

## SINGULAR DIRICHLET BOUNDARY VALUE PROBLEM FOR SECOND ORDER ODE

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*Cordially dedicated to Professor Ivan Kiguradze  
on his 70th birthday anniversary*

**Abstract.** This paper investigates the singular Dirichlet problem

$$-u'' = f(t, u, u'), \quad u(0) = 0, \quad u(T) = 0,$$

where  $f$  satisfies the Carathéodory conditions on the set  $(0, T) \times \mathbb{R}_0^2$  and  $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ .

The function  $f(t, x, y)$  can have time singularities at  $t = 0$  and  $t = T$  and space singularities at  $x = 0$  and  $y = 0$ . The existence principle for the above problem is given and its application is presented here. The paper provides conditions which guarantee the existence of a solution which is positive on  $(0, T)$  and which has the absolutely continuous first derivative on  $[0, T]$ .

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### 1. INTRODUCTION

Let  $[0, T] \subset \mathbb{R}$  and  $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ . We will investigate the solvability of the problem

$$-u'' = f(t, u, u'), \tag{1.1}$$

$$u(0) = 0, \quad u(T) = 0, \tag{1.2}$$

where  $f$  satisfies the Carathéodory conditions on  $(0, T) \times \mathbb{R}_0^2$  and  $f(t, x, y)$  can have time singularities at  $t = 0$ ,  $t = T$  and space singularities at  $x = 0$  and  $y = 0$ .

**Definition 1.1.** We say that  $f$  has a *time singularity* at  $t = 0$  ( $t = T$ ) if there exist  $x, y \in \mathbb{R}_0$  such that

$$\int_0^\varepsilon |f(t, x, y)| dt = \infty \quad \left( \int_{T-\varepsilon}^T |f(t, x, y)| dt = \infty \right)$$

for any sufficiently small  $\varepsilon > 0$ .

**Definition 1.2.** We say that  $f$  has a *space singularity* at  $x = 0$  ( $y = 0$ ) if there exists a set  $J \subset [0, T]$  with a positive Lebesgue measure such that the

condition

$$\limsup_{x \rightarrow 0} |f(t, x, y)| = \infty \quad \left( \limsup_{y \rightarrow 0} |f(t, x, y)| = \infty \right)$$

holds for a.e.  $t \in J$  and some  $y \in \mathbb{R}_0$  ( $x \in \mathbb{R}_0$ ).

In what follows we will use the notation:

$[a, b] \subset \mathbb{R}$ ;  $\Sigma \subset (a, b)$  is a finite set;  $\mathcal{M} \subset \mathbb{R}^2$ ;

$C[a, b]$  is the Banach space of functions continuous on  $[a, b]$  with the norm  $\|x\|_C = \max\{|x(t)|; t \in [a, b]\}$ ;

$C^1[a, b]$  is the Banach space of functions having continuous first derivatives on  $[a, b]$  with the norm  $\|x\|_{C^1} = \|x\|_C + \|x'\|_C$ ;

$AC^1[a, b]$  is the set of functions having absolutely continuous derivatives on  $[a, b]$ ;

$AC^1_{loc}((a, b) \setminus \Sigma)$  is the set of functions  $x \in AC^1[c, d]$  for each  $[c, d] \subset [a, b] \setminus \Sigma$ ;

$L[a, b]$  is the Banach space of functions Lebesgue integrable on  $[a, b]$  with the norm  $\|x\|_L = \int_a^b |x(t)| dt$ ;

$\text{Car}([a, b] \times \mathcal{M})$  is the set of functions  $f : [a, b] \times \mathcal{M} \rightarrow \mathbb{R}$  satisfying the Carathéodory conditions on  $[a, b] \times \mathcal{M}$ , i.e.,

$$f(\cdot, x, y) : [a, b] \rightarrow \mathbb{R} \text{ is measurable for all } (x, y) \in \mathcal{M};$$

$$f(t, \cdot, \cdot) : \mathcal{M} \rightarrow \mathbb{R} \text{ is continuous for a.e. } t \in [a, b];$$

for each compact set  $\mathcal{K} \subset \mathcal{M}$  there is a function  $m_{\mathcal{K}} \in L[a, b]$  such that

$$|f(t, x, y)| \leq m_{\mathcal{K}}(t) \text{ for a.e. } t \in [a, b] \text{ and all } (x, y) \in \mathcal{K}.$$

$\text{Car}((a, b) \times \mathcal{M})$  is the set of functions  $f \in \text{Car}([c, d] \times \mathcal{M})$  for each  $[c, d] \subset (a, b)$ ;

$\text{meas } \mathcal{A}$  is the Lebesgue measure of  $\mathcal{A} \subset \mathbb{R}$ .

We say that a sequence  $\{v_n\} \subset C[0, T]$  is equicontinuous on  $[0, T]$  if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that for each  $t_1, t_2 \in [0, T]$

$$|t_1 - t_2| < \delta \quad \Rightarrow \quad |v_n(t_1) - v_n(t_2)| < \varepsilon$$

for each  $n \in \mathbb{N}$ .

**Definition 1.3.** By a *solution* of problem (1.1), (1.2) we understand a function  $u \in AC^1[0, T]$  satisfying equation (1.1) a.e. on  $[0, T]$  and satisfying conditions (1.2).

In literature we can find an alternative approach to the solvability of singular problems where solutions are defined as continuous functions whose first derivatives can have discontinuities at some points in  $[0, T]$ . Here we will call such functions *w-solutions* and according to [4] or [11] we will define them as follows:

**Definition 1.4.** We say that  $u \in C[0, T]$  is a *w-solution* of (1.1), (1.2) if there exists a finite set  $\Sigma \subset (0, T)$  such that  $u \in AC^1_{loc}((0, T) \setminus \Sigma)$  satisfies equation (1.1) a.e. on  $[0, T]$  and satisfies conditions (1.2).

