{n-\nu+1}},$$

$$\begin{aligned}(\partial_z\partial_{\bar z})^{\nu+1}C_\nu={}&(-1)^{n-1}\frac{(n-\nu-1)!}{(\nu-1)!(n-2\nu-2)!}\frac{(z-\bar z)^{n-2\nu-2}(t-\bar z)^{\nu-1}}{(t-z)^{n-\nu}}\\&+(-1)^{n-1}\frac{(n-\nu)!}{(\nu-1)!(n-2\nu-1)!}\frac{(z-\bar z)^{n-2\nu-1}(t-\bar z)^{\nu-1}}{(t-z)^{n-\nu+1}}\\&+\frac{(n-\nu)!}{\nu!(n-2\nu-2)!}\frac{(z-\bar z)^{n-2\nu-2}(t-z)^{\nu}}{(t-\bar z)^{n-\nu+1}}\\&-\frac{(n-\nu)!}{(\nu-1)!(n-2\nu-1)!}\frac{(z-\bar z)^{n-2\nu-1}(t-z)^{\nu-1}}{(t-\bar z)^{n-\nu+1}},\end{aligned}$$

$$\begin{aligned}(\partial_z\partial_{\bar z})^{\nu+\rho}C_\nu={}&\sum_{\tau=0}^{\min\{\rho,n-2\nu-1\}}\sum_{\sigma=\max\{0,\rho-\nu\}}^{\min\{\rho-1,n-2\nu-1-\tau\}}(-1)^{n-\rho}\binom{\rho}{\tau}\binom{\rho-1}{\sigma}\\&\times\frac{(n-\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(\nu-\rho+\sigma)!}\frac{(z-\bar z)^{n-2\nu-\tau-\sigma-1}(t-\bar z)^{\nu+\sigma-\rho}}{(t-z)^{n-\nu+\rho-\tau}}\\&+\sum_{\tau=0}^{\min\{\rho-1,n-2\nu-1\}}\sum_{\sigma=\max\{0,\rho-\nu\}}^{\min\{\rho,n-2\nu-1-\tau\}}(-1)^{\rho-\sigma+\tau}\binom{\rho-1}{\tau}\binom{\rho}{\sigma}\\&\times\frac{(n-\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(\nu-\rho+\sigma)!}\frac{(z-\bar z)^{n-2\nu-\tau-\sigma-1}(t-z)^{\nu+\sigma-\rho}}{(t-\bar z)^{n-\nu+\rho-\tau}}\end{aligned}$$

for $2(\nu+\rho)\leqslant n-1$,

$$\begin{aligned}\partial_z^{\nu+\rho}\partial_{\bar z}^{\nu+\rho+1}C_\nu={}&\sum_{\tau=0}^{\min\{\rho,n-2\nu-1\}}\sum_{\sigma=\max\{0,\rho-\nu+1\}}^{\min\{\rho,n-2\nu-1-\tau\}}(-1)^{n-1-\rho}\binom{\rho}{\tau}\binom{\rho}{\sigma}\\&\times\frac{(n-\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(\nu-\rho+\sigma-1)!}\frac{(z-\bar z)^{n-2\nu-\tau-\sigma-1}(t-\bar z)^{\nu-\rho+\sigma-1}}{(t-z)^{n-\nu+\rho-\tau}}\\&+\sum_{\tau=0}^{\min\{\rho,n-2\nu-1\}}\sum_{\sigma=\max\{0,\rho-\nu\}}^{\min\{\rho,n-2\nu-1-\tau\}}(-1)^{\rho-\sigma+\tau}\binom{\rho}{\tau}\binom{\rho}{\sigma}\end{aligned}$$

$$\times \frac{(n-\nu+\rho-\tau)!}{(n-2\nu-\tau-\sigma-1)!(\nu-\rho+\sigma)!} \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-z)^{\nu-\rho+\sigma}}{(t-\bar{z})^{n-\nu+\rho-\tau+1}}$$

for $2(\nu+\rho) \leqslant n-1$.

