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BOUNDED SOLUTIONS OF SOME SECOND ORDER DIFFERENCE EQUATIONS

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Cordially dedicated to Professor Ivan Kiguradze
on his 70th birthday anniversary

Abstract. We give some existence results for bounded solutions of linear and nonlinear second order difference equations. In particular, a method of lower and upper solutions for bounded solutions of some nonlinear second order difference equations is obtained, and applied to Duffing-type difference equations.

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1. Introduction

If $m \in \mathbb{Z}$ and $(x_j)_{j\in J}$ is a finite or infinite sequence of real numbers such that $\{m-1, m, m+1\} \subset J$, we define as usual the first order difference operators

$$Dx_m = x_{m+1} - x_m \tag{1}$$

and

$$\Delta x_m = x_m - x_{m-1} \tag{2}$$

and the mixed second order difference operator

$$D(\Delta x_m) = \Delta(Dx_m) = x_{m+1} - 2x_m + x_{m-1}. \tag{3}$$

To given continuous functions $f_m : \mathbb{R} \to \mathbb{R}$ $(m \in J_0)$, with

$$J_0 = J \setminus \{\inf J, \sup J\},$$

one can associate the nonlinear difference equation of the second order

$$\Delta(Dx_m) + f_m(x_m) = 0 \quad (m \in J_0). \tag{4}$$

The existence and multiplicity of solutions of equation (4) satisfying some boundary conditions like the n-periodic ones (for which $J = \{0, \dots, n\}$),

$$x_0 = x_n, \quad Dx_0 = Dx_{n-1}$$

or the Dirichlet ones (for which $J = \{0, \dots, n\}$)

$$x_0 = 0 = x_n$$

have been considered in [1, 2]. In particular, the method of lower and upper solutions was developed for those problems, and applied to the obtention of some existence and multiplicity results.

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In this paper, we are interested in bounded solutions $x = (x_m)_{m\in\mathbb{Z}}$ of second order nonlinear difference equations of the form

$$\Delta(Dx_m) + cDx_m + f_m(x_m) = 0 \quad (m \in \mathbb{Z}).$$

under some conditions upon c and f_m. The results depend upon versions for difference equations of some results of Ortega [6] on bounded solutions of second order linear differential equations given in Section 3, and of a limiting lemma of Krasnosel'skiĭ [4] given in Section 4. In particular, a method of lower and upper solutions for the bounded solutions of some second order nonlinear difference equations is developed in Section 5. As an application, we give in Section 6 existence results for Duffing-type second order nonlinear difference equations.

The following notations are used in the paper. We write $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, and denote by $l^\infty_{\mathbb{Z}}$ the Banach space of bounded sequences $x = (x_m)_{m\in\mathbb{Z}}$ equipped with the norm

$$\|x\|_\infty = \sup_{m\in\mathbb{Z}} |x_m|.$$

For each $k \in \mathbb{N}^*$, we define the vector subspace $l^\infty_{\mathbb{Z},k}$ of $l^\infty_{\mathbb{Z}}$ by

$$l^\infty_{\mathbb{Z},k} = \{x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}} : x_j = 0 \quad \text{for} \quad |j| \geq k+2)\}. \tag{5}$$

Of course, $l^\infty_{\mathbb{Z},k}$ is isometric to the space $\mathbb{R}^{2k+3}$ with the norm

$$|x|_\infty = \max_{-1-k\leq j\leq k+1} |x_j|.$$

If we define the projector P_k in $l^\infty_{\mathbb{Z}}$ by

$$P_k(x) = (\ldots, 0, 0, \ldots, x_{-k-1}, \ldots, x_0, \ldots, x_{k+1}, 0, 0, \ldots) \tag{6}$$

we have $l^\infty_{\mathbb{Z},k} = P_k(l^\infty_{\mathbb{Z}})$ and $\|P_k\| = 1$.

2. First Order Linear Difference Equations

Let us first recall some elementary results on the first order linear equations

$$Dx_m + cx_m = h_m \quad (m \in \mathbb{Z}) \tag{7}$$

and

$$\Delta x_m + cx_m = h_m \quad (m \in \mathbb{Z}) \tag{8}$$

where D and Δ are respectively defined in (1) and (2), $c \in \mathbb{R}$ and $h = (h_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$. Elementary considerations easily lead to the following result.

