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ON THE GLOBAL AND LOCAL SOLUTION OF THE MULTIDIMENSIONAL DARBOUX PROBLEM FOR SOME NONLINEAR WAVE EQUATIONS

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Dedicated to the memory of Prof. Gaetano Fichera

Abstract. We consider a multidimensional analogue of the Darboux problem for wave equations with power nonlinearity. Depending on the spatial dimension of an equation, a power nonlinearity exponent and the sign in front of a nonlinear term, it is proved that the Darboux problem is globally solvable in some cases, but has no global solution in other cases though the local solvability of this problem remains in force.

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1. Statement of the Problem

Let us consider a nonlinear wave equation of the form

$$Lu := \frac{\partial^2 u}{\partial t^2} - \Delta u + mu = f(u) + F, \tag{1}$$

where f and F are given real functions, f being nonlinear, $f(0) = 0$, and u is an unknown real function, $m = const \geq 0$, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$.

Denote by $D : t > |x|$, $x_n > 0$, the half of the light cone of the future which is bounded by the part $S^0 = D \cap \{x_n = 0\}$ of the hyperplane $x_n = 0$ and by the half $S : t = |x|$, $x_n \geq 0$, of the characteristic conoid $C : t = |x|$ of equation (1). Assume $D_T = \{(x,t) \in D : t < T\}$, $S_T^0 = \{(x,t) \in S^0 : t \leq T\}$, $S_T = \{(x,t) \in S : t \leq T\}$, $T > 0$. When $T = \infty$, it is obvious that $D_\infty = D$, $S_\infty^0 = S^0$ and $S_\infty = S$.

We will consider the problem on defining, in the domain D_T, a solution $u(x,t)$ of equation (1) by the boundary conditions

$$u\big|_{S_T^0} = 0, \quad u\big|_{S_T} = g, \tag{2}$$

where g is a given real function on S_T.

Problem (1),(2) is a multidimensional variant of the first Darboux problem for the nonlinear equation (1) when one part of the problem data support is a characteristic manifold, while the remaining part is a manifold of time type [1, Ch. III, § 1.1^0].

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Problems pertaining to the existence or nonexistence of a global solution of the Cauchy problem for nonlinear equations of form (1) with the boundary conditions $u|_{t=0} = u_0$, $\frac{\partial u}{\partial t}\big|_{t=0} = u_1$ are considered in [2]–[17]. As for multidimensional variants of the first Darboux problem for linear hyperbolic equations of second order, they are well posed and their global solvability is proved in the corresponding function spaces [18]–[20].

In this paper, we discuss the concrete cases for the nonlinear function $f = f(u)$, where problem (1),(2) is globally solvable in some cases, but has no global solution in other cases though the local solvability of this problem remain in force.

2. Global Solvability of Problem (1),(2) in the Case of Linearity of the Form $f(u) = -\lambda|u|^p u$

For $f(u) = -\lambda|u|^p u$, where $\lambda \neq 0$ and $p > 0$ are given real numbers, equation (1) takes the form

$$Lu := \frac{\partial^2 u}{\partial t^2} - \Delta u + mu = -\lambda|u|^p u + F. \tag{3}$$

Note that equation (3) arises in relativistic quantum mechanics [21]–[24].

In this section, our consideration is limited to the case where the boundary conditions (2) are assumed homogeneous, i.e.

$$u\big|_{S_T^0} = 0, \quad u\big|_{S_T} = 0. \tag{4}$$

We assume that $\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T) = \Big\{u \in W_2^1(D_T) : u\big|_{S_T^0 \cup S_T} = 0\Big\}$, where $W_2^1(D_T)$ is the known Sobolev space with the norm

$$\|u\|^2_{W_2^1(D_T)} = \int\limits_D \left[u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2\right] dx\,dt,$$

and the boundary condition $u|_{S_T^0 \cup S_T} = 0$ should be understood in terms of trace theory [25, Ch. I, §§ 5, 6].

Remark 1. The embedding operator $I : \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T) \to L_q(D_T)$ is a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, where $n > 1$ [25, Ch.I, § 7]. Simultaneously, the Nemitski operator $K : L_q(D_T) \to L_2(D_T)$ acting by the formula $Ku := -\lambda|u|^p u$ is continuous and bounded if $q \geq 2(p+1)$ [26, Ch. V, § 17.5], [27, Ch. III, §§ 12.10; 12.11]. Therefore if $p < \frac{2}{n-1}$, i.e. $2(p+1) < \frac{2(n+1)}{n-1}$, then there exists a number q such that $1 < 2(p+1) \leq q < \frac{2(n+1)}{n-1}$ and hence the operator

$$K_0 = KI : \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T) \to L_2(D_T) \tag{5}$$

is continuous and compact. Moreover, from $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ it follows that $u \in L_{p+1}(D_T)$. As has been mentioned above, it is assumed that here and everywhere $p > 0$.

If $u \in C^2(\overline{D}_T)$ is a classical solution of problem (3),(4), then, after multiplying both parts of equation (3) by an arbitrary function $\varphi \in C^2(\overline{D}_T)$ that satisfies the condition $\varphi|_{t=T} = 0$ and applying integration by parts, we obtain

$$\int\limits_{S_T^0 \cup S_T} \frac{\partial u}{\partial N}\, \varphi\, ds - \int\limits_{D_T} u_t \varphi_t\, dx\, dt + \int\limits_{D_T} \nabla_x u\, \nabla_x \varphi\, dx\, dt + \int\limits_{D_T} m u \varphi\, dx\, dt$$
$$= -\lambda \int\limits_{D_T} |u|^p u \varphi\, dx\, dt + \int\limits_{D_T} F \varphi\, dx\, dt, \tag{6}$$

where $\frac{\partial}{\partial N} = \nu_0 \frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i}$ is a derivative with respect to the conormal, $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ is the unit outward normal to ∂D_T, $\nabla_x = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$. Since the hypersurface S_T is the characteristic manifold on which the operator $\frac{\partial}{\partial N}$ is an internal differential operator, by (4) we have $\frac{\partial}{\partial N}\big|_{S_T} = 0$. Therefore, assuming additionally that the function $\varphi|_{S_T^0} = 0$, from equality (6) we obtain

$$-\int\limits_{D_T} u_t \varphi_t\, dx\, dt + \int\limits_{D_T} \nabla_x u\, \nabla_x \varphi\, dx\, dt + \int\limits_{D_T} m u \varphi\, dx\, dt$$
$$= -\lambda \int\limits_{D_T} |u|^p u \varphi\, dx\, dt + \int\limits_{D_T} F \varphi\, dx\, dt. \tag{7}$$

Since, by virtue of Remark 1, from $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ it follows that $|u|^p u \in L_2(D_T)$, equality (7) can underlie the definition of a weak generalized solution of problem (3),(4) of the class $\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$.