Lemma 9. *For* $0 \leqslant 2\nu \leqslant n-2$, $t \in \mathbb{R}$ *and* $z \in H$

$$(\partial_z \partial_{\bar{z}})^\rho D_\nu = \sum_{\mu=2\nu+1}^{n-1} \sum_{\tau=0}^{\rho} (-1)^{\mu-\nu-1+\rho} \binom{\rho}{\tau} \times \frac{(\mu-\nu-1)!(\mu-2\nu+\rho-\tau-1)!}{\nu!(\mu-\tau-\rho)!(\mu-2\nu-1)!} (z-\bar{z})^{\mu-\tau-\rho} g_{\mu-2\nu+\rho-\tau}(z,t)$$

for $0 \leqslant \rho \leqslant \nu$,

$$\begin{aligned}\partial_z^\rho \partial_{\bar{z}}^{\rho+1} D_\nu = & \sum_{\mu=2\nu+1}^{n-1} \sum_{\tau=0}^{\rho} (-1)^{\mu-\nu+\rho} \binom{\rho}{\tau} \\ & \times \frac{(\mu-\nu-1)!(\mu-2\nu+\rho-\tau-1)!}{\nu!(\mu-\tau-\rho-1)!(\mu-2\nu-1)!} \frac{(z-\bar{z})^{\mu-\rho-\tau-1}}{(t-z)^{\mu-2\nu+\rho-\tau}} \\ & + \sum_{\mu=2\nu+1}^{n-1} \sum_{\tau=0}^{\rho+1} (-1)^{\tau+\nu-1} \binom{\rho+1}{\tau} \frac{(\mu-\nu-1)!(\mu-2\nu+\rho-\tau)!}{\nu!(\mu-\tau-\rho)!(\mu-2\nu-1)!} \\ & \times \frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-\bar{z})^{\mu-2\nu+\rho-\tau+1}}\end{aligned}$$

for $0 \leqslant \rho \leqslant \nu$,

$$\begin{aligned}\partial_z^\nu \partial_{\bar{z}}^{\nu+1} D_\nu = & \sum_{\mu=2\nu+1}^{n-1} \Bigg\{ \sum_{\tau=0}^{\nu} (-1)^\mu \binom{\nu}{\tau} \binom{\mu-\nu-1}{\nu} \frac{(z-\bar{z})^{\mu-\tau-\nu-1}}{(t-z)^{\mu-\nu-\tau}} \\ & + \sum_{\tau=0}^{\nu+1} (-1)^{\tau+\nu-1} \binom{\nu+1}{\tau} \binom{\mu-\nu-1}{\nu} \frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-\bar{z})^{\mu-\nu-\tau+1}} \Bigg\} \\ = & -g_1(z,t) - \sum_{\tau=0}^{\nu-1} \binom{\nu}{\tau} \frac{(z-\bar{z})^{\nu-\tau}}{(t-z)^{\nu-\tau+1}} \\ & + \sum_{\tau=0}^{\nu} (-1)^{\nu-\tau-1} \binom{\nu+1}{\tau} \frac{(z-\bar{z})^{\nu-\tau+1}}{(t-\bar{z})^{\nu-\tau+2}} \\ & + \sum_{\mu=2\nu+2}^{n-1} \Bigg\{ \sum_{\tau=0}^{\nu} (-1)^\mu \binom{\nu}{\tau} \binom{\mu-\nu-1}{\nu} \frac{(z-\bar{z})^{\mu-\tau-\nu-1}}{(t-z)^{\mu-\nu-\tau}} \\ & + \sum_{\tau=0}^{\nu+1} (-1)^{\tau+\nu-1} \binom{\nu+1}{\tau} \binom{\mu-\nu-1}{\nu} \frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-\bar{z})^{\mu-\nu-\tau+1}} \Bigg\} \\ = & \sum_{\mu=2\nu+1}^{n-1} \binom{\mu-\nu-1}{\nu} \Bigg\{ (-1)^\mu \frac{(z-\bar{z})^{\mu-2\nu-1}(t-\bar{z})^\nu}{(t-z)^{\mu-\nu}}\end{aligned}$$

$$+ \frac{(z-\bar{z})^{\mu-2\nu-1}(t-z)^{\nu+1}}{(t-\bar{z})^{\mu-\nu+1}}\Bigg\},$$

