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SLOWLY VARYING SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH RETARDED AND ADVANCED ARGUMENTS

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Dedicated to Professor I. Kiguradze
on the occasion of his seventieth birthday

Abstract. Regularity, in the sense of Karamata, (with nonoscillation as a consequence) and the precise asymptotic behaviour of solutions of two functional differential equations are studied.

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0. Introduction. We prove the existence of slowly varying (in the sense of Karamata) solutions, implying their nonoscillation, of the following functional differential equations with both retarded and advanced arguments:

$$x''(t) - p(t)x(g(t)) - q(t)x(h(t)) = 0, \tag{A}$$

$$x''(t) + p(t)x(g(t)) + q(t)x(h(t)) = 0. \tag{B}$$

In addition, the precise asymptotic behavior of such solutions of equation (A) is obtained.

It is assumed here that on $[a, \infty)$, for some $a > 0$, functions p, q, g and h are continuous and p, q are integrable.

For readers' convenience we recall that a measurable function $f : [0, \infty) \to (0, \infty)$ is said to be regularly varying and having an index $\rho \in \mathbf{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for any} \quad \lambda > 0.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. The symbol SV is used to denote $\mathrm{RV}(0)$ and a member of $\mathrm{SV} = \mathrm{RV}(0)$ is referred to as a slowly varying function. If $f(t) \in \mathrm{RV}(\rho)$, then $f(t) = t^\rho L(t)$ for some $L(t) \in \mathrm{SV}$, and so of fundamental importance in regular variation is the class of slowly varying functions. In this paper, among many basic properties of slowly varying functions, we emphasize the representation theorem which asserts that $L(t) \in \mathrm{SV}$ if and only if it is expressed in the form

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\varepsilon(s)}{s} ds \right\}, \quad t \geq t_0, \tag{0.1}$$

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for some $t_0 > 0$ and some measurable functions $c(t)$ and $\varepsilon(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \varepsilon(t) = 0.$$

The most comprehensive text on the theory and numerous applications of these function can be found [1].

The first result asserting the existence of SV solutions of a functional differential equation is proved in [5] via the Schauder–Tychonoff fixed point theorem and the same approach is used here.

Nonoscillation and oscillation of solutions of such equations of higher even order are studied in [4].

1. Results. These are concerned with the existence of SV solutions for both equations (A) and (B) and the asymptotic behaviour of such solutions of equation (A).

1.1. Existence. There holds

Theorem 1.1. *Suppose that for* $t \geq a$

$$\begin{aligned} &\text{(i)} && p(t) > 0 \text{ and } q(t) > 0;\\ &\text{(ii)} && g(t) \text{ is increasing, } g(t) < t \text{ and } \lim_{t\to\infty} g(t) = \infty;\\ &\text{(iii)} && h(t) \text{ is increasing and } h(t) > t;\\ &\text{(iv)} && \limsup_{t\to\infty} \tfrac{t}{g(t)} < \infty, \quad \limsup_{t\to\infty} \tfrac{h(t)}{t} < \infty. \end{aligned} \tag{1.1}$$

Then both equations (A) *and* (B) *possess a slowly varying solution if and only if*

$$\lim_{t\to\infty} t\int_t^\infty p(s)ds = \lim_{t\to\infty} t\int_t^\infty q(s)ds = 0. \tag{1.2}$$

1.2. Asymptotic behaviour. There hold

Theorem 1.2. *Put*

$$r(t) = p(t) + q(t);$$

then, equation (A) *possesses slowly varying solutions* $x(t)$ *such that for* $t \to \infty$

$$x(t) \to c > 0$$

if and only if

$$\int_a^\infty tr(t)dt < \infty. \tag{1.3}$$

Theorem 1.3. *If*

$$\int_a^\infty s^2 r(s)ds \int_{g(s)}^\infty r(u)du < \infty, \tag{1.4}$$

then for each slowly varying solution $x(t)$ *of equation* (A) *there holds for* $t \to \infty$,

$$x(t) \sim A \exp\left\{ -\int_a^t sr(s)ds \right\}, \quad A > 0. \tag{1.5}$$

2. Preliminaries. The following result [2], [3] is needed for the proofs.

Lemma 2.1. *Let* $f(t)$ *be continuous integrable and one-signed on some half-axis* $[a, \infty)$, $a > 0$. *Then the linear ordinary* (*without a functional argument*) *equation*

$$x''(t) + f(t)x(t) = 0 \tag{C}$$

has a slowly varying solution if and only if

$$\lim_{t\to\infty} t \int_t^\infty f(s)ds = 0. \tag{2.1}$$

We present here an outline of the proof of Lemma 2.1.

