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THE ROBIN FUNCTION AND ITS EIGENVALUES

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Abstract. The paper deals with the Robin function and eigenvalue problems generated by the Robin operator. First we show that Green's function of an n-fold connected domain is the Robin function of an appropriate simply connected domain. The main part of the paper deals with eigenvalue problems for the Robin operator: the mixed Stekloff eigenvalue problem and the membrane problem with mixed boundary conditions. Isoperimetric inequalities are proved for the sum of reciprocal eigenvalues.

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1. Introduction

Green's function is a powerful tool for solving boundary value problems of potential theory. Depending on a boundary problem there are three kinds of Green's function. The Green function of a mixed boundary-value problem for a harmonic function is sometimes called the Robin function.

Let $\Omega \ni \infty$ be a domain of connectivity n in the extended complex plane $\widehat{\mathbb{C}}$ with the boundary $\partial\Omega = C = \bigcup\limits_{i=1}^{n} C_i$, where C_i are simple closed Jordan curves.

Assume that m_{ij} disjoint closed arcs A_{ij}, $j = 1, \dots, m_{ij}$, are prescribed on C_i. It may be that $m_i = 1$ and $A_{i1} = C_i$, or that $m_i = 0$ and no arcs A_{ij} are prescribed on C_i. Let $A = \bigcup\limits_{i=1}^{n} \bigcup\limits_{j=1}^{m_{ij}} A_{ij}$, $B = C \setminus A$, with the understanding that $\bigcup\limits_{j=1}^{m_{ij}} A_{ij}$ is an empty set if $m_{ij} = 0$.

The Robin function $R_{\Omega,A}(z,\zeta)$ of the domain Ω with respect to the boundary set A is defined by the following properties:

(1) $R_{\Omega,A}(z,\zeta)$ is harmonic in Ω and continuous in $\overline{\Omega}$, except at $z = \zeta$, where $R_{\Omega,A}(z,\zeta) + \ln|z-\zeta|$ is harmonic, for $\zeta = \infty$ the property is modified to require that $R_{\Omega,A}(z,\zeta) - \ln|z|$ be harmonic in Ω,
(2) $R_{\Omega,A}(z,\zeta) = 0$ for all $z \in A$,
(3) $\frac{\partial R_{\Omega,A}(z,\zeta)}{\partial n}(z,\zeta) = 0$ for all $z \in B$, where n denotes the inner normal.

The Robin function may be viewed as a generalization of Green's function, to which it reduces when the set B is empty. Both Green's and the Robin function are conformally invariant. Starting with P. L. Duren and M. M. Schiffer [11] the Robin function has been investigated recently by many authors, we only

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mention here [12, 10]. A survey paper is given by P. L. Duren [9]. Let $G_\Omega(z,\zeta)$ be Green's function of the domain Ω. Then we have [9] for the Robin constant $\gamma(A)$

$$\gamma(A) = \lim_{z\to\infty}(G_\Omega(z,\infty) - \ln|z|). \tag{1.1}$$

The transfinite diameter $d(A)$ or the capacity of the set A is

$$d(A) = e^{-\gamma(A)}. \tag{1.2}$$

The Robin capacity $\delta_\Omega(A)$ of A with respect to Ω is defined by

$$\delta_\Omega(A) = e^{-\rho_\Omega(A)}, \tag{1.3}$$

where

$$\rho_\Omega(A) = \lim_{z\to\infty}(R_{\Omega,A}(z,\infty) - \ln|z|). \tag{1.4}$$

It is evident that the Robin function is a generalization of the Green function [11] but it is easy to see that the Green function of a domain $\widetilde{\Omega}$ is also the Robin function of an appropriate subdomain $\Omega \subset \widetilde{\Omega}$. What we show is that every Green's function of a finitely connected domain is also the Robin function of a simply connected domain. We obtain this simply connected domain from the given domain by cutting it along piecewise analytic arcs which connect the boundary components and on which the normal derivative of Green's function vanishes. A remarkable consequence is that the capacity of a more component set is equal to the Robin capacity of an appropriate set on the boundary of a simply connected domain. The main part of the paper is devoted to the study of the eigenvalue problems generated by the Robin operator. We wish to investigate the mixed Stekloff problem and the membrane problem with mixed boundary conditions (see below (3.1) and (3.2), respectively).