$$\begin{aligned}(\partial_z\partial_{\bar{z}})^{\nu+1}D_\nu &= \sum_{\mu=2\nu+1}^{n-1}\sum_{\tau=0}^{\nu+1}(-1)^\mu\binom{\nu+1}{\tau}\binom{\mu-\nu-1}{\nu}\\ &\quad\times(\mu-\tau-\nu)(z-\bar{z})^{\mu-\tau-\nu-1}g_{\mu-\nu-\tau+1}(z,t)\\ &= (-1)^{n-1}\frac{(n-\nu-1)!}{\nu!(n-2\nu-2)!}\,\frac{(z-\bar{z})^{n-2\nu-2}(t-\bar{z})^{\nu}}{(t-z)^{n-\nu}}\\ &\quad-\frac{(n-\nu-1)!}{\nu!(n-2\nu-2)!}\,\frac{(z-\bar{z})^{n-2\nu-1}(t-z)^{\nu}}{(t-\bar{z})^{n-\nu+1}},\end{aligned}$$

$$\begin{aligned}\partial_z^{\nu+1}\partial_{\bar{z}}^{\nu+2}D_\nu &= (-1)^n\frac{(n-\nu-1)!}{\nu!(n-2\nu-3)!}\,\frac{(z-\bar{z})^{n-2\nu-3}(t-\bar{z})^{\nu}}{(t-z)^{n-\nu}}\\ &\quad+(-1)^n\frac{(n-\nu-1)!}{(\nu-1)!(n-2\nu-2)!}\,\frac{(z-\bar{z})^{n-2\nu-2}(t-\bar{z})^{\nu-1}}{(t-z)^{n-\nu}}\\ &\quad+\frac{(n-1-\nu)!(n-2\nu-1)}{\nu!(n-2\nu-2)!}\,\frac{(z-\bar{z})^{n-2\nu-2}(t-z)^{\nu}}{(t-\bar{z})^{n-\nu+1}}\\ &\quad-\frac{(n-1-\nu)!(n-\nu+1)}{\nu!(n-2\nu-2)!}\,\frac{(z-\bar{z})^{n-2\nu-1}(t-z)^{\nu}}{(t-\bar{z})^{n-\nu+2}},\end{aligned}$$

$$\begin{aligned}(\partial_z\partial_{\bar{z}})^{\nu+\rho}D_\nu = &\sum_{\tau=0}^{\min\{\rho-1,n-2\nu-2\}}\ \sum_{\sigma=\max\{0,\rho-\nu-1\}}^{\min\{\rho-1,n-2\nu-2-\tau\}}(-1)^{n+\rho}\binom{\rho-1}{\tau}\binom{\rho-1}{\sigma}\\ &\times\frac{(n-\nu+\rho-\tau-2)!}{(n-2\nu-\tau-\sigma-2)!(\nu-\rho+\sigma+1)!}\,\frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-2}(t-\bar{z})^{\nu-\rho+\sigma+1}}{(t-z)^{n-\nu+\rho-\tau-1}}\\ &+\sum_{\tau=0}^{\min\{\rho-1,n-2\nu-2\}}\ \sum_{\sigma=\max\{0,\rho-\nu-1\}}^{\min\{\rho-1,n-2\nu-2-\tau\}}(-1)^{\rho+\tau-\sigma}\binom{\rho-1}{\tau}\binom{\rho-1}{\sigma}\frac{1}{n-\nu}\\ &\times\frac{(n-\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-2)!(\nu-\rho+\sigma+1)!}\,\frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-2}(t-z)^{\nu-\rho+\sigma+1}}{(t-\bar{z})^{n-\nu+\rho-\tau-2}}\end{aligned}$$

for $2(\nu+\rho)\leqslant n-2$,

$$\begin{aligned}\partial_z^{\nu+\rho}\partial_{\bar{z}}^{\nu+\rho+1}D_\nu = &\sum_{\tau=0}^{\min\{\rho-1,n-2\nu-2\}}\ \sum_{\sigma=\max\{0,\rho-\nu-1\}}^{\min\{\rho,n-2\nu-2-\tau\}}(-1)^{n-\rho+1}\binom{\rho-1}{\tau}\binom{\rho}{\sigma}\\ &\times\frac{(n-\nu+\rho-\tau-2)!}{(n-2\nu-\tau-\sigma-2)!(\nu-\rho+\sigma)!}\,\frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-2}(t-\bar{z})^{\nu-\rho+\sigma}}{(t-z)^{n-\nu+\rho-\tau-1}}\\ &+\sum_{\tau=0}^{\min\{\rho,n-2\nu-1\}}\ \sum_{\sigma=\max\{0,\rho-\nu-1\}}^{\min\{\rho-1,n-2\nu-1-\tau\}}(-1)^{\nu+\rho-\sigma}\binom{\rho}{\tau}\binom{\rho-1}{\sigma}\end{aligned}$$

$$\times \frac{n-2\nu-1}{n-\nu} \frac{(n-\nu+\rho-\tau)!}{(n-2\nu-\tau-\sigma-1)!(\nu-\rho+\sigma+1)!} \times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-z)^{\nu-\rho+\sigma+1}}{(t-\bar{z})^{n-\nu+\rho-\tau+1}}$$

for $2(\nu+\rho) \leqslant n-2$.