Let $x(t)$ be an SV solution. Since the derivative $x'(t)$ is monotone, $f(t)$ being one-signed, by [6, Proposition 9,c)] one has $tx'(t)/x(t) \to 0$ as $t \to \infty$ and the "only if" part is straightforward (cf. [6, Theorem 1.10]).

The proof of the "if" part follows that in [3]: Put

$$F(t) = t \int_t^\infty f(s)ds. \tag{2.2}$$

Choose $t_0 > a$ and $m > 0$ so that for $t \geq t_0$

$$|F(t)| \leq m < \frac{1}{4}, \tag{2.3}$$

which is possible because of (2.1).

Observe at this point that throughout the text, if not stated otherwise, all relations (e.g., inequalities) hold for $t \geq t_0$, which is therefore occasionally omitted.

Consider the set

$$V = \{v \in C_0[t_0, \infty) : \ 0 \leq v(t) \leq m, \ \ t \geq t_0\} \tag{2.4}$$

and the integral operator

$$\mathcal{F}v(t) = t \int_t^\infty \left(\frac{v(s) + F(s)}{s} \right)^2 ds, \tag{2.5}$$

where $C_0[t_0, \infty)$ denotes the set of all continuous functions on $[t_0, \infty)$ that tend to 0 as $t \to \infty$; $C_0[t_0, \infty)$ is a Banach space with the norm $\|v\|_0 = \sup_{t \geq t_0} |v(t)|$.

$\mathcal{F}$ is shown to be a contraction mapping on V, which is a closed subset of $C_0[t_0,\infty)$, and so there exists $v_0 \in V$ such that $v_0 = \mathcal{F}v_0$. i.e.,

$$v_0(t) = t\int_t^\infty \left(\frac{v_0(s)+F(s)}{s}\right)^2 ds. \tag{2.6}$$

Using this $v_0(t)$, we define the function $x_0(t)$ by

$$x_0(t) = \exp\left\{\int_{t_0}^t \frac{v_0(s)+F(s)}{s} ds\right\}, \tag{2.7}$$

which satisfies equation (C) and is SV due to (0.1) with $c(t)=1$ and since $v_0(s)$ and $F(s)$ tend to zero in view of (2.4) and (1.2).

Remark 2.1. If in equation (C) instead of $f(t)$ one has $-f(t)$ with $f(t)>0$, then in (2.6) and (2.7) one has the minus sign in front of $F(s)$. Then, any SV solution of (C) is decreasing ([6, §1.2]) and (2.7) implies

$$v_0(t) \le F(t). \tag{2.8}$$

If, on the other hand, $f(t)$ is positive, then SV solutions are increasing, which follows directly from (2.6) and (2.7).

Also, by (2.4) there holds

$$0 \le v_0(t) \le m. \tag{2.9}$$

Remark 2.2. If $x(t)$ is an SV function, then the representation theorem (0.1) implies that

$$\lim_{t\to\infty} \frac{x(g(t))}{x(t)} = \lim_{t\to\infty} \frac{x(h(t))}{x(t)} = 1.$$

For example,

$$\frac{x(h(t))}{x(t)} = \frac{c(h(t))}{c(t)} \exp\left\{\int_t^{h(t)} \frac{\varepsilon(s)}{s} ds\right\}.$$

Due to the properties of $\varepsilon(t)$ and $h(t)$ one has for $t\to\infty$

$$\int_t^{h(t)} \frac{|\varepsilon(s)|}{s} ds \le \sup_{s\ge t} \varepsilon(s) \log\frac{h(t)}{t} \to 0$$

and the statement follows for $h(t)$ and likewise for $g(t)$.

3. Proofs.

3.1. Proof of Theorem 1.1.

a) *Equation* (A).

The proof of the "only if" part is a direct consequence of Lemma 2.1 and the representation theorem for SV functions as envisaged in Remark 2.2.