A systematic study of the solvability of Dirichlet problems having both time and space singularities was initiated by Taliaferro [21]. Now, we can find a large group of works which focused their attention on the existence of  $w$ -solutions, e.g. [1]–[4], [10]–[15], and a less number of works which provide also conditions for the existence of solutions, e.g. [5], [7], [8], [16], [17], [22]. All the above works deal with differential equations where the nonlinearity  $f(t, x, y)$  has a space singularity at  $x = 0$  and/or time singularities at  $t = 0$ ,  $t = T$ . The first existence result for the Dirichlet problem where  $f(t, x, y)$  has singularities at both variables  $x$  and  $y$  was obtained by Staněk [20]. He assumed that  $f$  is strictly positive and its behaviour in a right neighbourhood of the singular point  $x = 0$  is controlled by a function  $\omega_0(x)$  which is integrable. Then we say that  $f$  has a weak space singularity at  $x = 0$ . Here, we extend the existence result of [20] to  $f$  with a strong space singularity at  $x = 0$ , i.e., we consider  $f$  which is controlled by a nonintegrable function  $\omega_0(x)$ . Our main result is Theorem 2.2.

In our proofs we will need the following Fredholm type existence theorem:

**Theorem 1.5** (Fredholm type existence theorem, [23]). *Let  $h \in \text{Car}([a, b] \times \mathbb{R}^2)$  and  $m \in L[a, b]$  be such that*

$$|h(t, x, y)| \leq m(t) \text{ for a.e. } t \in [a, b] \text{ and all } x, y \in \mathbb{R} .$$

*Then the problem*

$$-u'' = h(t, u, u'), \quad u(a) = u(b) = 0$$

*has a solution  $u \in AC^1[a, b]$ .*

We approximate the singular equation (1.1) by a sequence of regular equations

$$-u'' = f_n(t, u, u'), \tag{1.3}$$

where  $f_n \in \text{Car}([0, T] \times \mathbb{R}^2)$ ,  $n \in \mathbb{N}$ .

Having a sequence  $\{u_n\}$  of solutions of problems (1.3), (1.2) we need to prove the existence of its converging subsequence. A type of this convergence defines the properties of its limit  $u$  and, at the same time, it determines if  $u$  is a solution (or a  $w$ -solution) of the original problem (1.1), (1.2). The investigation of convergence is based on next two theorems.

**Theorem 1.6** (Arzelà–Ascoli theorem in  $C[a, b]$  and  $C^1[a, b]$ , [9]).  *$A \subset C[a, b]$  is relatively compact if and only if  $A$  is bounded in  $C[a, b]$  and functions in  $A$  are equicontinuous on  $[a, b]$ .*

*$B \subset C^1[a, b]$  is relatively compact if and only if  $B$  is bounded in  $C^1[a, b]$  and the first derivatives of functions in  $B$  are equicontinuous on  $[a, b]$ .*

**Theorem 1.7** (Fatou lemma, [19]). *Let  $\varphi_n \in L[a, b]$  for  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \varphi_n(t) = \varphi(t)$  a.e. on  $[a, b]$ . Assume that there exists  $c \in (0, \infty)$  such that*

$$\int_a^b |\varphi_n(t)| dt \leq c \text{ for each } n \in \mathbb{N} .$$

Then  $|\varphi| \in L[a, b]$  and  $\int_a^b |\varphi(t)| dt \leq c$ . (Clearly,  $\varphi \in L[a, b]$ , as well.)

The paper is organized as follows. In Section 2 we provide the existence principle (Theorem 2.2) giving the properties of approximating regular functions  $f_n$  in order to find a sequence of approximate solutions converging to a solution (or to a  $w$ -solution) of problem (1.1), (1.2). In Section 3 we apply the existence principle of Section 2 to get conditions which are imposed on  $f$  directly and yield the solvability of problem (1.1), (1.2).

## 2. EXISTENCE PRINCIPLE FOR SINGULAR DIRICHLET PROBLEMS

**Theorem 2.1.** Assume that  $f \in \text{Car}((0, T) \times \mathbb{R}_0^2)$ ,  $f_n \in \text{Car}([0, T] \times \mathbb{R}^2)$ ,  
 $f_n(t, x, y) = f(t, x, y)$  for a.e.  $t \in \Delta_n$  and

$$\text{each } |x| \geq \frac{1}{n}, |y| \geq \frac{1}{n}, n \in \mathbb{N}, \quad (2.1)$$

$$\text{where } \Delta_n = \left[ \frac{1}{n}, T - \frac{1}{n} \right] \cap [0, T];$$

there exists a bounded set  $\Omega \subset C^1[0, T]$  such that for each  
 $n \in \mathbb{N}$ , problem (1.3), (1.2) has a solution  $u_n \in \Omega$ . (2.2)

Then there exist  $u \in C[0, T]$  and a subsequence  $\{u_k\} \subset \{u_n\}$  such that

$$\lim_{k \rightarrow \infty} \|u_k - u\|_C = 0. \quad (2.3)$$

Assume in addition that there exists a finite set  $\Sigma = \{s_1, \dots, s_\nu\} \subset (0, T)$  such that

on each interval  $[a, b] \subset (0, T) \setminus \Sigma$  the sequence  $\{u'_k\}$  is equicontinuous. (2.4)

Then  $u \in C^1((0, T) \setminus \Sigma)$  and

$$\lim_{k \rightarrow \infty} u'_k(t) = u'(t) \text{ locally uniformly on } (0, T) \setminus \Sigma. \quad (2.5)$$

*Proof.* By (2.2) there exist  $r > 0$  and a sequence  $\{u_n\}$  of solutions of (1.3), (1.2) such that

$$\|u_n\|_{C^1} \leq r \text{ for each } n \in \mathbb{N}. \quad (2.6)$$

Therefore the sequence  $\{u_n\}$  is bounded in  $C[0, T]$  and equicontinuous on  $[0, T]$ . By Theorem 1.6 (Arzelà–Ascoli) we can choose a subsequence  $\{u_l\}$  such that

$$\lim_{l \rightarrow \infty} \|u_l - u\|_C = 0, \quad u \in C[0, T]. \quad (2.7)$$