The proof of the last two lemmas follow step by step as indicated in their formulations by direct differentiation using Lemma 1 in the same way as for Lemmas 6 and 7.

The boundary behavior of the function in (3.3) is checked via the property of the Poisson integral (3.1).

Lemma 10. *Let* $\gamma \in C(\mathbb{R};\mathbb{C})$ *be bounded and* $k, n \in \mathbb{N}$ *with* $k \leqslant n$. *Then*

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \gamma(t) \frac{(z-\bar{z})^n}{(t-z)^k} dt = 0.$$

Proof. Rewriting

$$\frac{(z-\bar{z})^n}{(t-z)^k} = \frac{(z-\bar{z})^{n-1}}{(t-z)^{k-2}} \frac{t-\bar{z}}{t-z} \frac{z-\bar{z}}{|t-z|^2} = (z-\bar{z})^{n+1-k} \left(\frac{t-\bar{z}}{t-z} - 1\right)^{k-2} \frac{t-\bar{z}}{t-z} \frac{z-\bar{z}}{|t-z|^2},$$

where the first factor on the right-hand side tends to zero with z tending to $\bar{z}$ while the two middle terms are bounded and the last is the Poisson kernel. □

Lemma 11. *Let* $\gamma \in C^m(\mathbb{R};\mathbb{C})$ *be bounded together with all its derivatives,* $m \in \mathbb{N}$. *Then for any* $n \in \mathbb{N}$, $z \in H$

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \gamma(t) \frac{(z-\bar{z})^n}{(t-z)^{m+n}} dt = 0.$$

Proof. Integrating by parts m times shows

$$\int_{-\infty}^{\infty} \gamma(t) \frac{(z-\bar{z})^n}{(t-z)^{m+n}} dt = (-1)^m \frac{(n-1)!}{(m+n-1)!} \int_{-\infty}^{\infty} \gamma^{(m)}(t) \left(\frac{z-\bar{z}}{t-z}\right)^n dt.$$

Thus via the preceding lemma the result follows. □

According to Lemma 11, the following limits vanish:

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sum_{\nu=1}^{n-1} \frac{(z-\bar{z})^\nu}{(t-\bar{z})^{\nu+1}} \gamma_0(t) = 0, \tag{3.8}$$

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(z-\bar{z})^{n-1}}{(t-\bar{z})^{n+1}} \gamma_0(t) dt = 0, \tag{3.9}$$

for $2 \leqslant n$,

$$\lim_{z\to\bar z}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sum_{\tau=0}^{\rho-1}(-1)^{\tau}\binom{\rho-1}{\tau}\frac{(n+\rho-\tau-1)!}{(n-\rho-\tau-1)!}\frac{(z-\bar z)^{n-\rho-\tau-1}}{(t-\bar z)^{n+\rho-\tau}}\gamma_0(t)dt=0 \quad (3.10)$$

for $2 \leqslant 2\rho \leqslant n-1$,

$$\lim_{z\to\bar z}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sum_{\tau=0}^{\rho}(-1)^{\tau}\binom{\rho}{\tau}\frac{(n+\rho-\tau)!}{(n-\rho-\tau-1)!}\frac{(z-\bar z)^{n-\rho-\tau-1}}{(t-\bar z)^{n+\rho-\tau+1}}\gamma_0(t)dt=0 \quad (3.11)$$

for $2 \leqslant 2\rho \leqslant n-2$,

$$\lim_{z\to\bar z}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\{\sum_{\mu=1}^{n-2}(-1)^{\mu}\frac{(z-\bar z)^{\mu}}{(t-z)^{\mu+1}}+\frac{(z-\bar z)^{n-1}}{(t-\bar z)^{n}}\right\}\hat{\gamma_0}(t)dt=0, \quad (3.12)$$

$$\lim_{z\to\bar z}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sum_{\tau=0}^{\rho-1}(-1)^{\rho}\binom{\rho-1}{\tau}\frac{(n+\rho-\tau-2)!}{(n-\rho-\tau-1)!} \times (z-\bar z)^{n-\rho-\tau-1}g_{n+\rho-\tau-1}(z,t)\hat{\gamma_0}(t)dt=0 \quad (3.13)$$