Rewrite equation (A) as

$$x''(t) = f_x(t)x(t), \tag{3.1}$$

where

$$f_x(t) = p(t)\frac{x(g(t))}{x(t)} + q(t)\frac{x(h(t))}{x(t)}. \tag{3.2}$$

In view of (1.1)(i) and Remark 2.2 the function f_x is integrable over some positive half-axis and the "only if" part follows by the application of Lemma 2.1 to (3.1) giving

$$\lim_{t\to\infty} t\int_t^\infty f_x(s)ds = 0,$$

which implies (1.2).

The "if" part:

Assume that (1.2) holds. To apply Lemma 2.1, we introduce the notation:

$$\begin{aligned} f(t)(= r(t)) &= p(t) + q(t), \\ F(t) &= -t\int_t^\infty f(s)ds. \end{aligned} \tag{3.3}$$

Choose $m > 0$ and $t_0 > a$ so large that $g(t_0) > a$ and such that for $t \geq t_0$,

$$|F(t)| \leq m < \frac{1}{8}, \tag{3.4}$$

and

$$2m\log\frac{t}{g(t)} \leq \log 2 \tag{3.5}$$

which is possible because of (1.1)(iv) and (1.2).

Let us now define Ξ to be the set of all positive continuous nonincreasing functions $\xi(t)$ on $|g(t_0), \infty)$ with the properties

$$\xi(t) = 1 \quad \text{for} \quad g(t_0) \leq t \leq t_0, \qquad \frac{\xi(g(t))}{\xi(t)} \leq 2 \quad \text{for} \quad t \geq t_0. \tag{3.6}$$

It is easy to see that Ξ is a nonempty closed, convex subset of $C[g(t_0), \infty)$ equipped with the usual metric topology of uniform convergence on compact subintervals of $[g(t_0),\infty)$ (cf. [5]). For any $\xi\in\Xi$ define

$$\begin{aligned} f_\xi(t) &= p(t)\frac{\xi(g(t))}{\xi(t)} + q(t)\frac{\xi(h(t))}{\xi(t)}\,, \\ F_\xi(t) &= -t\int_t^\infty f_\xi(s)ds. \end{aligned} \tag{3.7}$$

We consider a family of linear (ordinary) differential equations

$$x''(t) = f_\xi(t)x(t), \quad \xi\in\Xi. \tag{3.8}$$

By (3.6), (3.7), the decreasing nature of $\xi(t)$ and due to (3.4) one has

$$F_\xi(t) \le 2|F(t)| \le 2m < \frac{1}{4}. \tag{3.9}$$

We conclude by using Lemma 2.1 that for each $\xi\in\Xi$ equation (3.8) has an SV solution $X_\xi(t)$ having the representation

$$X_\xi(t) = \exp\left\{\int_{t_0}^t \frac{v_\xi(s)-F_\xi(s)}{s}\right\}, \tag{3.10}$$

where $v_\xi(t)$ is a solution of the integral equation

$$v_\xi(t) = t\int_t^\infty \left(\frac{v_\xi(s)-F_\xi(s)}{s}\right)^2 ds. \tag{3.11}$$

In virtue of (2.4) and (3.9) one has

$$0 \le v_\xi(t) \le 2m. \tag{3.12}$$

Further, let Φ denote the mapping which to each $\xi\in\Xi$ assigns the function $\Phi\xi$ defined by

$$\Phi\xi(t) = 1 \quad \text{for} \quad g(t_0)\le t\le t_0, \quad \Phi\xi(t) = X_\xi(t) \quad \text{for} \quad t\ge t_0. \tag{3.13}$$

We will show that Φ is a continuous self-map on Ξ which sends Ξ to a relatively compact subset of Ξ in the topology of $C[g(t_0),\infty)$.

(i) $\Phi(\Xi)\subset\Xi$: If $\xi\in\Xi$, then $\Phi\xi(t) = X_\xi(t)$ is clearly decreasing for $t\ge t_0$.