Definition 1. Let $F \in L_2(D_T)$ and $0 < p < \frac{2}{n-1}$. A function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ is called a weak generalized solution of the nonlinear problem (3),(4) in the domain D_T if the integral equality (7) is fulfilled for any function $\varphi \in W_2^1(D_T)$ such that $\varphi|_{t=T} = 0$, $\varphi|_{S_T^0} = 0$.

Assume that

$$\overset{\circ}{C}{}^2(\overline{D}_T, S_T^0 \cup S_T) = \Big\{ u \in C^2(\overline{D}_T) : u\big|_{S_T^0 \cup S_T} = 0 \Big\}.$$

Definition 2. Let $F \in L_2(D_T)$ and $0 < p < \frac{2}{n-1}$. A function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ is called a strong generalized solution of the nonlinear problem (3),(4) in the domain D_T if there exists a sequence of functions $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0 \cup S_T)$ such that $u_k \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ and $[Lu_k + \lambda |u_k|^p u_k] \to F$ in the space $L_2(D_T)$. Moreover, from Remark 1 it follows the sequence $\{\lambda |u_k|^p u_k\}$ converges to the function $\lambda |u|^p u$ in the space $L_2(D_T)$ as $u_k \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$.

Remark 2. One can easily verify that if $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ is a strong generalized solution of problem (3),(4), then it is automatically a weak generalized solution of this problem.

Definition 3. Let $0 < p < \frac{2}{n-1}$, $F \in L_{2,loc}(D)$ and $F \in L_2(D_T)$ for any $T > 0$. We say that problem (3),(4) is globally solvable if for any $T > 0$ this problem has a strong generalized solution in the domain D_T from the space $\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$.

Lemma 1. *Let $\lambda \geq 0$, $0 < p < \frac{2}{n-1}$ and $F \in L_2(D_T)$. Then for any strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ of problem (3),(4) in the domain D_T the following a priori estimate is valid:*

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)} \leq \sqrt{\frac{e}{2}}\, T \|F\|_{L_2(D_T)}\,. \tag{8}$$

Proof. Let $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ be a strong generalized solution of problem (3),(4). By virtue of Definition 2 there exists a sequence of functions $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0 \cup S_T)$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)} = 0, \quad \lim_{k\to\infty} \|Lu_k + \lambda |u_k|^p u + k - F\|_{L_2(D_T)} = 0\,. \tag{9}$$

Let us consider the function $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0 \cup S_T)$ as a solution of the problem

$$Lu_k + \lambda |u_k|^p u_k = F_k, \tag{10}$$

$$u_k\big|_{S_T^0} = 0, \quad u\big|_{S_T} = 0. \tag{11}$$

Here

$$F_k = Lu_k + \lambda |u_k|^p u_k \tag{12}$$

Multiplying both parts of equation (10) by $\frac{\partial u_k}{\partial t}$ and integrating over the domain D_τ, $0 < \tau \leq T$, we obtain

$$\frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial t}\right)^2 dx\,dt - \int\limits_{D_\tau} \Delta u_k \frac{\partial u_k}{\partial t}\,dx\,dt + \frac{m}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\, u_k^2\,dx\,dt$$
$$+ \frac{\lambda}{p+2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\,|u_k|^{p+2}\,dx\,dt = \int\limits_{D_\tau} F_k \frac{\partial u_k}{\partial t}\,dx\,dt. \tag{13}$$

Assume that $\Omega_\tau := D_T \cap \{t = \tau\}$, $0 < \tau < T$. It is obvious that $\partial D_\tau = S_\tau^0 \cup S_\tau \cup \Omega_\tau$. Using (11), the equality $\frac{\partial u}{\partial t}\big|_{S_T^0} = 0$ and also the equalities $\nu|_{\Omega_\tau} = (0, \dots, 0, 1)$, $\nu_{S_T^0} = (0, \dots, 0, -1, 0)$ and performing the integration by parts, we obtain

$$\int\limits_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial t}\right)^2 dx\,dt = \int\limits_{\partial D_\tau} \left(\frac{\partial u_k}{\partial t}\right)^2 \nu_0\,ds = \int\limits_{\Omega_\tau} \left(\frac{\partial u_k}{\partial t}\right)^2 dx + \int\limits_{S_\tau} \left(\frac{\partial u_k}{\partial t}\right)^2 \nu_0\,ds,$$

$$\int_{D_\tau} \frac{\partial}{\partial t}\, u_k^2\, dx\, dt = \int_{\partial D_\tau} u_k^2 \nu_0\, ds = \int_{\Omega_\tau} u_k^2\, dx,$$

$$\int_{D_\tau} \frac{\partial}{\partial t} |u_k|^{p+2}\, dx\, dt = \int_{\partial D_\tau} |u_k|^{p+2} \nu_0\, ds = \int_{\Omega_\tau} |u_k|^{p+2}\, dx,$$

$$\int_{D_\tau} \frac{\partial^2 u_k}{\partial x_i^2}\, \frac{\partial u_k}{\partial t}\, dx\, dt = \int_{\partial D_\tau} \frac{\partial u_k}{\partial x_i}\, \frac{\partial u_k}{\partial t}\, \nu_i\, ds - \frac{1}{2} \int_{D_\tau} \frac{\partial}{\partial t} \left(\frac{\partial u_k}{\partial x_i} \right)^2 dx\, dt$$