Following the basic ideas in [4, 5], we prove isoperimetric inequalities for sums of reciprocal eigenvalues and derive formulas for such sums. On the basis of these an existence proof for the Robin function could be established using a complete orthonormal system of eigenfunctions and the fact that the Robin function is a reproducing kernel. For an up-to-date treatment of eigenvalues and conformal mappings we refer to [6].

2. Green's Function and the Robin Function

Let $\Omega \ni \infty$ be a domain of the above kind, and let $\widetilde{\Omega}$ be the component of $\widehat{\mathbb{C}} \setminus A$ that contains infinity.

Let A be a subset of C as described above, then we have the following theorem which goes back to Duren and Schiffer [11].

Theorem 1. *Let $\Omega \ni \infty$ be a finitely connected domain containing the point at infinity and bounded by smooth Jordan curves $C_i, i = 1,\dots,n$. Let A be a subset of C as described above and let $\widetilde{\Omega}$ be the component of $\widehat{\mathbb{C}}\setminus A$ that contains infinity. Then*

$$d(A) = \delta_\Omega(A), \tag{2.1}$$

if and only if

$$G_{\widetilde{\Omega}}(z,\zeta) \equiv R_{\Omega,A}(z,\zeta). \tag{2.2}$$

Proof. Let $G_{\widetilde{\Omega}}(z,\zeta) \equiv R_{\Omega,A}(z,\zeta)$, then the assertion follows using (1.1), (1.2), (1.3) and (1.4).

We have [11] $\delta_\Omega(A) \leq d(A)$, and the equality occurs if and only if $B = C \setminus A$ lies on the slits where $\frac{\partial G_{\widetilde{\Omega}}}{\partial n} = 0$. This means that (2.2) holds. □

Theorem 2. *Let $\widetilde{\Omega} \ni \infty$ be an n-fold connected plane domain. Then there exists a set of piecewise analytic curves $B \subset \overline{\widetilde{\Omega}}$ such that*

$$\frac{\partial G_{\widetilde{\Omega}}(z,\zeta)}{\partial n} = 0 \quad \text{for } z \text{ on } B. \tag{2.3}$$

That is, if $\Omega = \widetilde{\Omega} \setminus B$ is a domain, then

$$R_{\Omega,A}(z,\zeta) \equiv G_{\widetilde{\Omega}}(z,\zeta), \quad A = \partial\widetilde{\Omega}. \tag{2.4}$$

In particular there exists a set of piecewise analytic arcs B connecting the boundary curves C_i in such a way that the exterior Ω of $\widehat{\mathbb{C}} \setminus (C \cup B)$ is simply connected and (2.4) holds.

Proof. The first assertion is evident if we consider analytic arcs orthogonal to the level lines of the Green function.

It is also easy to see that a set of piecewise analytic curves of the above kind exists. We prove this by complete induction. If we have a two-fold connected domain then we can map it onto a circle domain. The curve B is given by the straight line connecting both circles and orthogonal to both of them.

If we have a domain of connectivity $n > 2$ we first consider two boundary components and we can see in the same way that a piecewise analytic curve exists connecting these two boundary components on which the normal derivative of Green's function vanishes. We slit the domain along this curve and map the exterior of these two boundary components and the slit onto the exterior of a disk with two radial slits. The image of the original domain furnished with this slit is a domain with a connectivity $n-1$, which makes the proof complete. □

3. Eigenvalues of the Robin Function

Let Ω be a bounded n-fold connected domain in the plane $\mathbb{C}$ and on the boundary curves $C = \bigcup_{i=1}^{n} C_i$ the arcs A_{ij} be given according to the assumptions above.

Two eigenvalue problems are posed with the Robin function as a kernel function: the mixed Stekloff problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } A, \\ \frac{\partial u}{\partial n} &= \lambda u \quad \text{on } C \setminus A \end{aligned} \tag{3.1}$$

and the membrane problem with mixed boundary conditions

$$\begin{aligned} \Delta u + \mu u &= 0 \qquad \text{in} \quad \Omega, \\ u &= 0 \qquad \text{on} \quad A, \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on} \quad C \setminus A, \end{aligned} \tag{3.2}$$

where ∂n is the outward normal derivative and λ and μ are the eigenvalue parameters.

For both problems there exist infinitely many eigenvalues with finite multiplicity. We consider first the case where the domain Ω is doubly connected. Later in Section 3.2, we consider the simply connected case.