Lemma 1. *If $c \notin \{0, 2\}$ equation (7) has, for each $h = (h_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$, a unique solution $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ given by*

$$x_m = \begin{cases} \displaystyle\sum_{k=-\infty}^{m-1} (1-c)^{m-k-1} h_k & \text{if } c \in (0,2) \\ \displaystyle -\sum_{k=m}^{+\infty} (1-c)^{m-k-1} h_k & \text{if } c \in (-\infty, 0) \cup (2, +\infty) \end{cases} \quad (m \in \mathbb{Z}). \tag{9}$$

Proof. That x is solution is immediately checked. That $x \in l^\infty$ follows from the fact that the corresponding series is dominated by a convergent geometric one. The uniqueness follows from the structure of the solutions of the homogeneous equation. □

Lemma 2. *If $c \notin \{-2, 0\}$ equation (8) has, for each $h = (h_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$, a unique solution $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ given by*

$$x_m = \begin{cases} -\sum\limits_{k=-m+1}^{+\infty} (1+c)^{-m+k-1} h_k & \text{if } c \in (-2,0) \\ \sum\limits_{k=-\infty}^{m} (1+c)^{-m+k-1} h_k & \text{if } c \in (-\infty,-2) \cup (0,+\infty) \end{cases} \quad (m \in \mathbb{Z}). \tag{10}$$

Proof. Similar to that of Lemma 1. □

Given the sequence $h = (h_m)_{m\in\mathbb{Z}}$, let $H^D = (H^D_m)_{m\in\mathbb{Z}}$ be a D-primitive of h, i.e. a sequence such that

$$DH^D_m = h_m \quad (m \in \mathbb{Z}), \tag{11}$$

and let $H^\Delta = (H^\Delta_m)_{m\in\mathbb{Z}}$ be a Δ-primitive of h, i.e. a sequence such that

$$\Delta H^\Delta_m = h_m \quad (m \in \mathbb{Z}). \tag{12}$$

For example, one can take

$$H^D_m = \begin{cases} \sum\limits_{k=0}^{m-1} h_k & \text{if} \quad m \geq 1 \\ 0 & \text{if} \quad m = 0 \\ -\sum\limits_{k=m}^{-1} h_k & \text{if} \quad m \leq -1 \end{cases} \quad (m \in \mathbb{Z}), \tag{13}$$

$$H^\Delta_m = \begin{cases} \sum\limits_{k=1}^{m} h_k & \text{if} \quad m \geq 1 \\ 0 & \text{if} \quad m = 0 \\ -\sum\limits_{k=m+1}^{0} h_k & \text{if} \quad m \leq -1 \end{cases} \quad (m \in \mathbb{Z}). \tag{14}$$

Denote by BDP (resp. $B\Delta P$) the space of sequences $h = (h_m)_{m\in\mathbb{Z}}$ whose D-primitives H^D (resp. Δ-primitives H^Δ) belong to $l^\infty_{\mathbb{Z}}$. From the relations

$$H^D_{m+1} = H^\Delta_m + h_0 \quad (m \in \mathbb{Z}),$$

we see that $BDP = B\Delta P$, and we denote this unique space by $BP_{\mathbb{Z}}$. Clearly $BP_{\mathbb{Z}} \subset l^\infty_{\mathbb{Z}}$ and simple examples show that there exist elements of $l^\infty_{\mathbb{Z}}$ not in $BP_{\mathbb{Z}}$.

3. Some Second Order Linear Equations

Let us now consider the second order linear difference equations

$$\Delta(Dx_m) + cDx_m = h_m \quad (m \in \mathbb{Z}), \tag{15}$$

and

$$D(\Delta x_m) + c\Delta x_m = h_m \quad (m \in \mathbb{Z}), \tag{16}$$

where $c \in \mathbb{R}$.

The following results are versions for (15) and (16) of a result of Ortega [6] for differential equations.