Furthermore, since by (1.1)(ii), $g(t)$ is increasing and $g(t)<t$, there might exist an interval $t_0\le t<t_1$ where $g(t)<t_0$ and $g(t_1)=t_0$, and by (3.13), $\Phi\xi(g(t))=1$. Hence because of (3.10), (3.9), (3.4) and (3.5) one has for $t_0\le t\le t_1$,

$$\begin{aligned} \frac{\Phi\xi(g(t))}{\Phi\xi(t)} &= \exp\left\{-\int_{t_0}^t \frac{v_\xi(s)-F_\xi(s)}{s}ds\right\} \\ &\le \exp\left\{2m\log\frac{t_1}{g(t_1)}\right\} \le \exp\{\log 2\} = 2. \end{aligned}$$

Similarly, for $t \geq t_1$ one has

$$\frac{\Phi\xi(g(t))}{\Phi\xi(t)} = \exp\left\{ -\int_{g(t)}^{t} \frac{v_\xi(s) - F_\xi(s)}{s} ds \right\} \leq \exp\left\{ 2m \log \frac{t}{g(t)} \right\} \leq 2.$$

Thus (3.6) holds, which ensures that $\Phi\xi \in \Xi$.

(ii) $\Phi(\Xi)$ is relatively compact in $C[g(t_0), \infty)$: The inclusion $\Phi(\Xi) \subset \Xi$ implies that $\Phi(\Xi)$ is locally uniformly bounded on $[g(t_0), \infty)$. The inequality

$$0 \geq \frac{d}{dt}\Phi\xi(t) = \frac{d}{dt}X_\xi(t) = X_\xi(t)\frac{v_\xi(t) - F_\xi(t)}{t} \geq -\frac{2F(t)}{t},$$

holding for $t \geq t_0$ and all $\xi \in \Xi$, shows that $\Phi(\Xi)$ is locally equicontinuous on $[g(t_0), \infty)$. The desired relative compactness of $\Phi(\Xi)$ then follows from the Arzela–Ascoli lemma.

(iii) Φ is a continuous mapping: Let $\{\xi_n\}$ be a sequence in Ξ converging to $\eta \in \Xi$, which means that $\{\xi_n(t)\}$ converges to $\eta(t)$ uniformly on any compact subinterval of $[g(t_0), \infty)$. The continuity of Φ is assured if it is shown that the sequence $\{\Phi\xi_n(t)\}$ converges to $\Phi_\eta(t)$ uniformly on compact subintervals of $[g(t_0), \infty)$. If suffices to restrict our attention to the interval $[t_0, \infty)$. Using (3.10) and the mean value theorem, we obtain

$$\begin{aligned}
|\Phi\xi_n(t) - \Phi\eta(t)| &= |X_{\xi_n}(t) - X_\eta(t)| \\
&= \left| \exp\left\{ \int_{t_0}^{t} \frac{v_{\xi_n}(s) - F_{\xi n}(s)}{s} ds \right\} - \exp\left\{ \int_{t_0}^{t} \frac{v_\eta(s) - F_\eta(s)}{s} ds \right\} \right| \\
&\leq \int_{t_0}^{t} \frac{|v_{\xi_n}(s) - v_\eta(s)| + |F_{\xi_n}(s) - F_\eta(s)|}{s} ds.
\end{aligned}$$

Our task is accomplished if it is verified that the two sequences

$$\frac{1}{t}|v_{\xi_n}(t) - v_\eta(t)|, \quad \frac{1}{t}|F_{\xi_n}(t) - F_\eta(t)| \tag{3.14}$$

converge to 0 uniformly on compact subintervals of $[t_0, \infty)$. As a matter of fact, it can be shown more strongly that the convergence to 0 of the sequences in (3.14) is uniform on $[t_0, \infty)$. The uniform convergence of the second sequence in (3.14) is an immediate consequence of the Lebesgue dominated convergence theorem applied to the inequality

$$\begin{aligned}
&\frac{1}{t}|F_{\xi_n}(t) - F_\eta(t)| \\
&\quad \leq \int_{t}^{\infty} \left\{ p(s) \left| \frac{\xi_n(g(s))}{\xi_n(s)} - \frac{\eta(g(s))}{\eta(s)} \right| + q(s) \left| \frac{\xi_n(h(s))}{\xi_n(s)} - \frac{\eta(h(s))}{\eta(s)} \right| \right\} ds.
\end{aligned} \tag{3.15}$$