3.1. **Doubly connected domain.** Let Ω be a doubly connected domain in the plane with piecewise analytic boundary components A and B. We consider the following eigenvalue problem:

$$\begin{aligned} \Delta u &= 0 \qquad \text{in} \quad \Omega, \\ u &= 0 \qquad \text{on} \quad A, \\ \frac{\partial u}{\partial n} &= \lambda u \qquad \text{on} \quad B. \end{aligned} \tag{3.3}$$

We have the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ with the corresponding system of eigenfunctions $u_1, u_2, \dots$.

In the case of an annulus $A(1, R)$ with radii 1 and R, $1 < R < \infty$, the eigenvalues and eigenfunctions are known [11, p. 271] for $A = \{z : |z| = 1\}$ and $B = \{z : |z| = R\}$. We have the radial eigenfunction

$$u_1^{(0)} = \frac{\ln r}{\sqrt{R \ln^2 R \cdot 2\pi}} \quad \text{with} \quad \lambda_1^{(0)} = \frac{1}{R \ln R}, \quad \text{and with} \quad z = re^{i\phi}$$

$$\begin{matrix} u_{2n}^{(0)} \\ u_{2n+1}^{(0)} \end{matrix} = \frac{r^n - r^{-n}}{\sqrt{\pi R}(R^n - R^{-n})} \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \quad \text{with} \quad \lambda_{2n}^{(0)} = \lambda_{2n+1}^{(0)} = n \frac{R^{n-1} + R^{-n-1}}{R^n - R^{-n}},$$

for $n = 1, 2, \dots$. The eigenfunctions of the annulus are the eigenfunctions of the following Robin function [11, p. 272]:

$$R(\theta, \phi) = \frac{\ln R}{2\pi} + \sum_{n=1}^{\infty} \frac{R^n - R^{-n}}{\pi n (R^n + R^{-n})} \cos n(\theta - \phi)$$

on $B^{(0)} = \{z : |z| = R\}$. We follow closely [1, p. 97], [5].

Let f be a mapping that conformally maps $A(1, R)$ onto Ω. Then the eigenvalues of (3.3) satisfy the variational characterization based on the Robin function ($z = re^{i\phi}, \zeta = se^{i\theta}$)

Lemma 1.

$$\frac{1}{\lambda_k} = \max_h \int_{B^{(0)}} \int_{B^{(0)}} R(z, \zeta) h(z) h(\zeta) |f'(z)| |f'(\zeta)| \, ds_z \, ds_\zeta,$$

where the maximum is taken over all $h \in L_2(B^{(0)})$ *with* $\int_{B^{(0)}} h^2|f'|\,ds = 1$ *and* $\int_{B^{(0)}} hu_j|f'|\,ds = 0$, $j = 1, 2, \ldots, k-1$, *where* u_j *are the eigenfunctions of* (3.3) *transplanted onto* $A(1, R)$.

Proof. Let h be a function that satisfies the requirements of Lemma 1 and let u_j be the eigenfunction transplanted onto $A(1, R)$. We have [5, (2.9)]

$$R(z,\zeta)|f'(z)||f'(\zeta)| = \sum_{j=1}^{\infty} \frac{u_j(z)|f'(z)|u_j(\zeta)|f'(\zeta)|}{\lambda_j},$$

where the convergence is in the mean. So we can change the order of integration and summation to get using the Bessel inequality

$$\int\limits_{B^{(0)}} \int\limits_{B^{(0)}} R(z,\zeta)h(z)h(\zeta)|f'(z)||f'(\zeta)|\,ds_z\,ds_\zeta$$

$$= \sum_{j=k}^{\infty} \frac{\left(\int_{B^{(0)}} h(z)u_j(z)|f'(z)|\,ds_z\right)^2}{\lambda_j} \le \frac{\int_{B^{(0)}} h^2(z)|f'(z)|\,ds}{\lambda_k} = \frac{1}{\lambda_k}.$$

This equality holds for $h(z) = u_k(z)$. □

Now we give a variational characterization independent of the transplanted eigenfunctions.