Now assume also (2.4) and choose an interval  $[a, b] \subset (0, T) \setminus \Sigma$  arbitrarily. Then  $\{u'_l\}$  is equicontinuous on  $[a, b]$ . By (2.6) the sequence  $\{u_l\}$  is bounded in  $C^1[a, b]$ . Theorem 1.6 implies that we can choose a subsequence  $\{u_k\} \subset \{u_l\}$  such that

$$\lim_{k \rightarrow \infty} u'_k(t) = u'(t) \text{ uniformly on } [a, b].$$

By virtue of (2.7) the sequence  $\{u_k\}$  satisfies (2.3). Using the diagonalization method we can choose  $\{u_k\}$  such that (2.5) holds too. □

**Theorem 2.2** (Existence principle for problem (1.1), (1.2)). *Let all assumptions of Theorem 2.1 be fulfilled. Let the finite set  $\Sigma$  have the form:*

$$\Sigma = \{s \in (0, T) : u(s) = 0 \text{ or } u'(s) = 0 \text{ or } u'(s) \text{ does not exist.}\} \quad (2.8)$$

Then  $u \in AC_{loc}^1((0, T) \setminus \Sigma)$  is a  $w$ -solution of (1.1), (1.2).

Denote  $s_0 = 0$  and  $s_{\nu+1} = T$ . If  $u$  and  $\{u_k\}$  satisfy (2.3), (2.5) and, besides, there exist  $\eta \in (0, \frac{T}{2})$ ,  $\lambda_0, \mu_0, \lambda_1, \mu_1, \dots, \lambda_{\nu+1}, \mu_{\nu+1} \in \{-1, 1\}$  and  $\psi \in L[0, T]$  such that

$$\begin{aligned} \lambda_i f_k(t, u_k(t), u'_k(t)) &\geq \psi(t) \text{ for a.e. } t \in (s_i - \eta, s_i) \cap (0, T) \\ \mu_i f_k(t, u_k(t), u'_k(t)) &\geq \psi(t) \text{ for a.e. } t \in (s_i, s_i + \eta) \cap (0, T), \\ &\text{for all } i \in \{0, \dots, \nu + 1\}, \quad k \in \mathbb{N}, \end{aligned} \quad (2.9)$$

then  $u \in AC^1[0, T]$  is a solution of (1.1), (1.2).

*Proof.* **1.** Let (2.1), (2.2), (2.4) and (2.8) be true. Then for  $k \in \mathbb{N}$

$$\begin{aligned} -u''_k(t) &= f_k(t, u_k(t), u'_k(t)) \text{ for a.e. } t \in [0, T], \\ u_k(0) &= 0, \quad u_k(T) = 0, \end{aligned} \quad (2.10)$$

and by Theorem 2.1, there exists  $u \in C[0, T]$  such that (2.3) and (2.5) hold. By (2.3),  $u$  satisfies (1.2).

Define the sets

$$\begin{aligned} V_1 &= \{t \in (0, T) : f(t, \cdot, \cdot) : \mathbb{R}_0^2 \rightarrow \mathbb{R} \text{ is not continuous}\}, \\ V_2 &= \{t \in (0, T) : \text{the equality in (2.1) is not satisfied}\} \end{aligned}$$

and let

$$U = (0, T) \setminus (\Sigma \cup V_1 \cup V_2).$$

We see that

$$\text{meas}(\Sigma \cup V_1 \cup V_2) = 0. \quad (2.11)$$

Choose an arbitrary  $t \in U$ . Then there exists  $k_0 \in \mathbb{N}$ , such that for each  $k \in \mathbb{N}$ ,  $k \geq k_0$ :

$$t \in \Delta_k, \quad |u_k(t)| > \frac{1}{k}, \quad |u'_k(t)| > \frac{1}{k}$$

and

$$f_k(t, u_k(t), u'_k(t)) = f(t, u_k(t), u'_k(t)).$$

Since  $t$  is an arbitrary element of  $U$ , by (2.3), (2.5) and (2.11) we get

$$\lim_{k \rightarrow \infty} f_k(t, u_k(t), u'_k(t)) = f(t, u(t), u'(t)) \text{ a.e. on } [0, T]. \quad (2.12)$$

Now choose an arbitrary interval  $[a, b] \subset (0, T) \setminus \Sigma$  and integrate equation (2.10). We get

$$-u'_k(t) + u'_k(a) = \int_a^t f_k(s, u_k(s), u'_k(s)) ds \text{ for each } t \in [a, b]. \quad (2.13)$$

Moreover, there exists  $k^* \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ ,  $k \geq k^*$

$$|f_k(t, u_k(t), u'_k(t))| \leq m(t) \text{ for a.e. } t \in [a, b],$$

where

$$m(t) = \sup \left\{ |f(t, x, y)| : \frac{1}{k^*} \leq |x| \leq r, \frac{1}{k^*} \leq |y| \leq r \right\} \in L[a, b].$$

Since  $m \in L[a, b]$  we can apply Lebesgue convergence theorem on  $[a, b]$  and get  $f(\cdot, u(\cdot), u'(\cdot)) \in L[a, b]$ . Moreover,

$$\lim_{k \rightarrow \infty} \int_a^b f_k(s, u_k(s), u'_k(s)) \, ds = \int_a^b f(s, u(s), u'(s)) \, ds,$$

which by (2.13) yields.

$$-u'(t) + u'(a) = \int_a^t f(s, u(s), u'(s)) \, ds \text{ for each } t \in [a, b]. \quad (2.14)$$

Since  $[a, b]$  is an arbitrary interval in  $(0, T) \setminus \Sigma$ , we get that  $u \in AC_{loc}^1((0, T) \setminus \Sigma)$  is a  $w$ -solution of (1.1), (1.2).

**2.** Now assume that there exist  $\eta \in (0, \frac{T}{2})$ ,  $\lambda_0, \mu_0, \lambda_1, \mu_1, \dots, \lambda_{\nu+1}, \mu_{\nu+1} \in \{-1, 1\}$  and  $\psi \in L[0, T]$  such that (2.9) holds. Since  $u$  is a  $w$ -solution of (1.1), (1.2), it remains to prove that  $u \in AC^1[0, T]$ .