$$= \int_{\partial D_\tau} \frac{\partial u_k}{\partial x_i}\, \frac{\partial u_k}{\partial t}\, \nu_i\, ds - \frac{1}{2} \int_{D_\tau} \left(\frac{\partial u_k}{\partial x_i} \right)^2 \nu_0\, ds$$

$$= \int_{S_\tau} \frac{\partial u_k}{\partial x_i}\, \frac{\partial u_k}{\partial t}\, \nu_i\, ds - \frac{1}{2} \int_{S_\tau} \left(\frac{\partial u_k}{\partial x_i} \right)^2 \nu_0\, ds - \frac{1}{2} \int_{\Omega_\tau} \left(\frac{\partial u_k}{\partial x_i} \right)^2 dx.$$

Hence by virtue of (13) it follows that

$$\int_{D_\tau} F_k\, \frac{\partial u_k}{\partial t}\, dx\, dt$$

$$= \int_{S_\tau} \frac{1}{2\nu_0} \Bigg[\sum_{i=1}^{n} \left(\frac{\partial u_k}{\partial x_i}\, \nu_0 - \frac{\partial u_k}{\partial t}\, \nu_i \right)^2 + \left(\frac{\partial u_k}{\partial t} \right)^2 \left(\nu_0^2 - \sum_{j=1}^{n} \nu_j^2 \right) \Bigg] ds$$

$$+ \frac{1}{2} \int_{\Omega_\tau} \left[m u_k^2 + \left(\frac{\partial u_k}{\partial t} \right)^2 + \sum_{i=1}^{n} \left(\frac{\partial u_k}{\partial x_i} \right)^2 \right] dx + \frac{\lambda}{p+2} \int_{\Omega_\tau} |u_k|^{p+2}\, dx. \quad (14)$$

Since S_τ is a characteristic manifold, we have

$$\left(\nu_0^2 - \sum_{j=1}^{n} \nu_j^2 \right) \Bigg|_{S_\tau} = 0. \quad (15)$$

Taking into account that the operator $\left(\nu_0 \frac{\partial}{\partial x_i} - \nu_i \frac{\partial}{\partial t} \right)$, $1 \le i \le n$, is an internal differential operator on S_τ, by virtue of (11) we have

$$\left(\frac{\partial u_k}{\partial x_i}\, \nu_0 - \frac{\partial u_k}{\partial t}\, \nu_i \right) \Bigg|_{S_\tau} = 0, \quad i = 1, \dots, n. \quad (16)$$

With regard for (15), (16), from (14) we have

$$\int_{\Omega_\tau} \left[m u_k^2 + \left(\frac{\partial u_k}{\partial t} \right)^2 + \sum_{i=1}^{n} \left(\frac{\partial u_k}{\partial x_i} \right)^2 \right] dx + \frac{2\lambda}{p+2} \int_{\Omega_\tau} |u_k|^{p+2}\, dx = 2 \int_{D_\tau} F_k\, \frac{\partial u_k}{\partial t}\, dx\, dt,$$

which, by virtue of $\lambda \ge 0$, implies in turn that

$$\int_{\Omega_\tau} \left[m u_k^2 + \left(\frac{\partial u_k}{\partial t} \right)^2 + \sum_{i=1}^{n} \left(\frac{\partial u_k}{\partial x_i} \right)^2 \right] dx \le 2 \int_{D_\tau} F_k\, \frac{\partial u_k}{\partial t}\, dx\, dt. \quad (17)$$

Using the notation

$$w(\delta)=\int\limits_{\Omega_\delta}\left[mu_k^2+\left(\frac{\partial u_k}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u_k}{\partial x_i}\right)^2\right]dx$$

and taking into account that the inequality $2F_k\,\frac{\partial u_k}{\partial t}\le\varepsilon\big(\frac{\partial u_k}{\partial t}\big)^2+\frac{1}{\varepsilon}\,F_k^2$ holds for any $\varepsilon=const>0$, from (17) we obtain

$$w(\delta)\le\varepsilon\int\limits_0^\delta w(\sigma)\,d\sigma+\frac{1}{\varepsilon}\,\|F_k\|^2_{L_2(D_\delta)},\quad 0<\delta\le T. \tag{18}$$

If we take into account that the value $\|F_k\|^2_{L_2(D_\delta)}$ as a function of δ is nondecreasing, then, by virtue Gronwall's lemma [28, Ch. I, § 2], from (18) it follows that

$$\|w(\delta)\|\le\frac{1}{\varepsilon}\,\|F_k\|^2_{L_2(D_\delta)}\exp\delta\varepsilon.$$

Hence, since $\inf\limits_{\varepsilon>0}\frac{\exp\delta\varepsilon}{\varepsilon}=e\delta$ for $\varepsilon=\frac{1}{\delta}$, we obtain

$$w(\delta)\le e\delta\|F_k\|^2_{L_2(D_\delta)},\quad 0<\delta\le T, \tag{19}$$

which implies in turn that

$$\begin{aligned}\|u_k\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T^0\cup S_T)}&=\int\limits_{D_T}\left[mu_k^2+\left(\frac{\partial u_k}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u_k}{\partial x_i}\right)^2\right]dx\,dt\\&=\int\limits_0^T w(\delta)\,d\delta\le\frac{e}{2}\,T^2\|F_k\|^2_{L_2(D_T)}.\end{aligned} \tag{20}$$

Here we have used the fact that in the space $\overset{\circ}{W}{}^1_2(D_T,S_T^0\cup S_T)$ one of the equivalent norms is given by the expression

$$\left\{\int\limits_{D_T}\left[mu_k^2+\left(\frac{\partial u_k}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u_k}{\partial x_i}\right)^2\right]dx\,dt\right\}^{\frac12}$$

independently of the assumption whether $m=0$ or $m>0$. Indeed, by a standard reasoning, the equalities $u|_{S_T}=0$ and $u(x,t)=\int\limits_{\psi(x)}^{t}\frac{\partial u(x,\tau)}{\partial t}\,d\tau$, $(x,t)\in\overline{D}_T$, where $t-\psi(x)=0$ is an equation of the conical manifold S_T, imply the inequality [25, Ch. I, § 6]

$$\int\limits_{D_T}u^2(x,t)\,dx\,dt\le T^2\int\limits_{D_T}\left(\frac{\partial u}{\partial t}\right)^2dx\,dt.$$