for $2 \leqslant 2\rho \leqslant n-1$,

$$\lim_{z\to\bar z}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\{\sum_{\tau=0}^{\rho-1}(-1)^{\rho}\binom{\rho-1}{\tau}\frac{(n+\rho-\tau-2)!}{(n-\rho-\tau-2)!} \times \frac{(z-\bar z)^{n-\rho-\tau-2}}{(t-z)^{n+\rho-\tau-1}}+\sum_{\tau=0}^{\rho}(-1)^{n-\tau}\binom{\rho}{\tau}\frac{(n+\rho-\tau-1)!}{(n-\rho-\tau-1)!} \times \frac{(z-\bar z)^{n-\rho-\tau-1}}{(t-\bar z)^{n+\rho-\tau}}\right\}\hat{\gamma_0}(t)dt=0 \quad (3.14)$$

for $2 \leqslant 2\rho \leqslant n-2$,

$$\lim_{z\to z_0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Bigg\{\sum_{\tau=0}^{\min\{\rho-\nu,n-2\nu-1\}}\sum_{\sigma=\max\{0,\rho-2\nu\}}^{\min\{\rho-\nu-1,n-2\nu-1-\tau\}}(-1)^{n-\rho+\nu}\binom{\rho-\nu}{\tau} \times \binom{\rho-\nu-1}{\sigma}\frac{(n-2\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(2\nu-\rho+\sigma)!} \times \frac{(z-\bar z)^{n-2\nu-\tau-\sigma-1}(t-\bar z)^{2\nu+\sigma-\rho}}{(t-z)^{n-2\nu+\rho-\tau}}$$

$$+\sum_{\tau=0}^{\min\{\rho-\nu-1,n-2\nu-1\}}\sum_{\sigma=\max\{0,\rho-2\nu\}}^{\min\{\rho-\nu,n-2\nu-1-\tau\}}(-1)^{\rho-\nu-\sigma+\tau}\binom{\rho-\nu-1}{\tau} \times \binom{\rho-\nu}{\sigma}\frac{(n-2\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(2\nu-\rho+\sigma)!}$$

$$\times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-z)^{2\nu+\sigma-\rho}}{(t-\bar{z})^{n-2\nu+\rho-\tau}}\Big\}\gamma_\nu(t)dt = 0 \tag{3.15}$$

for $1 \leqslant \nu \leqslant \rho - 1$, $2 \leqslant 2\rho \leqslant n-1$,

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big\{\sum_{\tau=0}^{\rho-1}\binom{\rho}{\tau}(z-\bar{z})^{\rho-\tau}g_{\rho-\tau+1}(z,t) + \sum_{\mu=2\rho+1}^{n-1}\sum_{\tau=0}^{\rho}(-1)^\mu$$

$$\times \binom{\rho}{\tau}\Big\{\binom{\mu-\rho-1}{\rho-1}\frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-z)^{\mu-\rho-\tau+1}} + (-1)^{\mu-\rho-\tau}\binom{\mu-\rho}{\rho}$$

$$\times \frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-\bar{z})^{\mu-\rho-\tau+1}}\Big\}\Big\}\gamma_\rho(t)dt = 0 \tag{3.16}$$

for $2 \leqslant 2\rho \leqslant n-1$,

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big\{\sum_{\tau=0}^{\min\{\rho-\nu,n-2\nu-1\}}\sum_{\sigma=\max\{0,\rho-2\nu+1\}}^{\min\{\rho-\nu,n-2\nu-1-\tau\}}(-1)^{n-\rho+\nu}\binom{\rho-\nu}{\tau}$$

$$\times \binom{\rho-\nu}{\sigma}\frac{(n-2\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(2\nu-\rho+\sigma-1)!}$$

$$\times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-\bar{z})^{2\nu-\rho+\sigma-1}}{(t-z)^{n-2\nu+\rho-\tau}}$$

$$+ \sum_{\tau=0}^{\min\{\rho-\nu,n-2\nu-1\}}\sum_{\sigma=\max\{0,\rho-2\nu\}}^{\min\{\rho-\nu,n-2\nu-1-\tau\}}(-1)^{\rho-\nu-\sigma+\tau-1}\binom{\rho-\nu}{\tau}$$