Choose  $i \in \{0, \dots, \nu+1\}$  and denote  $(c_i, d_i) = (s_i - \eta, s_i) \cap (0, T)$ . For  $k \in \mathbb{N}$  and for a.e.  $t \in (c_i, d_i)$  we denote

$$h_k(t) = \lambda_i f_k(t, u_k(t), u'_k(t)) + |\psi(t)|, \quad h(t) = \lambda_i f(t, u(t), u'(t)) + |\psi(t)|.$$

Then  $h_k \in L[c_i, d_i]$  and according to (2.12) we have

$$\lim_{k \rightarrow \infty} h_k(t) = h(t) \text{ for a.e. } t \in [c_i, d_i].$$

Integrating (2.10) on  $[c_i, d_i]$  we get

$$\int_{c_i}^{d_i} f_k(s, u_k(s), u'_k(s)) \, ds = -u'_k(d_i) + u'_k(c_i).$$

Therefore, by (2.6) and (2.9)

$$\begin{aligned} \int_{c_i}^{d_i} |h_k(s)| \, ds &= \int_{c_i}^{d_i} h_k(s) \, ds = \lambda_i \int_{c_i}^{d_i} f_k(s, u_k(s), u'_k(s)) \, ds \\ &+ \int_{c_i}^{d_i} |\psi(s)| \, ds \leq |u'_k(d_i)| + |u'_k(c_i)| + \int_{c_i}^{d_i} |\psi(s)| \, ds \leq c, \end{aligned}$$

where  $c = 2r + \|\psi\|_L$ .

Theorem 1.7 (Fatou) implies that  $h \in L[c_i, d_i]$  and  $f(\cdot, u(\cdot), u'(\cdot)) \in L[c_i, d_i]$ .

If  $(c_i, d_i) = (s_i, s_i + \eta) \cap (0, T)$  we argue similarly.

Hence  $f(\cdot, u(\cdot), u'(\cdot)) \in L[0, T]$  and the equality in (2.14) is fulfilled for each  $t \in [0, T]$  and  $u \in AC^1[0, T]$ . We have proved that  $u$  is a solution of (1.1), (1.2). □

### 3. APPLICATION OF EXISTENCE PRINCIPLE

The main result of this section is Theorem 3.1, where we present the conditions sufficient for the existence of a solution of problem (1.1), (1.2) which is positive on  $(0, T)$ .

Now, let us state our assumptions on problem (1.1), (1.2). We are interested in the existence of a positive solution and hence we investigate problem (1.1), (1.2) on the set  $[0, T] \times [0, \infty) \times \mathbb{R}$ . Denote  $\mathcal{D} = (0, \infty) \times \mathbb{R}_0$ . We assume that  $f \in \text{Car}([0, T] \times \mathcal{D})$  has space singularities at  $x = 0$  and  $y = 0$ , particularly

$$\begin{cases} \limsup_{x \rightarrow 0^+} f(t, x, y) = \infty & \text{for a.e. } t \in [0, T] \text{ and for some } y \in \mathbb{R}_0, \\ \limsup_{y \rightarrow 0} f(t, x, y) = \infty & \text{for a.e. } t \in [0, T] \text{ and for some } x \in (0, \infty). \end{cases} \quad (3.1)$$

**Theorem 3.1.** *Let (3.1) hold and let  $c, \gamma, \delta \in (0, \infty)$ ,  $\alpha, \beta \in [0, 1]$ . Assume that there exist positive and nonincreasing functions  $\omega_0, \omega_1 \in C(0, \infty)$  and non-negative functions  $h_0, h_1, h_2 \in L[0, T]$  satisfying*

$$\int_0^T (t^\gamma + t^\delta) \omega_0(t) dt < \infty, \quad \int_0^T \omega_1(t) dt < \infty. \quad (3.2)$$

$$\begin{aligned} T\|h_1\|_L + \|h_2\|_L &< 1 && \text{for } \alpha, \beta = 1; \\ T\|h_1\|_L &< 1 && \text{for } \alpha = 1, \beta < 1; \\ \|h_2\|_L &< 1 && \text{for } \alpha < 1, \beta = 1; \end{aligned} \quad (3.3)$$

$$\begin{aligned} c \leq f(t, x, y) \leq t^\gamma(T - t)^\delta \omega_0(x) + \omega_1(|y|) + h_0(t) + h_1(t)x^\alpha + h_2(t)|y|^\beta \\ \text{for a.e. } t \in [0, T], \text{ and all } x \in (0, \infty), y \in \mathbb{R}_0. \end{aligned} \quad (3.4)$$

Then problem (1.1), (1.2) has a solution positive on  $(0, T)$ .

The proof of Theorem 3.1 is based on the Existence principle (Theorem 2.2), where the existence of a bounded set  $\Omega$  is necessary. Therefore we first prove a priori estimates for a class of functions which will be needed for the construction of such a set  $\Omega$ .

**Lemma 3.2.** *Let  $c > 0$ . Then there exists  $\eta > 0$  such that for each  $u \in AC^1[0, T]$  satisfying (1.2) and*

$$c \leq -u''(t) \quad \text{for a.e. } t \in [0, T], \quad (3.5)$$

the estimate  $\|u\|_C \geq \eta$  is valid.

*Proof.* Let  $G(t, s)$  be the Green function of problem (1.2),  $-u''(t) = 0$ .

Then

$$G(t, s) = \begin{cases} \frac{t(T-s)}{T}, & t \leq s, \\ \frac{s(T-t)}{T}, & s \leq t. \end{cases}$$

We define

$$\Phi(t, s) = \frac{G(t, s)}{t(T-t)} \quad \text{for } (t, s) \in (0, T) \times (0, T) .$$

For any  $s \in (0, T)$  we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} \Phi(t, s) &= \lim_{t \rightarrow 0^+} \frac{t(T-s)}{t(T-t)} \frac{1}{T} = \frac{T-s}{T^2} , \\ \lim_{t \rightarrow T^-} \Phi(t, s) &= \lim_{t \rightarrow T^-} \frac{s(T-t)}{t(T-t)} \frac{1}{T} = \frac{s}{T^2} , \end{aligned}$$

thus we can extend  $\Phi(t, s)$  continuously to  $[0, T]$  and for every  $s \in (0, T)$  we have  $\Phi(t, s) > 0$  for  $t \in [0, T]$ .