Proposition 1. *If $c \notin \{-2, 0\}$, equation (15) has a solution $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ if and only if $h \in BP_\mathbb{Z}$.*

Proof. Necessity. If $x \in l^\infty_\mathbb{Z}$ is a solution of (15), we have, for $m \geq 1$

$$\sum_{k=0}^{m-1} \Delta(Dx_k) + c\sum_{k=0}^{m-1} Dx_k = \sum_{k=0}^{m-1} h_k$$

and hence explicitely

$$(1+c)x_m - x_{m-1} - (1+c)x_0 + x_{-1} = H^D_m \quad (m \geq 1).$$

Similar expressions are obtained for $m \leq -1$, and the sequence defined by the left-hand member belongs to $l^\infty_\mathbb{Z}$.

Sufficiency Let us consider the first order equation

$$\Delta z_m + cz_m = H^D_m \quad (m \in \mathbb{Z}). \tag{17}$$

It follows from Lemma 2 that (17) has a unique solution $z = (z_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$. Now, it follows from (17) that

$$D(\Delta z_m) + cDz_m = DH^D_m \quad (m \in \mathbb{Z})$$

or, equivalently

$$\Delta(Dz_m) + cDz_m = h_m \quad (m \in \mathbb{Z}),$$

which shows that $(x_m)_{m\in\mathbb{Z}} = (Dz_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ is a solution of (15). □

Proposition 2. *If $c \notin \{0, 2\}$, equation (16) has a solution $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ if and only if $h \in BP_\mathbb{Z}$.*

Proof. It is entirely similar to that of Proposition 1. □

4. A Limiting Lemma

In this section, we state and prove an analog for difference equations of a result of M. A. Krasnosel'skií [4] for differential systems (Lemma 8.1, p. 149). For each $m \in \mathbb{Z}$, let $f_m : \mathbb{R} \to \mathbb{R}$ be continuous and let $c \in \mathbb{R}$.

Lemma 3. *Assume that, for each $n \in \mathbb{N}^*$, there exists $x^n = (x^n_m)_{-n-1\leq m\leq n+1} \in l^\infty_{\mathbb{Z},n}$ such that*

$$\Delta(Dx^n_m) + cDx^n_m + f_m(x^n_m) = 0 \quad (-n \leq m \leq n), \tag{18}$$

and assume that there exist $\alpha = (\alpha_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ and $\beta = (\beta_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ such that

$$\alpha_m \leq x^n_m \leq \beta_m \quad (m \in \mathbb{Z}, \quad n \in \mathbb{N}^*). \tag{19}$$

Then there exists $\widehat{x} = (\widehat{x}_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ *such that*

$$\Delta(D\widehat{x}_m) + cD\widehat{x}_m + f_m(\widehat{x}_m) = 0 \quad (m \in \mathbb{Z}), \tag{20}$$

and

$$\alpha_m \le \widehat{x}_m \le \beta_m \quad (m \in \mathbb{Z}). \tag{21}$$

Proof. We use Bolzano–Weierstrass theorem and Cantor diagonal process. By assumption, the sequence $(P_1 x^n)_{n\ge1}$ is bounded in $l^\infty_{\mathbb{Z},1}$ and hence $(x^n)_{n\ge1}$ contains a subsequence $(x^{n,1})_{n\ge1}$ such that $(P_1x^{n,1})_{n\ge1}$ converges. Now the sequence $(P_2x^{n,1})_{n\ge2}$ is bounded in $l^\infty_{\mathbb{Z},2}$ and hence $(x^n)_{n\ge2}$ contains a subsequence $(x^{n,2})_{n\ge2}$ such that $(P_2x^{n,2})_{n\ge2}$ converges. Continuing in the same way, if the subsequence $(x^{n,k})_{n\ge k}$ such that $(P_kx^{n,k})_{n\ge k}$ converges is given, the sequence $(P_{k+1}x^{n,k})_{n\ge k+1}$ is bounded in $l^\infty_{\mathbb{Z},k+1}$ and hence $(x^n)_{n\ge k+1}$ contains a subsequence $(x^{n,k+1})_{n\ge k+1}$ such that $(P_{k+1}x^{n,k+1})_{n\ge k+1}$ converges. Let us now consider the corresponding diagonal subsequence $(x^{n,n})_{n\in\mathbb{N}^*}$. If $q \in \mathbb{N}^*$ is given, $(x^{n,n})_{n\ge q}$ is a subsequence of $(x^{n,q})_{n\ge q}$ and, as $(P_qx^{n,q})_{n\ge q}$ converges, say to $\widehat{x}^q \in l^\infty_{\mathbb{Z},q}$, the same is true for $(P_qx^{n,n})_{n\ge q}$. On the other hand, as