Now, passing to the limit in inequality (20) as $k\to\infty$, we obtain (8), which proves the lemma. □

Theorem 1. *Let $\lambda > 0$, $0 < p < \frac{2}{n-1}$, $F \in L_{2,loc}(D)$ and $F \in L_2(D_T)$ for any $T > 0$. Then problem (3), (4) is globally solvable, i.e. for any $T > 0$ this problem has a strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ in the domain D_T.*

Proof. Before proceeding to the discussion whether the nonlinear problem (3), (4) is solvable, we will consider the solvability for the linear case where it is assumed that in equation (3) the parameter $\lambda = 0$, i.e. for the problem

$$\begin{cases} Lu(x,t) = F(x,t), \quad (x,t) \in D_T, \\ u\big|_{S_T^0} = 0, \quad u\big|_{S_T} = 0. \end{cases} \tag{21}$$

In that case, analogously to the above, we introduce, for $F \in L_2(D_T)$, the notion of a strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ of problem (21) for which there exists a sequence of functions $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0 \cup S_T)$ such that $\lim\limits_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)} = 0$, $\lim\limits_{k\to\infty} \|Lu_k - F\|_{L_2(D_T)} = 0$. It should be noted here that by virtue of Lemma 1 for $\lambda = 0$ the a priori estimate (8) holds for a strong generalized solution of problem (21), too.

Since the space $C_0^\infty(D_T)$ of finite, infinitely differentiable in D_T, functions is dense in $L_2(D_T)$, for given $F \in L_2(D_T)$ there exists a sequence of functions $F_k \in C_0^\infty(D_T)$ such that $\lim\limits_{k\to\infty} \|F_k - F\|_{L_2(D_T)} = 0$. If we continue the function F_k in an odd manner with respect to the variable x_n into the domain $D_T^- := \{(x,t) \in R^{n+1} : x_n < 0, |x| < t < T\}$ and after that continue the resulting function by zero beyond the domain $D_T \cup D_T^-$ and denote it by the previous symbol, then for fixed k we obtain $F_k \in C^\infty(R_+^{n+1})$ for which the support is $\operatorname{supp} F_k \subset D_\infty \cup D_\infty^-$, where $R_+^{n+1} = R^{n+1} \cap \{t \geq 0\}$. Denote by u_k a solution of the Cauchy problem

$$Lu_k = F_k, \quad u_k\big|_{t=0} = 0, \quad \frac{\partial u_k}{\partial t}\bigg|_{t=0} = 0, \tag{22}$$

which, as is known, exists, is unique and belongs to the space $C^\infty(R_+^{n+1})$ [29, Ch. V, § 6]. Moreover, since $\operatorname{supp} F_k \subset D_\infty \cup D_\infty^- \subset \{(x,t) \in R^{n+1} : t > |x|\}$ and $u_k|_{t=0} = 0$, $\frac{\partial u_k}{\partial t}\big|_{t=0} = 0$, by taking into account the geometry of the solution dependence domain of the linear wave equation $Lu = F$ we have $\operatorname{supp} u_k \subset \{(x,t) \in R^{n+1} : t > |x|\}$ and in particular $u_k|_{S_T} = 0$. On the other hand, the function $\widetilde{u}_k(x_1, \dots, x_n, t) = -u_k(x_1, \dots, -x_n, t)$ is also a solution of the Cauchy problem (22), since the function F_k is odd with respect to the variable x_n. Hence by virtue of the uniqueness of a solution of the Cauchy problem we have $\widetilde{u}_k = u_k$, i.e. $u_k(x_1, \dots, -x_n, t) = -u_k(x_1, \dots, x_n, t)$ and thereby the function u_k is also even with respect to the variable x_n. This in turn implies that $u_k|_{x_n=0} = 0$, which together with the condition $u_k|_{S_T} = 0$ implies that by preserving the previous notation for the restriction of the function u_k to the domain D_T we obtain $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0 \cup S_T)$. Furthermore, by virtue of (8) and

(22) the inequality

$$\|u_k - u_l\|_{\overset{\circ}{W}{}_2^1(D_T,S_T^0\cup S_T)} \leq \sqrt{\frac{e}{2}}\,T\|F_k - F\|_{L_2(D_T)} \tag{23}$$

is valid because the a priori estimate (8) holds for a strong generalized solution of the linear problem (21), too.

Since the sequence $\{F_k\}$ is fundamental in $L_2(D_T)$, by virtue of (23) the sequence $\{u_k\}$, too, is fundamental in the total space $\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$. Thus there exists a function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ such that $\lim\limits_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T^0\cup S_T)} = 0$ and since $Lu_k = F_k \to F$ in the space $L_2(D_T)$, this function is, by definition, a strong generalized solution of problem (21). The uniqueness of this solution from the space $\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ follows from the a priori estimate (8). Therefore for this solution u of problem (21) we can write $u = L^{-1}F$, where $L^{-1} : L_2(D_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ is a linear continuous operator whose norm admits, by virtue of (8), an estimate

$$\|L^{-1}\|_{L_2(D_T)\to \overset{\circ}{W}{}_2^1(D_T,S_T^0\cup S_T)} \leq \sqrt{\frac{e}{2}}\,T. \tag{24}$$

Note that, by virtue of (24) and Remark 1, Definition 2 and Remark 2, for $F \in L_2(D_T)$, $0 < p < \frac{2}{n-1}$, a function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ is a strong generalized solution of problem (3),(4) if and only if u is a solution of the functional equation

$$u = L^{-1}\left(-\lambda|u|^p u + F\right) \tag{25}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$.