We can define

$$F(t) = \int_0^T \Phi(t, s) ds \quad \text{for } t \in [0, T] .$$

For every  $t \in [0, T]$  there exists  $d_0 > 0$  such that  $d_0 \leq cF(t)$ . From the equation  $-u'' = -u''$  we have

$$\begin{aligned} u(t) &= - \int_0^T G(t, s) u''(s) ds \geq \int_0^T G(t, s) c ds \\ &= t(T-t)c \int_0^T \Phi(t, s) ds = t(T-t)cF(t) \geq t(T-t)d_0 , \\ \|u\|_C &\geq u\left(\frac{T}{2}\right) \geq \frac{T^2 d_0}{4} = \eta . \end{aligned} \quad \square$$

**Lemma 3.3.** *Let  $c, \gamma, \delta > 0$ ,  $\alpha, \beta \in [0, 1]$ , the functions  $\omega_0, \omega_1, h_0, h_1, h_2$  satisfy assumptions (3.2) and (3.3). Then there exists  $r > 1$  such that for each  $u \in AC^1[0, T]$  satisfying (1.2), (3.5) and*

$$\begin{aligned} -u''(t) &\leq (\omega_0(1) + \omega_0(u(t))) t^\gamma (T-t)^\delta + \omega_1(1) + \omega_1(|u'(t)|) + h_0(t) \\ &\quad + h_1(t) ((u(t))^\alpha + 1) + h_2(t) (|u'(t)|^\beta + 1) \end{aligned} \quad (3.6)$$

the estimate  $\|u\|_{C^1} \leq r$  is valid.

*Proof.* Condition (3.5) implies that  $u$  is nonnegative, concave and that there exists  $t_0 \in [0, T]$ , such that  $u'(t_0) = 0$ .

By Lemma 3.2 there exists  $\eta > 0$  such that

$$\eta \frac{t}{T} \leq \eta \frac{t}{t_0} \leq u(t) \quad \text{for } t \in [0, t_0] , \quad (3.7)$$

$$\eta \frac{T-t}{T} \leq \eta \frac{T-t}{T-t_0} \leq u(t) \quad \text{for } t \in [t_0, T] . \quad (3.8)$$



Since  $\omega_0$  is a nonincreasing function, we have

$$\begin{aligned} \int_0^T t^\gamma (T-t)^\delta \omega_0(u(t)) dt &\leq \int_0^{t_0} t^\gamma (T-t)^\delta \omega_0(u(t)) dt + \int_{t_0}^T t^\gamma (T-t)^\delta \omega_0(u(t)) dt \\ &\leq \int_0^{t_0} t^\gamma (T-t)^\delta \omega_0\left(\frac{\eta t}{T}\right) dt + \int_{t_0}^T t^\gamma (T-t)^\delta \omega_0\left(\frac{\eta(T-t)}{T}\right) dt \\ &\leq T^\delta \int_0^{t_0} t^\gamma \omega_0\left(\frac{\eta t}{T}\right) dt + T^\gamma \int_{t_0}^T (T-t)^\delta \omega_0\left(\frac{\eta(T-t)}{T}\right) dt. \end{aligned}$$

Without loss of generality we can assume that  $\eta < T$  and, by (3.2), using the substitution  $z = \frac{\eta t}{T}$  into the first integral and  $T-z = \frac{\eta(T-t)}{T}$  into the second integral we get

$$\begin{aligned} &\int_0^T t^\gamma (T-t)^\delta \omega_0(u(t)) dt \\ &\leq T^\delta \frac{T}{\eta} \int_0^{\frac{\eta t_0}{T}} \left(\frac{Tz}{\eta}\right)^\gamma \omega_0(z) dz + T^\gamma \frac{T}{\eta} \int_{T-\frac{\eta(T-t_0)}{T}}^T \left(\frac{T}{\eta}(T-z)\right)^\delta \omega_0(T-z) dz \\ &\leq \frac{T^{\delta+1}}{\eta} \left(\frac{T}{\eta}\right)^\gamma \int_0^{\frac{\eta t_0}{T}} z^\gamma \omega_0(z) dz + \frac{T^{\gamma+1}}{\eta} \left(\frac{T}{\eta}\right)^\delta \int_{T-\frac{\eta(T-t_0)}{T}}^T (T-z)^\delta \omega_0(T-z) dz \\ &\leq \frac{T^{\gamma+\delta+1}}{\eta^2} \int_0^T (z^\gamma + z^\delta) \omega_0(z) dz = A < \infty, \end{aligned}$$

where the constant  $A$  is independent of the function  $u$ .

Integrating  $c \leq -u''$  on  $[t_0, t]$  we get

$$c(t_0 - t) \leq u'(t) = |u'(t)| \text{ for } t \in [0, t_0], \quad (3.9)$$

$$c(t - t_0) \leq -u'(t) = |u'(t)| \text{ for } t \in [t_0, T]. \quad (3.10)$$

Since  $\omega_1$  is nonincreasing, we have

$$\begin{aligned} \int_0^T \omega_1(|u'(t)|) dt &= \int_0^{t_0} \omega_1(|u'(t)|) dt + \int_{t_0}^T \omega_1(|u'(t)|) dt \\ &\leq \int_0^{t_0} \omega_1(c(t_0 - t)) dt + \int_{t_0}^T \omega_1(c(t - t_0)) dt. \end{aligned}$$

Without loss of generality we can assume  $c < 1$  and, by (3.2), substituting  $z = c(t_0 - t)$  into the first integral and  $z = c(t - t_0)$  into second the integral, we get

$$\begin{aligned} \int_0^T \omega_1(|u'(t)|) dt &\leq \int_0^{t_0} \omega_1(c(t_0 - t)) dt + \int_{t_0}^T \omega_1(c(t - t_0)) dt \\ &= -\frac{1}{c} \int_{ct_0}^0 \omega_1(z) dz + \frac{1}{c} \int_0^{c(T-t_0)} \omega_1(z) dz = B < \infty, \end{aligned}$$

where  $B$  is independent of  $u$ , too.