Lemma 2.

$$\frac{1}{\lambda_k} = \max_{L_{k-1}} \min \int\limits_{B^{(0)}} \int\limits_{B^{(0)}} R(z,\zeta)|f'(z)||f'(\zeta)|h(z)h(\zeta)\,ds_z\,ds_\zeta,$$

where the maximum is taken over all $(k-1)$*-dimensional spaces* $L_{k-1} \subset L_2(B^{(0)})$ *and the minimum is taken over all* $h \in L_{k-1}$ *with* $\int_{B^{(0)}} h^2|f'|\,ds = 1$.

Proof. We define

$$\Lambda_k = \max_{L_{k-1}} \min \int\limits_{B^{(0)}} \int\limits_{B^{(0)}} R(z,\zeta)|f'(z)||f'(\zeta)|h(z)h(\zeta)\,ds_z\,ds_\zeta$$

under the restrictions of Lemma 2. In every space L_{k-1} there exists a function h with $\int_{B^{(0)}} hu_j|f'|\,ds = 0$ for $j = 1, 2, \ldots, k-1$ and $\int_{B^{(0)}} h^2|f'|\,ds = 1$. Using Lemma 1 we obtain

$$\Lambda_k \le \min \int\limits_{B^{(0)}} \int\limits_{B^{(0)}} R(z,\zeta)|f'(z)||f'(\zeta)|h(z)h(\zeta)\,ds_z\,ds_\zeta$$

$$\le \int\limits_{B^{(0)}} \int\limits_{B^{(0)}} R(z,\zeta)|f'(z)||f'(\zeta)|h(z)h(\zeta)\,ds_z\,ds_\zeta \le \frac{1}{\lambda_k}. \tag{3.4}$$

We choose the space $L_{k-1} = \operatorname{span}\{u_1(z), \ldots, u_{k-1}(z)\}$. So we have

$$\Lambda_k \ge \min \int\limits_{B^{(0)}} \int\limits_{B^{(0)}} R(z,\zeta)|f'(z)||f'(\zeta)|h(z)h(\zeta)\,ds_z\,ds_\zeta = \frac{1}{\lambda_k}, \tag{3.5}$$

where the minimum is taken over all functions $v \in L_{k-1}$ with $\int_{B^{(0)}} v^2|f'|\,ds = 1$.

With (3.4) and (3.5) we get Lemma 2. □

Finally, we prove

Theorem 3.

$$\sum_{j=1}^{n} \frac{1}{\lambda_j} \geq \sum_{j=1}^{n} \int\limits_{B^{(0)}} \int\limits_{B^{(0)}} R(z,\zeta)|f'(z)||f'(\zeta)|v_j(z)v_j(\zeta)\,ds_z\,ds_\zeta,$$

with the conditions $\int_{B^{(0)}} |f'|v_i v_j\,ds = \delta_{ij}$, $i, j = 1,\ldots,n$ *and* $v_j \in L_2(B^{(0)})$.

Proof. In every n-dimensional linear space $L_n \subset L_2(B^{(0)})$ there exists a set of functions $\{v_j\}_{j=1}^n$ with $\int_{B^{(0)}} v_i v_j |f'|\,ds = \delta_{ij}$ and $\int_{B^{(0)}} v_k u_j |f'|\,ds = 0$ for $i, j = 1,\ldots,n$ and $k = j+1,\ldots,n$. We obtain

$$\frac{1}{\lambda_j} \geq \int\limits_{B^{(0)}} \int\limits_{B^{(0)}} R(z,\zeta)|f'(z)||f'(\zeta)|h(z)h(\zeta)\,ds_z\,ds_\zeta$$

and thus Lemma 1 implies Theorem 3. □

We define $b_{j,k} = \int_{B^{(0)}} u_k^{(0)} v_j |f'|\,ds$ so that

$$\int\limits_{B^{(0)}} \int\limits_{B^{(0)}} R(z,\zeta)|f'(z)||f'(\zeta)|v_j(z)v_j(\zeta)\,ds_z\,ds_\zeta$$

$$= \int\limits_{B^{(0)}} \int\limits_{B^{(0)}} \sum_{k=1}^{\infty} \frac{u_j^{(0)}(z)u_j^{(0)}(\zeta)}{\lambda_j^{(0)}} |f'(z)||f'(\zeta)|v_j(z)v_j(\zeta)\,ds_z\,ds_\zeta = \sum_{k=1}^{\infty} \frac{b_{j,k}^2}{\lambda_k^{(0)}}$$

with suitable functions v_j, see Theorem 3. We choose $v_n = \sum\limits_{j=1}^{n} c_{n,j} u_j^{(0)}$ with $\int_{B^{(0)}} |f'|v_i \cdot v_j\,ds = \delta_{ij}$ and demand $c_{n,n} > 0$. So $c_{n,j}$ are determined and we get