Rewrite equation (25) as follows:

$$u = Au := L^{-1}(K_0 u + F), \tag{26}$$

where the operator $K_0 : \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T) \to L_2(D_T)$ from (5) is continuous and compact according to Remark 1. Therefore the operator $A : \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ is also continuous and compact by virtue of (24). At the same time, according to Lemma 1, for any parameter $\mu \in [0, 1]$ and for any solution of an equation with parameter $u = \mu Au$ we have the a priori estimate $\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq c\|F\|_{L_2(D_T)}$, where the positive constant c does not depend on u, μ and F. Thus, by the Lere–Shauder theorem [30, Ch. VIII, § 35.5], equation (26) and therefore problem (3),(4), too, have at least one solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$. Theorem 1 is proved. □

3. Nonexistence of Global Solvability of Problem (1),(2) in the Case of Nonlinearity of the Form $f(u)=\lambda|u|^{p+1}$

Below we will consider the case where the coefficients in problem (1),(2) are $m=0$ and $f(u)=\lambda|u|^{p+1}$ with λ and p being given positive numbers, i.e. the problem

$$\square u := \frac{\partial^2 u}{\partial t^2} - \Delta u = \lambda|u|^{p+1} + F, \tag{27}$$

$$u\big|_{S_T^0} = 0, \quad u\big|_{S_T} = g \tag{28}$$

in the domain D_T, $T>0$, where g is a given real function on S_T that by virtue of (28) satisfies the compatibility condition $g|_{\partial S_T}=0$.

Remark 3. Assuming that $F\in L_2(D_T)$, $g\in W_2^1(S_T)$ and $0<p<\frac{2}{n-1}$, analogously to Definitions 1 and 2 with regard to a weak and a strong generalized solution of problem (3),(4) in the domain S_T and taking into account Remark 1, we introduce the notions of a weak and a strong generalized solution of problem (27),(28):

(i) a function $u\in W_2^1(D_T)$ is called a weak generalized solution of the nonlinear problem (27),(28) in the domain D_T if for any function $\varphi\in W_2^1(D_T)$ such that $\varphi|_{t=T}=0$, $\varphi|_{S_T^0}=0$ the following integral equality is valid:

$$-\int\limits_{D_T} u_t\varphi_t\,dx\,dt + \int\limits_{D_T} \nabla_x u\,\nabla_x\varphi\,dx\,dt = \lambda\int\limits_{D_T} |u|^{p+1}\varphi\,dx\,dt + \int\limits_{D_T} F\varphi\,dx\,dt - \int\limits_{S_T} \frac{\partial g}{\partial N}\,\varphi\,ds, \tag{29}$$

where $\frac{\partial}{\partial N}=\nu_0\frac{\partial}{\partial t}-\sum\limits_{i=1}^n \nu_i\frac{\partial}{\partial x_i}$ is an internal derivative with respect to the conormal which is an internal differential operator on S_T, since the conical manifold S_T is characteristic, $\nu=(\nu_1,\dots,\nu_n,\nu_0)$ is the outward unit normal to ∂D_T, $\nabla_x=\big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\big)$;

(ii) a function $u\in W_2^1(D_T)$ is called a strong generalized solution of the nonlinear problem (27),(28) in the domain D_T if there exists a sequence of functions $u_k\in \overset{\circ}{C}{}^2(\overline{D}_T,S_T^0)=\{u\in C^2(\overline{D}_T): u|_{S_T^0}=0\}$ such that $u_k\to u$ in the space $W_2^1(D_T)$, $[\square u_k-\lambda|u_k|^{p+1}]\to F$ in the space $L_2(D_T)$ and $u_k|_{S_T}\to g$ in the space $W_2^1(S_T)$.

Remark 4. In a standard manner [25, Ch. II, § 5] one can prove that a weak generalized solution $u\in W_2^1(D_T)$ of problem (27),(28) satisfies the homogeneous boundary conditions (28) in the sense of trace theory.

It is obvious that a strong generalized solution $u\in W_2^1(D_T)$ of problem (27),(28) is also a weak generalized solution of this problem.

Let us introduce into consideration a function $\varphi^0=\varphi^0(x,t)$ such that

$$\varphi^0\in C^2(\overline{D}_T),\quad \varphi^0\big|_{D_{T=1}}>0,\quad \varphi^0\big|_{x_n=0}=0,\quad \varphi^0\big|_{t\geq 1}=0 \tag{30}$$

and

$$\varkappa_0=\int\limits_{D_{T=1}}\frac{|\Box\varphi^0|^{\alpha'}}{|\varphi^0|^{\alpha'-1}}\,dx\,dt<+\infty,\quad \alpha'=1+\frac{1}{p}\,. \tag{31}$$

It can be easily verified that for sufficiently large k and m we can take the function

$$\varphi^0(x,t)=\begin{cases}x_n^k(1-t)^m, & (x,t)\in D_{T=1},\\ 0, & t\geq 1,\end{cases}$$

as a function φ^0 satisfying conditions (30) and (31).

If it is assumed $\varphi_T(x,t)=\varphi^0\big(\frac{x}{T}\,,\frac{t}{T}\big)$, $T>0$, then by virtue of (30) we readily see that

$$\varphi_T\in C^2(\overline{D}_T),\quad \varphi_T\big|_{D_T}>0,\quad \varphi_T\big|_{x_n=0}=0,\quad \varphi_T\big|_{t=T}=0. \tag{32}$$

Assuming that the functions F, g and φ^0 are fixed, we introduce into consideration the function of one variable T

$$\gamma(T)=\int\limits_{D_T}F\varphi_T\,dx\,dt+\int\limits_{S_T}g\,\frac{\partial\varphi_T}{\partial N}\,ds-\int\limits_{S_T}\varphi_T\,\frac{\partial g}{\partial N}\,ds,\quad T>0. \tag{33}$$

We have the following theorem on the nonexistence of global solvability of problem (27),(28).