We set  $\max\{|u'(t)|; t \in [0, T]\} = \max\{|u'(0)|; |u'(T)|\} = |u'(\tau_0)| = \rho$ . Then

$$-\rho T \leq u(t) \leq \rho T \text{ for } t \in [0, T].$$

Let  $C = \omega_0(1) \int_0^T t^\gamma (T-t)^\delta dt + \omega_1(1)T$ . Integrating (3.6) from  $\tau_0$  to  $t_0$  we get

$$\begin{aligned} |u'(\tau_0)| = \rho &\leq C + \left| \int_{\tau_0}^{t_0} t^\gamma (T-t)^\delta \omega_0(u(s)) ds \right| + \left| \int_{\tau_0}^{t_0} \omega_1(|u'(s)|) ds \right| \\ &+ \left| \int_{\tau_0}^{t_0} h_0(s) + h_1(s)(|u(s)|^\alpha + 1) + h_2(s)(|u'(s)|^\beta + 1) ds \right| \end{aligned}$$

and

$$\rho \leq A + B + C + \left| \int_{\tau_0}^{t_0} h_0(s) + h_1(s)(|u(s)|^\alpha + 1) + h_2(s)(|u'(s)|^\beta + 1) ds \right|.$$

Hence, since  $|u(s)|^\alpha \leq |\rho T|^\alpha$ ,  $|u'(s)|^\beta \leq \rho^\beta$  we have

$$\rho \leq A + B + C + \left| \int_{\tau_0}^{t_0} h_0(s) + h_1(s)((\rho T)^\alpha + 1) + h_2(s)(\rho^\beta + 1) ds \right|$$

and, consequently,

$$\rho \leq A + B + C + \|h_0\|_L + ((\rho T)^\alpha + 1)\|h_1\|_L + (\rho^\beta + 1)\|h_2\|_L. \quad (3.11)$$

By contradiction we show, that there exists a constant  $r^* > 0$  (independent of  $u$ ) such that  $\rho < r^*$  for every  $u$ . Assume that there is a sequence  $\{u_n\}$  satisfying (1.2), (3.5) and (3.6) and the corresponding sequence  $\{\rho_n\}$  is not bounded.

- Let  $\alpha, \beta < 1$ , then from (3.11) we get

$$1 \leq \frac{A + B + C + \|h_0\|_L + \|h_1\|_L + \|h_2\|_L}{\rho_n} + \frac{T^\alpha \|h_1\|_L}{\rho_n^{(1-\alpha)}} + \frac{\|h_2\|_L}{\rho_n^{(1-\beta)}}$$

and for  $n \rightarrow \infty$  we get

$$1 \leq 0,$$

which is a contradiction.

- Let  $\alpha = 1$ ,  $\beta < 1$ , then for  $n \rightarrow \infty$  from (3.11) we get

$$1 \leq \frac{A + B + C + \|h_0\|_L + \|h_1\|_L + \|h_2\|_L}{\rho_n} + T\|h_1\|_L + \frac{\|h_2\|_L}{\rho_n^{(1-\beta)}},$$

$$1 \leq T\|h_1\|_L.$$

By (3.3) we have  $T\|h_1\|_L < 1$  and we have a contradiction.

- Let  $\alpha < 1$ ,  $\beta = 1$ , then for  $n \rightarrow \infty$  from (3.11) we get

$$1 \leq \frac{A + B + C + \|h_0\|_L + \|h_1\|_L + \|h_2\|_L}{\rho_n} + \frac{T^\alpha \|h_1\|_L}{\rho_n^{(1-\alpha)}} + \|h_2\|_L,$$

$$1 \leq \|h_2\|_L.$$

By (3.3) we have  $\|h_2\|_L < 1$  and we have a contradiction too.

- Let  $\alpha = \beta = 1$ , then

$$\rho_n \leq A + B + C + \|h_0\|_L + \|h_1\|_L + \|h_2\|_L + \rho_n T\|h_1\|_L + \rho_n \|h_2\|_L,$$

$$1 \leq \frac{A + B + C + \|h_0\|_L + \|h_1\|_L + \|h_2\|_L}{\rho_n} + T\|h_1\|_L + \|h_2\|_L.$$

By (3.3) we have  $T\|h_1\|_L + \|h_2\|_L < 1$  and thus

$$1 \leq T\|h_1\|_L + \|h_2\|_L < 1;$$

we again have a contradiction.

Hence there exists  $r^* > 0$  such that  $\rho < r^*$  for each  $u$  satisfying (1.2), (3.5) and (3.6). Since  $\|u\|_{C^1} \leq \rho T + \rho$ , we set  $r = r^* T + r^* + 1$ .  $\square$

*Proof of Theorem 3.1.*

*Step 1. Construction of an auxiliary singular problem.*

Let  $r \in (1, \infty)$  be given by Lemma 3.3. For a.e.  $t \in [0, T]$  and for all  $x, y, z \in \mathbb{R}$  define the auxiliary functions

$$\sigma(z) = \begin{cases} z & \text{for } |z| \leq r, \\ r \operatorname{sign} z & \text{for } |z| > r \end{cases}$$

and

$$g(t, x, y) = f(t, |\sigma(x)|, \sigma(y)).$$

We apply Theorems 2.1 and 2.2 to the auxiliary singular problem

$$-u'' = g(t, u, u'), \quad u(0) = 0, \quad u(T) = 0 \quad (3.12)$$

and we prove that problem (3.12) has a solution  $u$  such that

$$0 < u(t) \leq r \text{ for } t \in (0, T) \text{ and } \|u'\|_C \leq r. \quad (3.13)$$

Then  $u$  is also a solution of problem (1.1), (1.2).