Turning to the first sequence in (3.4), we obtain from (3.11)

$$\frac{1}{t}|v_{\xi_n}(t)-v_\eta(t)| \le \int_t^\infty \left| \left(\frac{v_{\xi_n}(s)-F_{\xi_n}(s)}{s}\right)^2 - \left(\frac{v_\eta(s)-F_\eta(s)}{s}\right)^2 \right| ds$$
$$\le \int_t^\infty w_n(s)\frac{|v_{\xi_n}(s)-v_\eta(s)|+|F_{\xi_n}(s)-F_\eta(s)|}{s^2}\, ds\,, \tag{3.16}$$

where

$$w_n(t) = v_{\xi n}(t)+v_\eta(t)+F_{\xi_n}(t)+F_\eta(t).$$

Since by (3.4)

$$w_n(t) \le 8m < 1,$$

we have the following inequality from (3.16)

$$\frac{1}{t}|v_{\xi_n}(t)-v_\eta(t)| \le \theta\int_t^\infty \frac{|v_{\xi_n}(s)-v_\eta(s)|}{s^2}\, ds + \theta\int_t^\infty \frac{R_n(s)}{s^2}\, ds, \tag{3.17}$$

where $\theta = 8m < 1$ and

$$R_n(t) = |F_{\xi_n}(t)-F_\eta(t)|.$$

Putting

$$z_n(t) = \int_t^\infty \frac{|v_{\xi_n}(s)-v_\eta(s)|}{s^2}\, ds \tag{3.18}$$

we rewrite (3.17) as

$$(t^\theta z_n(t))' \ge -\frac{\theta}{t^{1-\theta}}\int_t^\infty \frac{R_n(s)}{s^2} ds. \tag{3.19}$$

Since $t^\theta z_n(t) \to 0$ as $t \to \infty$ and since the right-hand side of (3.19) is integrable over $[t_0, \infty)$, integrating from t to ∞ yields

$$z_n(t) \le \frac{1}{t^\theta}\int_t^\infty \frac{R_n(s)}{s^{2-\theta}}\, ds. \tag{3.20}$$

By combining (3.17) with (3.20) we find that

$$\frac{1}{t}|v_{\xi_n}(t)-v_\eta(t)| \le \frac{\theta}{t^\theta}\int_t^\infty \frac{R_n(s)}{s^{2-\theta}}\, d\theta + \theta\int_t^\infty \frac{R_n(s)}{s^2}\, ds \le \frac{2\theta}{t^\theta}\int_t^\infty \frac{|F_{\xi n}(s)-F_\eta(s)|}{s^2}\, ds$$

for $t \ge t_0$, from which the uniform convergence $|v_{\xi_n}(t)-v_\eta(t)|/t \to 0$ immediately follows. This establishes the continuity of the mapping Φ.

Thus all the hypotheses of the Schauder–Tychonoff fixed point theorem have been fulfilled, and so Φ has a fixed point $\xi_0 \in \Xi : \xi_0 = \Phi\xi_0$. This means in particular that $\xi_0(t) = X_{\xi 0}(t)$ for $t \geq t_0$, which implies that $\xi_0(t)$ is a slowly varying function and satisfies the differential equation

$$\xi_0''(t) = (p_{\xi_0}(t) + q_{\xi_0}(t))\xi_0(t),$$

or equivalently

$$\xi_0''(t) = p(t)\xi_0(g(t)) + q(t)\xi_0(h(t)).$$

The existence of an SV-solution for equation (A) has thus been established.

b) *Equation* (B).

The proof of "only if" part is analogous to the previous case because of (1.1),(i) and Remark 2.2.

The "if" part also follows strictly the idea of the proof of the previous case. Hence we restrict ourselves to emphasizing the necessary complementary reasoning.

Suppose that (1.2) is satisfied. This time choose $t_0 > a$ so that $g(t_0) > a$ and such that, for $t \geq t_0$, (3.4) and

$$4m \log \frac{h(t)}{t} \leq \log 2 \tag{3.21}$$

hold.

We denote by H the set of all continuous nondecreasing functions $\eta(t)$ on $[g(t_0), \infty)$ such that

$$\eta(t) = 1 \quad \text{for} \quad g(t_0) \leq t \leq t_0, \qquad \frac{\eta(h(t))}{\eta(t)} \leq 2 \quad \text{for} \quad t \geq t_0.$$

This time we form a family of linear ordinary differential equations of the form

$$x''(t) + f_\eta(t)x(t) = 0, \quad \eta \in H, \tag{3.22}$$

where

$$f_\eta(t) = p(t)\frac{\eta(g(t))}{\eta(t)} + q(t)\frac{\eta(h(t))}{\eta(t)}\,.$$