Theorem 2. *Let $F\in L_{2,loc}(D)$, $g\in W^1_{2,loc}(S)$ and $F\in L_2(D_T)$, $g\in W^1_2(S_T)$ for any $T>0$. If $0<p<\frac{2}{n-1}$ and*

$$\varliminf_{T\to\infty}\gamma(T)>0, \tag{34}$$

then there exists a positive number $T_0=T_0(F,g)$ such that, for $T>T_0$, problem (27), (28) *cannot have a weak generalized solution $u\in W^1_2(D_T)$ in the domain D_T.*

Proof. Let $u\in W^1_2(D_T)$ be a weak generalized solution of problem (27), (28) in the domain D_T, i.e. the integral equality (29) be fulfilled. By virtue of (32) we can take, in equality (29), the function φ in the role of the test function φ_T. Integrating by parts the left-hand side of equality (29), we obtain

$$\int\limits_{D_T}u\Box\varphi_T\,dx\,dt=\lambda\int\limits_{D_T}|u|^{p+1}\varphi_T\,dx\,dt$$
$$+\int\limits_{D_T}F\,\varphi_T\,dx\,dt+\int\limits_{S_T}g\,\frac{\partial\varphi_T}{\partial N}\,dx-\int\limits_{S_T}\varphi_T\,\frac{\partial g}{\partial N}\,ds. \tag{35}$$

With (33) taken into account, equality (35) can be rewritten as

$$\lambda \int\limits_{D_T} |u|^{p+1}\varphi_T \, dx\, dt = \int\limits_{D_T} u\Box\varphi_T \, dx\, dt - \gamma(T). \tag{36}$$

If in the Young inequality with parameter $\varepsilon > 0$ for $\alpha = p+1$

$$ab \le \frac{\varepsilon}{\alpha}\, a^{\alpha} + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\, b^{\alpha'}, \quad a, b \ge 0, \quad \alpha' = \frac{\alpha}{\alpha-1} = 1 + \frac{1}{p}$$

we take $a = |u|\varphi_T^{\frac{1}{\alpha}}$, $b = \frac{|\Box\varphi_T|}{\varphi_T^{\frac{1}{\alpha}}}$, then keeping in mind that $\frac{\alpha'}{\alpha} = \alpha' - 1 = \frac{1}{p}$, we obtain

$$|u\Box\varphi_T| = |u|\varphi_T^{\frac{1}{\alpha}} \cdot \frac{|\Box\varphi_T|}{\varphi_T^{\frac{1}{\alpha}}} \le \frac{\varepsilon}{\alpha}\, |u|^{\alpha}\varphi_T + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\, \frac{|\Box\varphi_T|^{\alpha'}}{\varphi_T^{\alpha'-1}}\, . \tag{37}$$

By virtue of (37), from (36) we have

$$\Big(\lambda - \frac{\varepsilon}{\alpha}\Big) \int\limits_{D_T} |u|^{\alpha}\varphi_T \, dx\, dt \le \frac{1}{\alpha'\varepsilon^{\alpha'-1}} \int\limits_{D_T} \frac{|\Box\varphi_T|^{\alpha'}}{\varphi_T^{\alpha'-1}}\, dx\, dt - \gamma(T),$$

whence for $\varepsilon < \lambda\alpha$ we obtain

$$\int\limits_{D_T} |u|^{\alpha}\varphi_T \, dx\, dt \le \frac{\alpha}{(\lambda\alpha - \varepsilon)\alpha'\varepsilon^{\alpha'-1}} \int\limits_{D_T} \frac{|\Box\varphi_T|^{\alpha'}}{\varphi_T^{\alpha'-1}}\, dx\, dt - \frac{\alpha}{\lambda\alpha - \varepsilon}\, \gamma(T). \tag{38}$$

Taking into account that $\alpha' = \frac{\alpha}{\alpha-1}$, $\alpha = \frac{\alpha'}{\alpha'-1}$ and $\min\limits_{0<\varepsilon<\lambda\alpha} \frac{\alpha}{(\lambda\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} = \frac{1}{\lambda^{\alpha'}}$ for $\varepsilon = \lambda$, from (38) it follows that

$$\int\limits_{D_T} |u|^{\alpha}\varphi_T \, dx\, dt \le \frac{1}{\lambda^{\alpha'}} \int\limits_{D_T} \frac{|\Box\varphi_T|^{\alpha'}}{\varphi_T^{\alpha'-1}}\, dx\, dt - \frac{\alpha'}{\lambda}\, \gamma(T). \tag{39}$$

Since $\varphi_T(x,t) = \varphi^0\big(\frac{x}{T}, \frac{t}{T}\big)$, by virtue of (30), (31) it easy to verify after the substitution of the variables $t = Tt'$, $x = Tx'$ that

$$\int\limits_{D_T} \frac{|\Box\varphi_T|^{\alpha'}}{\varphi_T^{\alpha'-1}}\, dx\, dt = T^{n+1-2\alpha'} \int\limits_{D_{T=1}} \frac{|\Box\varphi^0|^{\alpha'}}{(\varphi^0)^{\alpha'-1}}\, dx'\, dt' = T^{n+1-2\alpha'}\varkappa_0 < +\infty. \tag{40}$$

By (32) and (40), from inequality (39) we obtain

$$0 \le \int\limits_{D_T} |u|^{\alpha}\varphi_T \, dx\, dt \le \frac{1}{\lambda^{\alpha'}}\, T^{n+1-2\alpha'}\varkappa_0 - \frac{\alpha'}{\lambda}\, \gamma(T). \tag{41}$$

If $p < \frac{2}{n-1}$, i.e. If $n + 1 - 2\alpha' < 0$, where $\alpha' = 1 + \frac{1}{p}$, then by virtue of (31) we have $\lim\limits_{T\to\infty} \frac{1}{\lambda^{\alpha'}}\, T^{n+1-2\alpha'}\varkappa_0 = 0$. Hence by virtue of (34) there exists a positive number $T_0 = T_0(F, g)$ such that, for $T > T_0$, the right-hand part of (41) is negative, while the left-hand part of this inequality is nonnegative. Thus if there exists a weak generalized solution $u \in W_2^1(D_T)$ of problem (27),(28) in the domain D_T, then necessarily $T \le T_0$, which proves Theorem 2. □

Remark 5. We give some sufficient conditions imposed on the functions F and g, which guarantee the fulfilment of condition (34):

(i) $F = const > 0$, $g = const$;

(ii) $F \in L_{2,loc}(D)$, $g \in W^1_{2,loc}(S)$ and $F \in L_2(D_T)$, $g \in W^1_2(S_T)$ for any $T > 0$, and also $\operatorname{diam}\operatorname{supp} g < +\infty$ and $F \geq 0$, $F(x,t) \geq ct^{-k}$ for $t \geq 1$, where $c = const > 0$, $0 < k = const < n+1$.