*Step 2. Construction of approximate regular problems.*

Since  $f$  has no time singularities at  $t = 0$  and  $t = T$ , we can put  $\Delta_n = [0, T]$  for  $n \in \mathbb{N}$ . Now, choosing an arbitrary  $n \in \mathbb{N}$ , for a.e.  $t \in [0, T]$  and all  $x, y \in \mathbb{R}$  we obtain

$$g_n(t, x, y) = \begin{cases} g(t, |x|, y) & \text{if } |x| \geq \frac{1}{n}, \\ g\left(t, \frac{1}{n}, y\right) & \text{if } |x| < \frac{1}{n} \end{cases}$$

and

$$f_n(t, x, y) = \begin{cases} g_n(t, x, y) & \text{if } |y| \geq \frac{1}{n}, \\ \frac{n}{2} \left[ g_n\left(t, x, \frac{1}{n}\right) \left(y + \frac{1}{n}\right) - g_n\left(t, x, -\frac{1}{n}\right) \left(y - \frac{1}{n}\right) \right] & \text{if } |y| < \frac{1}{n}. \end{cases}$$

We see that  $f_n \in \text{Car}([0, T] \times \mathbb{R}^2)$  satisfies

$$\begin{aligned} f_n(t, x, y) &= g(t, x, y) \text{ for a.e. } t \in [0, T] \\ &\text{and all } x \in \left[ \frac{1}{n}, \infty \right), \quad |y| \in \left[ \frac{1}{n}, \infty \right). \end{aligned} \quad (3.14)$$

Further we have

$$c \leq f_n(t, x, y) \leq t^\gamma (T-t)^\delta \omega_0\left(\frac{1}{n}\right) + \omega_1\left(\frac{1}{n}\right) + h_0(t) + h_1(t)r^\alpha + h_2(t)r^\beta = m_n(t)$$

for a.e.  $t \in [0, T]$ .

Since  $m_n \in L[0, T]$ , Theorem 1.5 yields a solution  $u_n$  of the problem

$$-u'' = f_n(t, u, u'), \quad u(0) = 0, \quad u(T) = 0 \quad (3.15)$$

for each  $n \in \mathbb{N}$ .

*Step 3. Convergence of a sequence  $\{u_n\}$  of approximate solutions.*

By (3.4) and (3.14) we get

$$\begin{aligned} c \leq -u_n''(t) &\leq t^\gamma (T-t)^\delta (\omega_0(u_n(t)) + \omega_0(1)) + \omega_1(1) + \omega_1(|u_n'(t)|) \\ &+ h_0(t) + h_1(t) (u_n(t)^\alpha + 1) + h_2(t) (|u_n'(t)|^\beta + 1) \end{aligned}$$

for a.e.  $t \in [0, T]$  and all  $n \in \mathbb{N}$ .

Therefore, due to Lemma 3.3,

$$\|u_n\|_{C^1} \leq r \text{ for each } n \in \mathbb{N}. \quad (3.16)$$

Define the set

$$\Omega = \{x \in C^1[0, T] : \|x\|_{C^1} \leq r\}.$$

By Theorem 2.1 there exist  $u \in C[0, T]$  and a subsequence  $\{u_k\} \subset \{u_n\}$  such that (2.3) holds.

Further we have

$$u_n(0) = 0, \quad u_n(T) = 0 \text{ and } u_n''(t) < 0 \text{ for a.e. } t \in [0, T].$$

Therefore  $u_n > 0$  on  $(0, T)$  and  $u_n$  has a unique maximum point  $t_n \in (0, T)$ . By Lemma 3.2, there is  $\eta \in (0, \frac{rT}{2})$  such that

$$\begin{aligned} u_n(t_n) > \eta, \quad u_n(t) &\geq \begin{cases} \frac{\eta t}{T} & \text{for } t \in [0, t_n], \\ \frac{\eta(T-t)}{T} & \text{for } t \in [t_n, T], \end{cases} \\ u'_n(t_n) = 0, \quad c(t_n - t) &\leq u'_n(t) \quad \text{for } t \in [0, t_n], \\ c(t - t_n) &\leq -u'_n(t) \quad \text{for } t \in [t_n, T], \quad n \in \mathbb{N}. \end{aligned} \quad (3.17)$$

By (3.16) and the Mean Value Theorem for  $u_n$  on the intervals  $[0, t_n]$  and  $[t_n, T]$  we get

$$0 < \frac{\eta}{r} \leq t_n \leq T - \frac{\eta}{r} < T, \quad n \in \mathbb{N},$$

and we can choose a subsequence  $\{u_k\}$  in such a way that it satisfies (2.3) and  $\lim_{k \rightarrow \infty} t_k = t_0 \in (0, T)$ . Then

$$u(t) \geq \begin{cases} \frac{\eta t}{T} & \text{for } t \in [0, t_0] \\ \frac{\eta(T-t)}{T} & \text{for } t \in [t_0, T]. \end{cases} \quad (3.18)$$

Put  $\Sigma = \{t_0\}$  and choose an arbitrary interval  $[a, b] \subset (0, T) \setminus \Sigma$ . For example, let  $[a, b] \subset (0, t_0)$ . Then there exists  $k_0 \in \mathbb{N}$  such that for  $k \geq k_0$  we have

$$\begin{aligned} |t_k - t_0| &\leq \frac{1}{2}(t_0 - b), \quad [a, b] \subset \left(\frac{1}{k}, t_k\right), \\ u_k(t) &\geq \frac{\eta a}{T} = m_0, \quad u'_k(t) \geq c(t_k - t) \geq c(t_k - b) \geq \frac{1}{2}(t_0 - b) = m_1 \end{aligned}$$

for  $t \in [a, b]$ . If we choose  $[a, b] \subset (t_0, T)$ , we argue similarly.