$$\times \binom{\rho-\nu}{\sigma}\frac{(n-2\nu+\rho-\tau)!}{(n-2\nu-\tau-\sigma-1)!(2\nu-\rho+\sigma)!}$$

$$\times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-z)^{2\nu-\rho+\sigma}}{(t-\bar{z})^{n-2\nu+\rho-\tau+1}}\Big\}\gamma_\nu(t)dt = 0 \tag{3.17}$$

for $1 \leqslant \nu \leqslant \rho$, $2 \leqslant 2\rho \leqslant n-2$,

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big\{\sum_{\tau=0}^{\min\{\rho-\nu-1,n-2\nu-2\}}\sum_{\sigma=\max\{0,\rho-2\nu-1\}}^{\min\{\rho-\nu-1,n-2\nu-2-\tau\}}(-1)^{n+\rho-\nu}$$

$$\times \binom{\rho-\nu-1}{\tau}\binom{\rho-\nu-1}{\sigma}\frac{(n-2\sigma+\rho-\tau-2)!}{(n-2\nu-\tau-\sigma-2)!(2\nu-\rho+\sigma+1)!}$$

$$\times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-2}(t-\bar{z})^{2\nu-\rho+\sigma+1}}{(t-z)^{n-2\nu+\rho-\tau-1}}$$

$$+ \sum_{\tau=0}^{\min\{\rho-\nu-1,n-2\nu-2\}}\sum_{\sigma=\max\{0,\rho-2\nu-1\}}^{\min\{\rho-\nu-1,n-2\nu-2-\tau\}}(-1)^{\rho-\nu+\tau-\sigma}\binom{\rho-\nu-1}{\tau}$$

$$\times \binom{\rho-\nu-1}{\sigma}\frac{1}{n-\nu}\,\frac{(n-2\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-2)!(2\nu-\rho+\sigma+1)!}$$

$$\times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-2}(t-z)^{2\nu-\rho+\sigma+1}}{(t-\bar{z})^{n-2\nu+\rho-\tau-2}}\Bigg\}\hat{\gamma_\nu}(t)dt = 0 \tag{3.18}$$

for $1 \leqslant \nu \leqslant \rho - 1$, $2 \leqslant 2\rho \leqslant n-1$,

$$\begin{aligned}\lim_{z\to\bar{z}} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \Bigg\{ &\sum_{\tau=0}^{\rho-1}\binom{\rho}{\tau}\frac{(z-\bar{z})^{\rho-\tau}}{(t-z)^{\rho-\tau+1}} \\ &+ \sum_{\tau=0}^{\rho}(-1)^{\rho-\tau}\binom{\rho+1}{\tau}\frac{(z-\bar{z})^{\rho-\tau+1}}{(t-\bar{z})^{\rho-\tau+2}} \\ &+ \sum_{\mu=2\rho+2}^{n-1}\Bigg\{\sum_{\tau=0}^{\rho}(-1)^{\mu+1}\binom{\rho}{\tau}\binom{\mu-\rho-1}{\rho}\frac{(z-\bar{z})^{\mu-\tau-\rho-1}}{(t-z)^{\mu-\rho-\tau}} \\ &+ \sum_{\tau=0}^{\rho+1}(-1)^{\tau+\rho}\binom{\rho+1}{\tau}\binom{\mu-\rho-1}{\rho}\frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-\bar{z})^{\mu-\rho-\tau+1}}\Bigg\}\Bigg\}\hat{\gamma_\rho}(t) = 0\end{aligned} \tag{3.19}$$