4. Local Solvability of Problem (1), (2) in the Case of Nonlinearity of the Form $f(u) = \lambda|u|^{p+1}$

Remark 6. In proving Theorem 1, it was shown that the linear problem (21), which, for $m = 0$, coincides with the corresponding linear problem (27), (28), has, for $\lambda = 0$ and $g = 0$, a unique solution $u = L^{-1}F$, where $L^{-1} : L_2(D_T) \to \overset{\circ}{W}{}^1_2(D_T, S^0_T \cup S_T)$ is a linear continuous operator whose norm admits estimate (24). It should also be noted that, analogously to Remark 1, for $0 < p < \frac{2}{n-1}$ the operator

$$K_1 : \overset{\circ}{W}{}^1_2(D_T, S^0_T \cup S_T) \to L_2(D_T) \quad (K_1 u = \lambda|u|^{p+1}) \tag{42}$$

is continuous and compact. Thus for $g = 0$ the nonlinear problem (27), (28) is equivalent to the functional equation

$$u = Au + u_0 \tag{43}$$

in the space $\overset{\circ}{W}{}^1_2(D_T, S^0_T \cup S_T)$, where with (42) taken into account

$$A = L^{-1}K_1, \quad u_0 = L^{-1}F \in \overset{\circ}{W}{}^1_2(D_T, S^0_T \cup S_T). \tag{44}$$

Remark 7. Let $B(0,d) := \{u \in \overset{\circ}{W}{}^1_2(D_T, S^0_T \cup S_T) : \|u\|_{\overset{\circ}{W}{}^1_2(D_T,S^0_T\cup S_T)} \leq d\}$ be a closed (convex) ball in the Hilbert space $\overset{\circ}{W}{}^1_2(D_T, S^0_T \cup S_T)$ with radius $d > 0$ and center at a zero element. Since by virtue of Remark 6 the operator $A : \overset{\circ}{W}{}^1_2(D_T, S^0_T \cup S_T) \to \overset{\circ}{W}{}^1_2(D_T, S^0_T \cup S_T)$ is continuous and compact for $0 < p < \frac{2}{n-1}$, by the Shauder principle in order to prove the solvability of equation (43) it is sufficient to show that the operator A_1 acting by the formula $A_1u = Au + u_0$ transverse the ball $B(0,d)$ into itself for some $d > 0$ [30, Ch. VIII, § 35.3]. To this end, below we will derive the needed estimate for the value $\|Au\|_{\overset{\circ}{W}{}^1_2(D_T,S^0_T\cup S_T)}$.

We further use the reasoning from [31]. If $u \in \overset{\circ}{W}{}^1_2(D_T, S^0_T \cup S_T)$, then we denote by $\widetilde{u}$ the function which continues in an even manner the function u across the plane $t = T$. It is obvious that $\widetilde{u} \in \overset{\circ}{W}{}^1_2(D^*_T)$, where $D^*_T : |x| < t < 2T - |x|$, $x_n > 0$.

Using the inequality [32, Ch. X, § 1]

$$\int_\Omega |v|\, d\Omega = (\operatorname{mes}\Omega)^{1-\frac{1}{q}} \|v\|_{q,\Omega}, \quad q \geq 1,$$

and taking into account the equalities

$$\|\widetilde{u}\|^q_{L_q(D_T^*)} = 2\|u\|^q_{L_q(D_T)}, \quad \|\widetilde{u}\|^2_{\overset{\circ}{W}{}^1_2} = 2\|u\|^2_{\overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T)}$$

from the well known multiplicative inequality [25, Ch. I, § 7]

$$\|v\|_{q,\Omega} \le \beta \|\nabla v\|^{\widetilde{\alpha}}_{m,\Omega} \|v\|^{1-\widetilde{\alpha}}_{r,\Omega} \quad \forall\, v \in \overset{\circ}{W}{}^1_2(\Omega), \quad \Omega \subset R^{n+1},$$

$$\widetilde{\alpha} = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{r} - \frac{1}{\widetilde{m}}\right)^{-1}, \quad \widetilde{m} = \frac{(n+1)m}{n+1-m}$$

for $\Omega = D_T^* \subset R^{n+1}$, $v = \widetilde{u}$, $r = 1$, $m = 2$ and $1 < q \le \frac{2(n+1)}{n-1}$, where $\beta = const > 0$ does not depend on v and T, we obtain the inequality

$$\|u\|_{L_q(D_T)} \le c_0 (\operatorname{mes} D_T)^{\frac{1}{q} + \frac{1}{n+1} - \frac{1}{2}} \|u\|_{\overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T)} \quad \forall u \in \overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T), \quad (45)$$

where $c_0 = const > 0$ does not depend on u.

Since $\operatorname{mes} D_T = \frac{\omega_n}{2(n+1)} T^{n+1}$, where ω_n is the volume of the unit ball in R^n, for $q = 2(p+1)$ inequality (45) implies

$$\|u\|_{L_{2(p+1)}(D_T)} \le c_0 \widetilde{\ell}_{p,n} T^{(n+1)\left(\frac{1}{2(p+1)} + \frac{1}{n+1} - \frac{1}{2}\right)} \|u\|_{\overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T)}$$

$$\forall u \in \overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T), \quad (46)$$

where $\widetilde{\ell}_{p,n} = \left(\frac{\omega_n}{2(n+1)}\right)^{\left(\frac{1}{2(p+1)} + \frac{1}{n+1} - \frac{1}{2}\right)}$.