$$P_qP_{q+1}x^{n,n} = P_qx^{n,n} \quad (n \ge q+1),$$

we have $P_q\widehat{x}^{q+1} = \widehat{x}^q$, and there exists a sequence $\widehat{x} = (\widehat{x}_m)_{m\in\mathbb{Z}}$ such that

$$\widehat{x}^q = P_q\widehat{x} \quad (q \in \mathbb{N}^*).$$

Consequently, for each $q \in \mathbb{N}^*$,

$$P_qx^{n,n} \to P_q\widehat{x},$$

or, equivalently,

$$x^{n,n}_m \to \widehat{x}_m \quad (-q-1 \le m \le q+1).$$

From the inequalities

$$\alpha_m \le x^{n,n}_m \le \beta_m \quad (m \in \mathbb{Z}),$$

we deduce (21), and, in particular $\widehat{x} \in l^\infty_{\mathbb{Z}}$. From the relations

$$\Delta(Dx^{n,n}_m) + cDx^{n,n}_m + f_m(x^{n,n}_m) = 0 \quad (-n-1 \le m \le n+1),$$

we deduce (20) by letting $n \to \infty$. □

5. The Method of Lower and Upper Solutions

To develop a method of lower and upper solutions for some nonlinear second order difference equations, we need some preliminary results on the same method for Dirichlet problems, which slightly generalize (with slightly different notations) some results of [2].

Let $n \in \mathbb{N}^*$ fixed and $(x_{-n-1}, \dots, x_{n+1}) \in \mathbb{R}^{2n+3}$. Let c, ξ_{-n-1}, $\xi_{n+1} \in \mathbb{R}$ and $f_m : \mathbb{R} \to \mathbb{R}$ $(-n \le m \le n)$ be continuous functions. We study the existence of solutions for the Dirichlet boundary value problem

$$\begin{aligned} \Delta(Dx_m) + cDx_m + f_m(x_m) &= 0 \quad (-n \le m \le n), \\ x_{-n-1} &= \xi_{-n-1}, \\ x_{n+1} &= \xi_{n+1}. \end{aligned} \tag{22}$$

If $\alpha, \beta \in \mathbb{R}^{2n+3}$, we write $\alpha \le \beta$ if $\alpha_m \le \beta_m$ for all $-n-1 \le m \le n+1$.

Definition 1. $\alpha = (\alpha_{-n-1}, \ldots, \alpha_{n+1})$ (resp. $\beta = (\beta_{-n-1}, \ldots, \beta_{n+1})$) is called a *lower solution* (resp. *upper solution*) for (22) if

$$\begin{aligned} \alpha_{-n-1} \le \xi_{-n-1} \quad &(\text{resp.} \quad \xi_{-n-1} \le \beta_{-n-1}), \\ \alpha_{n+1} \le \xi_{n+1} \quad &(\text{resp.} \quad \xi_{n+1} \le \beta_{n+1}) \end{aligned} \tag{23}$$

and the inequalities

$$\begin{gathered} \Delta(D\alpha_m) + cD\alpha_m + f_m(\alpha_m) \ge 0 \\ (\text{resp.} \quad \Delta(D\beta_m) + cD\beta_m + f_m(\beta_m) \le 0) \quad (-n \le m \le n) \end{gathered} \tag{24}$$

hold.