As before, we use the notation

$$F_\eta(t) = t\int_t^\infty f_\eta(s)ds.$$

It is obvious that $F_\eta(t) \leq 2F(t)$ for all $\eta \in H$. Therefore we have

$$F_\eta(t) \leq 2F(t) < \frac{1}{4}\,, \tag{3.23}$$

so that for each $\eta \in H$ equation (3.22) possesses an SV-solution having the form

$$X_\eta(t) = \exp\left\{\int_{t_0}^t \frac{v_\eta(s) + F_\eta(s)}{s}ds\right\},$$

where $v_\eta(t)$ is a solution of the integral equation

$$v_\eta(t) = t \int_t^\infty \left(\frac{v_\eta(s) + F_\eta(s)}{s} \right)^2 ds,$$

satisfying

$$0 \le v_\eta(t) \le 2m. \tag{3.24}$$

Let us now define Ψ to be the mapping which to each $\eta \in H$ assigns the function $\Psi\eta(t)$ given by

$$\Psi\eta(t) = 1 \quad \text{for} \quad g(t_0) \le t \le t_0, \quad \Psi\eta(t) = X_\eta(t) \quad \text{for} \quad t \ge t_0.$$

(i) $\Psi(H) \subset H$: If $\eta \in H$, then $\Psi\eta(t) = X_\eta(t)$ is continuous and increasing. Furthermore, we have

$$\frac{\Psi\eta(h(t))}{\Psi\eta(t)} = \exp\left\{ \int_t^{h(t)} \frac{v_\eta(s) + F_\eta(s)}{s} ds \right\} \le \exp\left\{ 4m \log \frac{h(t)}{t} \right\} \le 2,$$

where we have made use of inequalities (3.23), (3.24) and (3.21). This shows that $\Psi\eta \in H$, implying that Ψ is a self-map on H.

(ii) $\Psi(H)$ is relatively compact in $C[g(t_0), \infty)$: This is again a consequence of the local boundedness of $\Psi(H)$ on $[g(t_0), \infty)$ following from $\Psi(H) \subset H$ combined with the local equicontinuity of $\Psi(H)$ which is assured by the following inequality holding for all $\eta \in H$ and $t \ge t_0$,

$$0 \le \frac{d}{dt}\Psi\eta(t) = \frac{d}{dt} X_\eta(t) = \exp\left\{ \int_{t_0}^{t} \frac{v_\eta(s) + F_\eta(s)}{s} ds \right\} \cdot \frac{v_\eta(t) + F_\eta(t)}{t}$$
$$\le M t^\alpha, \quad \text{where} \quad M > 0, \ \ \alpha > 0.$$

(iii) Ψ is a continuous mapping. The proof proceeds exactly as in the corresponding part of Theorem A. Let $\{\eta_n(t)\}$ be a sequence of functions in H converging to $\zeta(t)$ uniformly on compact subintervals of $[g(t_0), \infty)$. We need to prove that $\{\Psi\eta_n(t)\}$ converges to $\Psi\zeta(t)$ uniformly on compact subintervals of $[g(t_0), \infty)$. Since we have

$$|\Psi\eta_n(t) - \Psi\zeta(t)| = |X_{\eta_n}(t) - X_\zeta(t)|$$
$$= \left| \exp\left\{ \int_{t_0}^{t} \frac{v_{\eta_n}(s) + F_{\eta_n}(s)}{s} ds \right\} - \exp\left\{ \int_{t_0}^{t} \frac{v_\zeta(s) + F_\zeta(s)}{s} ds \right\} \right|$$
$$\le 4m \int_{t_0}^{t} \frac{|v_{\eta_n}(s) - v_\zeta(s)| + |F_{\eta_n}(s) - F_\eta(s)|}{s} ds,$$

it is sufficient to demonstrate that the two sequences

$$\frac{1}{t}\,|v_{\eta_n}(t) - v_\zeta(t)|, \quad \frac{1}{t}\,|F_{\eta_n}(t) - F_\zeta(t)|$$

converge to 0 uniformly on any compact subinterval of $[t_0, \infty)$. But this can be done in exactly the same way as the sequences in (3.14) were dealt with at the end of the proof for equation (B). The convergence of the second sequence is almost trivial. As regards the first sequence, we have