For the value $\|K_1 u\|_{L_2(D_T)}$, where $u \in \overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T)$ and the operator K_1 is given by equality (42), by virtue of (46) we obtain the estimate

$$\|K_1 u\|_{L_2(D_T)} \le \lambda \left[\int\limits_{D_T} |u|^{2(p+1)}\, dx\, dt\right]^{\frac{1}{2}} = \lambda \|u\|^2_{L_{2(p+1)}(D_T)}$$

$$\le \lambda \ell_{p,n} T^{(p+1)(n+1)\left(\frac{1}{2(p+1)} + \frac{1}{n+1} - \frac{1}{2}\right)} \|u\|^{p+1}_{\overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T)}, \quad (47)$$

where $\ell_{p,n} = [c_0 \widetilde{\ell}_{p,n}]^{p+1}$.

Now, for $Au = L^{-1} K_1 u$, from (24) and (47) follows the estimate

$$\|Au\|_{\overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T)} \le \|L^{-1}\|_{L_2(D_T) \to \overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T)} \|K_1 u\|_{L_2(D_T)}$$

$$\le \sqrt{\frac{e}{2}}\, \lambda \ell_{p,n} T^{1+(p+1)(n+1)\left(\frac{1}{2(p+1)} + \frac{1}{n+1} - \frac{1}{2}\right)} \|u\|^{p+1}_{\overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T)}$$

$$\forall u \in \overset{\circ}{W}{}^1_2(D_T, S_T^0 \cup S_T). \quad (48)$$

Note that $\frac{1}{2(p+1)} + \frac{1}{n+1} - \frac{1}{2} > 0$ for $p < \frac{2}{n-1}$.

Consider the equation

$$az^{p+1} + b = z \quad (49)$$

with respect to the unknown z, where

$$a=\sqrt{\frac{e}{2}}\,\lambda\ell_{p,n}T^{1+(p+1)(n+1)\left(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2}\right)},\quad b=\sqrt{\frac{e}{2}}\,T\|F\|_{L_2(D_T)}\,. \tag{50}$$

For $T>0$ it is obvious that $a>0$ and $b\geq 0$. A simple analysis analogous to that carried out for $p=2$ in [30, Ch. VIII, § 35.4] shows that (i) in the case $b=0$ equation (49) has, along with the zero root $z_1=0$, the unique positive root $z_2=a^{-\frac{1}{p}}$; (ii) if $b>0$, then for $0<b<b_0$, where

$$b_0=\left[(p+1)^{-\frac{1}{p}}-(p+1)^{-\frac{p+1}{p}}\right]a^{-\frac{1}{p}}\,, \tag{51}$$

equation (49) has two positive roots z_1 and z_2, $0<z_1<z_2$. For $b=b_0$ these roots coincide and we have one positive root $z_1=z_2=z_0=[(p+1)a]^{-\frac{1}{p}}$; (iii) for $b>b_0$ equation (49) has no nonnegative roots.

Note that for $0<b<b_0$ we have the inequalities $z_1<z_0=[(p+1)a]^{-\frac{1}{p}}<z_2$.

By virtue of (50) and (51) the condition $b\leq b_0$ is equivalent to the condition

$$\sqrt{\frac{e}{2}}\,T\|F\|_{L_2(D_T)} \leq\left[\sqrt{\frac{e}{2}}\lambda\ell_{p,n}T^{1+(p+1)(n+1)\left(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2}\right)}\right]^{-\frac{1}{p}}\left[(p+1)^{-\frac{1}{p}}-(p+1)^{-\frac{p+1}{p}}\right]$$

or to

$$\|F\|_{L_2(D_T)}\leq\gamma_{n,\lambda,p}T^{-\alpha_n},\quad \alpha_n>0, \tag{52}$$

where

$$\gamma_{n,\lambda,p}=\left[(p+1)^{-\frac{1}{p}}-(p+1)^{-\frac{p+1}{p}}\right]\left(\lambda\ell_{p,n}\right)^{-\frac{1}{p}}\exp\left[-\frac{1}{2}\left(1+\frac{1}{p}\right)\right],$$

$$\alpha_n=1+\frac{1}{p}\left[1+(p+1)(n+1)\left(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2}\right)\right].$$

By the absolute continuity of the Lebesgue integral we have $\lim\limits_{T\to 0}\|F\|_{L_2(D_T)}=0$. Since at the same time time $\lim\limits_{T\to 0}T^{-\alpha_n}=+\infty$, there exists a number $T_1=T_1(F)$, $0<T_1<+\infty$, such that inequality (52) is fulfilled for

$$0<T\leq T_1(F). \tag{53}$$

Now we will show that if condition (53) is fulfilled, then the operator $A_1:\overset{\circ}{W}{}^1_2(D_T,S^0_T\cup S_T)\to\overset{\circ}{W}{}^1_2(D_T,S^0_T\cup S_T)$ acting by the formula $A_1u=Au+u_0$ transfers the ball $B(0,z_2)$ from Remark 7 into itself, where z_2 is the largest positive root of equation (49). Indeed, if $u\in B(0,z_2)$, then by virtue of (48)–(50) we have

$$\|A_1u\|_{\overset{\circ}{W}{}^1_2(D_T,S^0_T\cup S_T)}\leq a\|u\|^{p+1}_{\overset{\circ}{W}{}^1_2(D_T,S^0_T\cup S_T)}+b\leq az_2^{p+1}+b=z_2.$$

Therefore the following theorem is valid according to Remarks 6 and 7.

Theorem 3. *Let* $F \in L_{2,loc}(D)$*,* $g = 0$*,* $0 < p < \frac{2}{n-1}$ *and condition* (53) *be fulfilled for the value* T*. Then problem* (27)*,* (28) *has at least one strong generalized solution* $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ *in the domain* D_T*.*

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