Theorem 1. *If $c \ge 0$ and if (22) has a lower solution $\alpha = (\alpha_{-n-1}, \ldots, \alpha_{n+1})$ and an upper solution $\beta = (\beta_{-n-1}, \cdots, \beta_{n+1})$ such that $\alpha \le \beta$, then (22) has a solution $x = (x_{-n-1}, \cdots, x_{n+1})$ such that $\alpha \le x \le \beta$.*

Proof. I. A modified problem.
Let $\gamma_m : \mathbb{R} \longrightarrow \mathbb{R}$ $(-n \le m \le n)$ be the continuous functions defined by

$$\gamma_m(x) = \begin{cases} \beta_m & \text{if} \quad x > \beta_m, \\ x & \text{if} \quad \alpha_m \le x \le \beta_m, \\ \alpha_m & \text{if} \quad x < \alpha_m, \end{cases}$$

and define $F_m = f_m \circ \gamma_m$ $(-n \le m \le n)$. We consider the modified problem

$$\begin{aligned} \Delta(Dx_m) + cDx_m + F_m(x_m) - [x_m - \gamma_m(x_m)] &= 0 \quad (-n \le m \le n), \\ x_{-n-1} &= \xi_{-n-1}, \\ x_{n+1} &= \xi_{n+1}, \end{aligned} \tag{25}$$

and show that if $x = (x_{-n-1}, \cdots, x_{n+1})$ is a solution of (25) then $\alpha \le x \le \beta$ and hence x is a solution of (22). Suppose by contradiction that there is some $-n-1 \le i \le n+1$ such that $\alpha_i - x_i > 0$ so that

$$\alpha_m - x_m = \max_{-n-1 \le j \le n+1} (\alpha_j - x_j) > 0.$$

Using the inequalities (23), we obtain that $-n \le m \le n-$. Hence

$$\begin{aligned} &\Delta(D(\alpha_m - x_m)) = (\alpha_{m+1} - x_{m+1}) - 2(\alpha_m - x_m) + (\alpha_{m-1} - x_{m-1}) \le 0, \\ &\quad \alpha_{m+1} - x_{m+1} \le \alpha_m - x_m, \end{aligned}$$

so that

$$\gamma_m(x_m) = \alpha_m, \quad \Delta(D\alpha_m) \le \Delta(Dx_m), \quad D\alpha_m \le Dx_m.$$

Consequently,

$$\begin{aligned} 0 &= \Delta(Dx_m) + cDx_m + F_m(x_m) - [x_m - \gamma_m(x_m)] \\ &\geq \Delta(D\alpha_m) + cD\alpha_m + f_m(\alpha_m) + \alpha_m - x_m \\ &\geq \alpha_m - x_m > 0, \end{aligned}$$

a contradiction. Analogously we can show that $x \leq \beta$.

II. Abstract formulation of problem (25)*.*
We define a continuous mapping $G : \mathbb{R}^{2n+3} \to \mathbb{R}^{2n+3}$ by

$$\begin{aligned} G_{-n-1}(x) &= x_{-n-1} - \xi_{-n-1}, \\ G_m(x) &= \Delta(Dx_m) + cDx_m + F_m(x_m) - [x_m - \gamma_m(x_m)] \\ &\qquad (-n \leq m \leq n), \\ G_{n+1}(x) &= x_{n+1} - \xi_{n+1}. \end{aligned} \tag{26}$$

It is clear that the solutions of (25) are the zeros of G in $\mathbb{R}^{2n+3}$. In order to use the Brouwer degree [3, 5] to study those zeros, we introduce the homotopy $\mathcal{G} : [0,1] \times \mathbb{R}^{2n+3} \to \mathbb{R}^{2n+3}$ defined by

$$\begin{aligned} \mathcal{G}_{-n-1}(\lambda, x) &= x_{-n-1} - \lambda\xi_{-n-1}, \\ \mathcal{G}_m(\lambda, x) &= (1-\lambda)[\Delta(Dx_m) + cDx_m - x_m] + \lambda G_m(x) \\ &= \Delta(Dx_m) + cDx_m - x_m + \lambda[F_m(x_m) + \gamma_m(x_m)] \\ &\qquad (-n \leq m \leq n), \\ \mathcal{G}_{n+1}(\lambda, x) &= x_{n+1} - \lambda\xi_{n+1}. \end{aligned} \tag{27}$$

Notice that $\mathcal{G}(1, \cdot) = G$ and that $\mathcal{G}(0, \cdot)$ is linear.