for $2 \leqslant 2\rho \leqslant n-2$,

$$\begin{aligned}\lim_{z\to\bar{z}} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \Bigg\{ &\sum_{\tau=0}^{\min\{\rho-\nu-1,n-2\nu-2\}}\ \sum_{\sigma=\max\{0,\rho-2\nu-1\}}^{\min\{\rho-\nu,n-2\nu-2-\tau\}}(-1)^{n-\rho} \\ &\times \binom{\rho-\nu-1}{\tau}\binom{\rho-\nu}{\sigma}\frac{(n-2\nu+\rho-\tau-2)!}{(n-2\nu-\tau-\sigma-2)!(2\nu-\rho+\sigma)!} \\ &\times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-2}(t-\bar{z})^{2\nu-\rho+\sigma}}{(t-z)^{n-2\nu+\rho-\tau-1}} \\ &+ \sum_{\tau=0}^{\min\{\rho-\nu,n-2\nu-1\}}\ \sum_{\sigma=\max\{0,\rho-2\nu-1\}}^{\min\{\rho-\nu-1,n-2\nu-1-\tau\}}(-1)^{\rho-\sigma-1}\binom{\rho-\nu}{\tau} \\ &\times \binom{\rho-\nu-1}{\sigma}\frac{n-2\nu-1}{n-\nu}\,\frac{(n-2\nu+\rho-\tau)!}{(n-2\nu-\tau-\sigma-1)!(2\nu-\rho+\sigma+1)!} \\ &\times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-z)^{2\nu-\rho+\sigma+1}}{(t-\bar{z})^{n-2\nu+\rho-\tau+1}}\Bigg\}\hat{\gamma_\nu}(t)dt = 0\end{aligned} \tag{3.20}$$

for $1 \leqslant \nu \leqslant \rho - 1$, $2 \leqslant 2\rho \leqslant n-2$.

Thus rewriting (3.3) as

$$\begin{aligned}w(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Bigg\{A\gamma_0(t) + B\hat{\gamma_0}(t) + \sum_{\nu=1}^{[\frac{n-1}{2}]}C_\nu\gamma_\nu(t) + \sum_{\nu=1}^{[\frac{n}{2}]-1}D_\nu\hat{\gamma_\nu}(t)\Bigg\}dt \\ -\frac{1}{\pi}\int_H G_n(z,\zeta)f(\zeta)d\xi d\eta\end{aligned}$$

and using (3.8)–(3.20) and, finally, applying Lemma 10 for $t_0 \in \mathbb{R}$, we obtain

$$\lim_{z\to t_0} w(z) = -\lim_{z\to t_0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} A\gamma_0(t)dt.$$

Rewriting A as

$$A = -\frac{z-\bar{z}}{|t-z|^2} + \sum_{\nu=1}^{n-1} \frac{(z-\bar{z})^\nu}{(t-\bar{z})^{\nu+1}}$$

and using (3.10), we have

$$\lim_{t\to t_0} w(z) = \gamma_0(t)$$

because of (3.8).

Similarly,

$$\lim_{z\to t_0} w_{\bar{z}}(z) = -\lim_{z\to t_0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \big\{A_{\bar{z}}\gamma_0(t) + B_{\bar{z}}\hat{\gamma}_0(t)\big\}dt = \hat{\gamma}_0(t_0)$$

as (3.9) and (3.12) are satisfied.

Moreover, for $2 \leqslant 2\rho \leqslant n-1$,

$$\lim_{z\to t_0} (\partial_z\partial_{\bar{z}})^\rho w(z) = -\lim_{z\to t_0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Big\{(\partial_z\partial_{\bar{z}})^\rho A\gamma_0(t) + (\partial_z\partial_{\bar{z}})^\rho B\hat{\gamma}_0(t)$$
$$+ \sum_{\nu=1}^{\rho} (\partial_z\partial_{\bar{z}})^\rho C_\nu\gamma_\nu(t) + \sum_{\nu=1}^{\rho-1} (\partial_z\partial_{\bar{z}})^\rho D_\nu\hat{\gamma}_\nu(t)\Big\}dt = \gamma_\rho(t_0)$$

by (3.10), (3.13), (3.15), (3.16) and (3.18).

Also for $2 \leqslant 2\rho \leqslant n-2$,

$$\lim_{z\to t_0} \partial_z^\rho\partial_{\bar{z}}^{\rho+1} w(z) = -\lim_{z\to t_0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Big\{\partial_z^\rho\partial_{\bar{z}}^{\rho+1} A\gamma_0(t) + \partial_z^\rho\partial_{\bar{z}}^{\rho+1} B\hat{\gamma}_0(t)$$
$$+ \sum_{\nu=1}^{\rho} \partial_z^\rho\partial_{\bar{z}}^{\rho+1} C_\nu\gamma_\nu(t) + \sum_{\nu=1}^{\rho} \partial_z^\rho\partial_{\bar{z}}^{\rho+1} D_\nu\hat{\gamma}_\nu(t)\Big\}dt = \hat{\gamma}_\rho(t_0)$$

as (3.11), (3.14), (3.17), (3.19) and (3.20) hold. □

Remark 3. Only in the case $n = 1$ the continuity of the boundary data is necessary and sufficient for the solvability of the above Dirichlet problem [3].

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