$$\frac{1}{t}|v_{\eta_n}(t) - v_\zeta(t)| \le \theta \int_t^\infty \frac{|v_{\eta_n}(s) - v_\zeta(s)|}{s^2}\, ds + \theta \int_t^\infty \frac{R_n(s)}{s^2}\, ds,$$

where $\theta = 4m < 1$ and

$$R_n(t) = |F_{\eta_n}(t) - F_\zeta(t)|,$$

and then derive the inequality

$$\frac{1}{t}\,|v_{\eta_n}(t) - v_\zeta(t)| \le \frac{\theta}{t^\theta} \int_t^\infty \frac{R_n(s)}{s^{2-\theta}}\, ds + \theta \int_t^\infty \frac{R_n(s)}{s^2}\, ds,$$

from which the uniform convergence of $|v_{\eta_n}(t) - v_\zeta(t)|/t \to 0$ on compact subintervals of $[t_0, \infty)$ readily follows. This sketches the proof of the continuity of Ψ.

Therefore, the Schauder–Tychonoff fixed point theorem ensures the existence of $\eta_0 \in H$ such that $\eta_0 = \Psi\eta_0$. Since $\eta_0(t) = X\eta_0(t)$, $t \ge t_0$, $\eta_0(t)$ satisfies the differential equation

$$\eta_0''(t) + (p_{\eta_0}(t) + q_{\eta_0}(t))\eta_0(t) = 0,$$

which is nothing else but

$$\eta_0''(t) + p(t)\eta_0(g(t)) + q(t)\eta_0(h(t)) = 0.$$

We conclude therefore that $\eta_0(t)$ provides an SV-solution of equation (B) on $[t_0, \infty)$, and the proof is completed.

3.2. Proof of Theorem 1.2. Let $x(t)$ be an SV solution of (A) such that $x(t) \to c > 0$, as $t \to \infty$. Since $x(t)$ is decreasing and convex, it is such that $x'(t) \to 0$, as $t \to \infty$. Integrating both sides of (A) over (t, ∞), one obtains

$$x'(t) = \int_t^\infty (p(s)x(g(s)) + q(s)x(h(s)))ds$$

and, after another integration over (t, ∞) and integrating by parts,

$$x(t) = c - \int_t^\infty (s-t)\left\{p(s)\frac{x(g(s))}{x(s)} + q(s)\frac{x(h(s))}{x(s)}\right\} x(s)ds. \tag{3.25}$$

Now, due to (1.2) and Remark 2.2, condition (1.3) follows.

Conversely, let (1.3) hold. This implies (1.2) so that $x(t)$ is SV. Then, with the obvious notation (3.25) is written as

$$x(t) = c - \int_t^\infty (s-t) f_x(s)x(s)ds$$

or by the mean value theorem, since $x(s)$ is decreasing, we have

$$1 = \frac{c}{x(t)} - \int_t^{\xi} (s-t) f_x(s)ds.$$

If $t \to \infty$, then the integral tends to zero due to (1.2), (1.3) and Remark 2.2, whence $c > 0$. (For $c = 0$ this reasoning would imply $1 = 0$.)

Remark 3.1. Theorem 1.2 also holds for equation (B). The proof of the "only if" part is analogous to the previous case. But the proof of the "if" part requires again the use of a fixed point argument as in Theorem 1.1 and is omitted here and the result is framed as a remark only.

3.3. Proof of Theorem 1.3. By [6, Theorem 2.2] with $n = 1$ for each SV solution $x(t)$ of equation (A) written as

$$x''(t) = f_x(t)x(t) \tag{3.26}$$

with

$$f_x(t) = p(t)\frac{x(g(t))}{x(t)} + q(t)\frac{x(h(t))}{x(t)} \tag{3.27}$$

and

$$F_x(t) = t \int_t^{\infty} f_x(s)ds \tag{3.28}$$

one has for $t \to \infty$,

$$x(t) \sim A \exp\left\{ - \int_a^t \frac{F_x(s)}{s} ds \right\}, \quad A > 0, \tag{3.29}$$

provided that

$$\int_a^{\infty} \int_t^{\infty} \left(\frac{F_x(s)}{s} \right)^2 ds\, dt < \infty. \tag{3.30}$$

By substituting F_x from (3.28) into (3.29), integrating by parts and using (2.1) with f_x replacing f, one obtains for $t \to \infty$,

$$x(t) \sim A \exp\left\{ - \int_a^t s f_x(s)ds \right\}. \tag{3.31}$$

By the same argument, this time integrating by parts twice, condition (3.30) becomes

$$\int_a^{\infty} s^2 f_x(t) \int_t^{\infty} f_x(s)dsdt < \infty. \tag{3.32}$$

Notice that due to Remark 2.2 one has

$$f_x(t) = O(r(t)). \tag{3.33}$$

Since also $g(t) < t$, condition (1.3) implies (3.32) so that the asymptotic formula (3.21) holds.