III. A priori estimates for the possible zeros of $\mathcal{G}$*.*
Let R be any number such that

$$R > \max\Big\{ \max_{-n \leq m \leq n} \sup_{x \in \mathbb{R}} |F_m(x) + \gamma_m(x)|, |\xi_{-n-1}|, |\xi_{n+1}| \Big\}, \tag{28}$$

and let $(\lambda, x_{-n-1}, \cdots, x_{n+1}) \in [0,1] \times \mathbb{R}^{2n+3}$ be a possible zero of $\mathcal{G}$. One has $|x_{-n-1}| < R$ and $|x_{n+1}| < R$. If $0 \leq x_m = \max_{-n-1 \leq j \leq n+1} x_j$ is reached for some $-n \leq m \leq n$, then $\Delta(Dx_m) \leq 0$, and $Dx_m \leq 0$. Hence,

$$0 \geq \Delta(Dx_m) + cDx_m = x_m - \lambda[F_m(x_m) + \gamma_m(x_m)],$$

which implies

$$x_m \leq \sup_{x \in \mathbb{R}} |F_m(x) + \gamma_m(x)| < R.$$

Analogously it can be shown that $-R < \min_{-n \leq j \leq n} x_j$, and hence

$$\max_{-n-1 \leq j \leq n+1} |x_j| < R \tag{29}$$

for each possible zero (λ, x) of $\mathcal{G}$.

IV. The existence of a zero for G.
Using the results of Parts II, III and the invariance under homotopy of the

Brouwer degree, we see that the Brouwer degree $d[\mathcal{G}(\lambda,\cdot), B_R(0), 0]$ is well defined and independent of $\lambda \in [0,1]$. But $\mathcal{G}(0,\cdot)$ is a linear mapping whose set of solutions is bounded, and hence equal to $\{0\}$. Consequently, $|d[\mathcal{G}(0,\cdot), B_R(0), 0]| = 1$, so that $|d[G, B_R(0), 0]| = 1$ and the existence property of the Brouwer degree implies the existence of at least one zero of G, and hence of a solution of (22). □

Remark 1. One can prove a result similar to Theorem 1 for the Dirichlet problem

$$\begin{aligned} \Delta(Dx_m) + c\Delta x_m + f_m(x_m) = 0 \quad (-n+1 \le m \le n-1), \\ x_{-n-1} = \xi_{-n-1}, \\ x_{n+1} = \xi_{n+1}, \end{aligned} \tag{30}$$

when $c \le 0$.

Let now $f_m : \mathbb{R} \to \mathbb{R}$ $(m \in \mathbb{Z})$ be continuous functions. We study the existence of bounded solutions of the nonlinear second order difference equation

$$\Delta(Dx_m) + cDx_m + f_m(x_m) = 0 \quad (m \in \mathbb{Z}). \tag{31}$$

If $\alpha, \beta \in l^\infty_{\mathbb{Z}}$, we write $\alpha \le \beta$ if $\alpha_m \le \beta_m$ for all $m \in \mathbb{Z}$.

Definition 2. $\alpha = (\alpha_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ (resp. $\beta = (\beta_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$) is called a *lower solution* (resp. *upper solution*) for (31) if the inequalities

$$\begin{aligned} \Delta(D\alpha_m) + cD\alpha_m + f_m(\alpha_m) \ge 0 \\ (\text{resp.} \quad \Delta(D\beta_m) + cD\beta_m + f_m(\beta_m) \le 0) \quad (m \in \mathbb{Z}) \end{aligned} \tag{32}$$

hold.

Theorem 2. *If $c \ge 0$ and (31) has a lower solution $\alpha = (\alpha_m)_{m\in\mathbb{Z}}$ and an upper solution $\beta = (\beta_m)_{m\in\mathbb{Z}}$ such that $\alpha \le \beta$, then (31) has a solution $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ such that $\alpha \le x \le \beta$.*

Proof. For each $n \in \mathbb{N}^*$, let us consider the Dirichlet problem

$$\begin{aligned} \Delta(Dx_m) + cDx_m + f_m(x_m) = 0 \quad (-n \le m \le n) \\ x_{-n-1} = \alpha_{-n-1}, \\ x_{n+1} = \alpha_{n+1}. \end{aligned} \tag{33}$$