Thus, for a.e.  $t \in [a, b]$

$$|f_k(t, u_k(t), u'_k(t))| \leq \psi(t) \in L[a, b],$$

where

$$\psi(t) = \sup \{|f(t, x, y) : m_0 \leq x \leq r, \quad m_1 \leq |y| \leq r\}.$$

We have proved that on each  $[a, b] \subset (0, T) \setminus \Sigma$  there exists  $\psi \in L[a, b]$  such that

$$|u''_k(t)| \leq \psi(t) \text{ for a.e. } t \in [a, b] \text{ and all } k \in \mathbb{N}, \quad k \geq k_0.$$

By virtue of the absolute continuity of the Lebesgue integral we see that the sequence  $\{u'_k\}$  satisfies (2.4) and, by Theorem 2.1,  $u \in C^1((0, T) \setminus \Sigma)$  and (2.5) is valid.

*Step 4. The function  $u$  is a solution of problem (1.1), (1.2).*

Conditions (2.5) and (3.17) imply

$$\begin{aligned} c(t_0 - t) &\leq u'(t) \quad \text{for } t \in (0, t_0), \\ c(t - t_0) &\leq -u'(t) \quad \text{for } t \in (t_0, T). \end{aligned} \quad (3.19)$$

Since  $u'_k$  is decreasing on  $[0, T]$  for each  $k \geq k_0$ ,  $u'$  is nonincreasing on  $(0, t_0)$  and on  $(t_0, T)$ . Therefore there exist the limits

$$\lim_{t \rightarrow t_0^-} u'(t), \quad \lim_{t \rightarrow t_0^+} u'(t)$$

and

$$\lim_{t \rightarrow 0^+} u'(t) \geq ct_0 > 0, \quad \lim_{t \rightarrow T^-} u'(t) \leq -c(T - t_0) < 0.$$

We summarize that  $u'(t) > 0$  on  $[0, t_0)$  and  $u'(t) < 0$  on  $(t_0, T]$ . So,  $t_0$  is the unique point where  $u'(t_0) = 0$  or  $u'(t_0)$  does not exist. By (3.18),  $u$  is positive on  $(0, T)$ . Hence  $\Sigma = \{t_0\}$  satisfies (2.8) and, by Theorem 2.2,  $u \in AC_{loc}^1((0, T) \setminus \Sigma)$  is a  $w$ -solution of problem (3.12). Finally, by (3.4), we have  $f_k(t, u_k(t), u'_k(t)) \geq 0$  for a.e.  $t \in [0, T]$  and all  $k \in \mathbb{N}$ . Hence (2.9) holds and, by Theorem 2.2,  $u$  is a solution of problem (3.12). Having in mind that  $u > 0$  on  $(0, T)$  and  $\|u_k\|_{C^1} \leq r$  hold, we get by (2.3) and (2.5) that estimate (3.13) is satisfied and thus  $u$  is a solution of problem (1.1), (1.2).  $\square$

**Example 3.4.** Let  $h_1, h_2 \in L[0, T]$  be nonnegative. For a.e.  $t \in [0, T]$  and all  $x, y \in \mathbb{R}$  define the function

$$f(t, x, y) = 1 + \frac{t^{\frac{3}{2}}(T - t)^{\frac{3}{2}}}{x^2} + h_1(t)\sqrt{x} + \frac{1}{\sqrt{|y|}}(1 + h_2(t)|y|).$$

The second term of  $f$  has a space singularity at  $x = 0$  and the last one a singularity at  $y = 0$ . We can check that  $f$  satisfies the conditions of Theorem 3.1 with  $h_0(t) \equiv c = 1$ ,  $\alpha = \beta = \frac{1}{2}$ ,  $\gamma = \delta = \frac{3}{2}$ ,  $\omega_0(x) = \frac{1}{x^2}$  and  $\omega_1(|y|) = \frac{1}{\sqrt{|y|}}$ .

**Example 3.5.** Let  $T = 1$ . For a.e.  $t \in [0, 1]$  and all  $x, y \in \mathbb{R}$  define the function

$$f(t, x, y) = \sqrt{1 - t} \left( 1 + \frac{t^2}{x} \right) + \frac{3}{\sqrt[3]{|y|}} + \frac{1}{6\sqrt{t}}(x + |y|).$$

The first term has space singularity at  $x = 0$  and the second one at  $y = 0$ . We see that  $f$  satisfies the conditions of Theorem 3.1 if we put  $\alpha = \beta = 1$ ,  $\gamma = 2$ ,  $\delta = \frac{1}{2}$ ,  $\omega_0(x) = \frac{1}{x}$ ,  $\omega_1(|y|) = \frac{3}{\sqrt[3]{|y|}}$ ,  $h_0(t) = \sqrt{1 - t}$ ,  $h_1(t) = \frac{1}{6\sqrt{t}}$ ,  $h_2(t) = \frac{1}{6\sqrt{t}}$ , and choose  $c > 0$  sufficiently small.

**Example 3.6.** Let  $T = 2\pi$ . We define the function  $f$  for a.e.  $t \in [0, 2\pi]$  and all  $x, y \in \mathbb{R}$  by

$$f(t, x, y) = t^{\frac{5}{3}} \left( e + 10 \frac{\sqrt[3]{(2\pi - t)^4}}{x^2} \right) + \frac{e}{\sqrt[5]{|y|}} + t^3 \sqrt[6]{x} + \frac{5t^4 + 2t}{10000} |y|.$$

The function  $f$  has space singularities at  $x = 0$  and  $y = 0$ . We can check that  $f$  satisfies the assumptions of Theorem 3.1 for  $\alpha = \frac{1}{6}$ ,  $\beta = 1$ ,  $\gamma = \frac{8}{5}$ ,  $\delta = \frac{4}{3}$ ,  $\omega_0(x) = \frac{10}{x^2}$ ,  $\omega_1(|y|) = \frac{e}{\sqrt[5]{|y|}}$ ,  $h_0(t) = et^{\frac{5}{3}}$ ,  $h_1(t) = t^3$ ,  $h_2(t) = \frac{5t^4 + 2t}{10000}$  and sufficiently small  $c$ .

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