On the other hand, from (2.7), Remark 2.2 and (1.4) and due to the positivity of $v(t)$ one has

$$\frac{x(g(t))}{x(t)} = 1 + O\left\{ \int_{g(t)}^{t} \int_{s}^{\infty} f_x(u)duds \right\}. \tag{3.34}$$

Similarly, from (2.7), Remark 2.2 and due to $v(t) \le F(t)$ one obtains

$$\frac{x(h(t))}{x(t)} = 1 + O\left\{ \int_{t}^{h(t)} \int_{s}^{\infty} f_x(u)duds \right\}. \tag{3.35}$$

Since the inner integral in (3.34) and (3.35) is decreasing, in view of (1.1)(iv) and (3.33) inequalities (3.34) and (3.35) are respectively reduced to

$$\frac{x(g(t))}{x(t)} = 1 + O\left(t \int_{g(t)}^{\infty} r(s)ds \right), \qquad \frac{x(h(t))}{x(t)} = 1 + O\left(t \int_{t}^{\infty} r(s)ds \right).$$

Whence, due to $g(t) < t$ and (3.27)

$$f_x(s) = r(s) + O\left(sr(s) \int_{g(s)}^{\infty} r(u)du \right).$$

Consequently, in view of condition (1.4), the asymptotic formula (3.31) gives (1.5) – the desired one.

4. Examples and remarks.

Remark 4.1. It should be noted that an SV solution of (A) (respectively (B)) satisfies as $t \to \infty$ either $x(t) \to c > 0$ or

$$x(t) \to 0 \quad \text{(respectively } \; x(t) \to \infty). \tag{4.1}$$

It follows therefore that in case (1.3) is violated the SV-solution $x(t)$ guaranteed by Theorem 1.1 necessarily satisfies (4.1).

Example 4.1. Consider the differential equation

$$x''(t) = p(t)x\left(\frac{t}{e}\right) + q(t)x(et), \tag{4.2}$$

where $p(t)$ and $q(t)$ are defined by

$$p(t) = \frac{\lambda}{2t^2 \log t}\left(1 - \frac{1}{\log t}\right)^{\lambda}\left(1 + \frac{\lambda+1}{\log t}\right),$$

$$q(t) = \frac{\lambda}{2t^2 \log t}\left(1 + \frac{1}{\log t}\right)^{\lambda}\left(1 + \frac{\lambda+1}{\log t}\right),$$

λ being a positive constant.

Since $p(t)$ and $q(t)$ satisfy condition (1.2), equation (4.2) has an SV-solution $x(t)$ by Theorem 1.1. Condition (1.3) is not fulfilled, and so by Remark 4.1 the solution $x(t)$ must tend to 0 as $t \to \infty$. In fact, $x(t) = (\log t)^{-\lambda}$ is one of such solutions.

Example 4.2. Consider the equation

$$x''(t) + p(t)x\left(\frac{t}{e}\right) + q(t)x(et) = 0, \tag{4.3}$$

where $p(t)$ and $q(t)$ are given by

$$p(t) = \frac{\mu}{2t^2 \log t}\left(1 - \frac{1}{\log t}\right)^{-\mu}\left(1 - \frac{\mu - 1}{\log t}\right),$$
$$q(t) = \frac{\mu}{2t^2 \log t}\left(1 + \frac{1}{\log t}\right)^{-\mu}\left(1 - \frac{\mu - 1}{\log t}\right),$$

μ being a positive constant. Clearly, condition (1.2) is satisfied, and so there exists an SV-solution $x(t)$ of equation (4.3) by Theorem 1.1. This solution $x(t)$ grows to infinity as $t \to \infty$, since $p(t)$ and $q(t)$ do not satisfy (1.3). One of such solutions is $x(t) = (\log t)^{\mu}$.

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