It follows from Theorem 1 that (33) has at least one solution $\widetilde{x}^n = (\widetilde{x}^n_m)_{m\in\mathbb{Z}}$ such that

$$\alpha_m \le \widetilde{x}^n_m \le \beta_m \quad (-n-1 \le m \le n+1).$$

Consequently, the sequence $x^n = (x^n_m)_{m\in\mathbb{Z}} \in l^\infty_{\mathbb{Z},n}$ defined by

$$x^n_m = \begin{cases} \widetilde{x}^n_m & \text{if } |m| \le n+1, \\ 0 & \text{if } |m| > n+1 \end{cases}$$

satisfies conditions (19) and (20) of Lemma 3 and the conclusion follows. □

The case of constant lower and upper solutions gives the following simple existence condition.

Corollary 1. *Assume that* $c \geq 0$ *and that there exist two real numbers* $\alpha \leq \beta$ *such that*

$$f_m(\beta) \leq 0 \leq f_m(\alpha) \quad (m \in \mathbb{Z}). \tag{34}$$

Then equation (31) *has at least one solution* $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ *such that*

$$\alpha \leq x_m \leq \beta \quad (m \in \mathbb{Z}).$$

Remark 2. One can prove a variant of Theorem 2 for the bounded solutions of equation

$$\Delta(Dx_m) + c\Delta x_m + g_m(x_m) = 0 \quad (m \in \mathbb{Z})$$

when $c \leq 0$.

6. An Existence Condition for Duffing-Type Difference Equations

Let $c \in \mathbb{R}$, $g_m : \mathbb{R} \to \mathbb{R}$ be continuous functions ($m \in \mathbb{Z}$), and $h = (h_m)_{m\in\mathbb{Z}}$ be a sequence. We consider the existence of solutions $x = (x_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ of the second order nonlinear difference equation

$$\Delta(Dx_m) + cDx_m + g_m(x_m) = h_m \quad (m \in \mathbb{Z}). \tag{35}$$

The following result is a nonlinear extension of Lemma 1.

Theorem 3. *Assume that* $c > 0$ *and that there exists* $r > 0$ *such that, for each* $m \in \mathbb{Z}$, $g_m(s) \geq 0$ *for all* $s \leq -r$ *and* $g_m(s) \leq 0$ *for all* $s \geq r$. *Then equation* (35) *has at least one solution for each* $h = (h_m)_{m\in\mathbb{Z}} \in BP_\mathbb{Z}$.

Proof. From Proposition 1, there exists a solution $u = (x_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ of equation

$$\Delta(Dx_m) + cDx_m = h_m \quad (m \in \mathbb{Z}). \tag{36}$$

Letting $x_m = x_m + z_m$, the problem is reduced to find a solution $z = (z_m)_{m\in\mathbb{Z}} \in l^\infty_\mathbb{Z}$ of equation

$$\Delta(Dz_m) + cDz_m + g_m(x_m + z_m) = 0 \quad (m \in \mathbb{Z}). \tag{37}$$

Take $\alpha = -r - \|u\|_\infty$ and $\beta = r + \|u\|_\infty$. Then, for each $m \in \mathbb{Z}$,

$$x_m + \alpha = -r + x_m - \|u\|_\infty \leq -r,$$
$$x_m + \beta = r + \|u\|_\infty + x_m \geq r,$$

so that

$$g_m(x_m + \alpha) \geq 0 \geq g_m(x_m + \beta) \quad (m \in \mathbb{Z}).$$

The result follows from Corollary 1. □

Remark 3. A similar result holds for the equation

$$\Delta(Dx_m) + c\Delta x_m + g_m(x_m) = 0 \quad (m \in \mathbb{Z})$$

when $c < 0$.

Example 1. If $a \leq 0$, $b \geq 0$, $c > 0, \gamma > 0$, $\delta \geq 0$, the equation

$$\Delta(Dx_m) + cDx_m + \frac{ax_m|x_m|^{\gamma-1}}{1+b|x_m|^\delta} = h_m \quad (m \in \mathbb{Z})$$

has a solution $x = (x_m)_{m\in\mathbb{Z}}$ for each $h = (h_m)_{m\in\mathbb{Z}} \in BP_{\mathbb{Z}}$.

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