Lemma 3. *With the definitions above and* $d_{i,j} = \int_{B^{(0)}} |f'|u_i^{(0)} u_j^{(0)}\,ds$ *we have*

$$\sum_{j=1}^{k} b_{j,k}^2 = d_{k,k}.$$

Proof. With $\vec{v} = \{v_j\}_{j=1}^n$, $\vec{u}^{(0)} = \{u_j^{(0)}\}_{j=1}^n$ and $C = \{c_{j,m}\}_{j,m=1}^n$ one has $\vec{v} = C \cdot \vec{u}^{(0)}$ and $\vec{u}^{(0)} = C^{-1} \cdot \vec{v} = G \cdot \vec{v}$ with $u_k^{(0)} = \sum_{m=1}^k g_{k,m} v_m$. It follows

$$b_{j,k} = \int\limits_{B^{(0)}} |f'|v_j \sum_{m=1}^{k} g_{k,m} v_m\,ds = g_{k,j}.$$

We have

$$d_{k,k} = \int\limits_{B^{(0)}} |f'|u_k^{(0)} u_k^{(0)}\,ds = \int\limits_{B^{(0)}} |f'| \sum_{j=1}^{k} g_{k,j} v_j \sum_{j=1}^{k} g_{k,j} v_j\,ds = \sum_{j=1}^{k} g_{k,j}^2,$$

which is the desired result. □

In the case of the radial eigenfunction $u_1^{(0)}$ one gets

$$d_{1,1} = \int\limits_{B^{(0)}} \frac{1}{2\pi R}|f'(z)|\, ds_z = \frac{\alpha_0}{2},$$

where $|B^{(0)}| = \pi R\alpha_0$ is the length of $B^{(0)}$.

Otherwise, for two eigenfunctions we have the same eigenvalue

$$d_{k,k} + d_{k+1,k+1} = \int\limits_{B^{(0)}} \frac{1}{\pi R}|f'(z)|\, ds_z = \alpha_0.$$

Now, with suitable numbering of the eigenvalues, we get the following isoperimetric result.

Theorem 4. *For the eigenvalues of problem* (3.3) *in* Ω *we have*

$$\sum_{j=1}^{n} \frac{1}{\lambda_j} \geq \frac{\alpha_0}{2} \sum_{j=1}^{n} \frac{1}{\lambda_j^{(0)}}, \alpha_0 = \frac{|B^{(0)}|}{\pi R}$$

for any n. *The equality holds if* Ω *is an annulus.*

Because $\{\frac{1}{\lambda_j}\}_{j=1}^{\infty}$ is a decreasing sequence of real numbers, the theorem implies (see [17, p. 64],[16])

Corollary 1. *Let* Φ *be a convex and increasing function. For any* n *we have*

$$\sum_{j=1}^{n} \Phi\Big(\frac{1}{\lambda_j}\Big) \geq \sum_{j=1}^{n} \Phi\Big(\frac{\alpha_0}{2}\frac{1}{\lambda_j^{(0)}}\Big).$$

It is remarkable that for $\Phi(x) = x^2$ we can give a formula for all reciprocals [3].

Theorem 5. *Let* f *be a conformal mapping of an annulus* $A(1, R)$ *onto the doubly connected domain* Ω *with* $\int_{B^{(0)}} |f'(z)|\, ds_z < \infty$*, then we have*

$$\int\limits_{B^{(0)}} \int\limits_{B^{(0)}} R^2(z,\zeta)|f'(z)||f'(\zeta)|\, ds_z\, ds_\zeta = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}.$$

For the conformal mapping f we have on $B^{(0)} = \{z : |z| = R\}$

$$|f'(\phi)| \sim \frac{\alpha_0}{2} + \sum_{j=1}^{\infty}(\alpha_j \cos j\phi + \beta_j \sin j\phi)\,. \tag{3.6}$$

So we get the following

Theorem 6. *Let* f *be a conformal mapping of an annulus* $A(1, R)$ *onto the doubly connected domain* Ω *with* $\int_{B^{(0)}} |f'(z)|\, ds_z < \infty$*, then for the eigenvalues of* Ω *there holds*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = \frac{\alpha_0^2}{4} \cdot R^2 \cdot B_0 + \sum_{n=1}^{\infty} B_n \cdot R^2 \cdot (\alpha_n^2 + \beta_n^2) \tag{3.7}$$

with the notation

$$A_n = \frac{1}{n} \cdot \frac{R^n - R^{-n}}{R^n + R^{-n}} \qquad \textit{for} \quad R > 1,$$
$$A_n = \frac{1}{n} \cdot \frac{R^{-n} - R^n}{R^{-n} + R^n} \qquad \textit{for} \quad R < 1,$$
$$B_0 = 2\sum_{n=1}^{\infty} A_n^2 + \ln^2 R,$$
$$B_n = \sum_{m=1}^{\infty} A_{n+m}A_m + \frac{1}{2}\sum_{m=1}^{n-1} A_{n-m}A_m + \ln R \cdot A_n \qquad \textit{for} \quad R > 1,$$
$$B_n = \sum_{m=1}^{\infty} A_{n+m}A_m + \frac{1}{2}\sum_{m=1}^{n-1} A_{n-m}A_m - \ln R \cdot A_n \qquad \textit{for} \quad R < 1,$$

and coefficients α_n, β_n *according to* (3.6).

With formula (3.7) we get some interesting numerical results. So we obtain for the annulus $A(1,R)$

R	$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}$	R	$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}$
$\frac{11}{10}$	0.3932855337^{39}_{09}	$\frac{1}{10}$	$0.0851314170 6^{30}_{29}$
$\frac{12}{10}$	0.89533141977^{7}_{6}	$\frac{2}{10}$	$0.22324210238 7^{3}_{2}$
$\frac{13}{18}$	1.51207975498^{8}_{7}	$\frac{3}{10}$	$0.37051078181 1^{5}_{4}$
$\frac{14}{10}$	$2.2489956669 5^{80}_{79}$	$\frac{4}{10}$	$0.5000889856 2^{40}_{39}$
$\frac{15}{10}$	3.11113830918^{4}_{3}	$\frac{5}{10}$	$0.59095331538 3^{4}_{3}$
$\frac{16}{10}$	4.10321710773^{3}_{2}	$\frac{6}{10}$	$0.62713136711 9^{4}_{3}$
$\frac{17}{10}$	5.22963760787^{3}_{2}	$\frac{7}{10}$	$0.59600787009 8^{3}_{2}$
$\frac{18}{10}$	6.49454077602^{5}_{4}	$\frac{8}{10}$	$0.48702093189 4^{7}_{6}$
$\frac{19}{10}$	7.90183934493^{9}_{8}	$\frac{9}{10}$	$0.2910356691 9^{98}_{73}$

Theorem 7. *Let* $A(1,R)$ *be an annulus with radii* $r_1 = 1$ *and* $r_2 = R > 1$. *Then the sum* $\sum\limits_{j=1}^{\infty} \frac{1}{\lambda_j^2}$ *is strongly increasing in* R.

Proof. The result of Theorem 7 follows from the strong monotonicity of $A_n = \frac{1}{n} \cdot \frac{R^n - R^{-n}}{R^n + R^{-n}}$ in R for every fixed n. □

Let us consider the conformel mapping $f_r(z) = z + \frac{1}{r^2 z}$ which maps the annulus onto an elliptic ring D_r. We have the following numerical results:

	$R=\frac{10}{5}$	$R=\frac{9}{5}$	$R=\frac{8}{5}$	$R=\frac{7}{5}$	$R=\frac{6}{5}$
$r\to\infty$	9.455253046_{1}^{2}	6.494540776_{0}^{1}	4.10321710773_{2}^{3}	2.2489956695_{79}^{80}	0.89533141977_{6}^{7}
$r=\frac{10}{5}$	9.49056697_{89}^{99}	6.53193452_{24}^{45}	4.14148996_{058}^{255}	2.2851581_{0814}^{1000}	0.92222223_{787}^{953}
$r=\frac{9}{5}$	9.5090774_{392}^{404}	6.5515351_{275}^{301}	4.16155128_{042}^{285}	2.30411308_{721}^{950}	0.93631729_{434}^{639}
$r=\frac{8}{5}$	9.54147039_{39}^{54}	6.5858357_{699}^{732}	4.19665802_{569}^{877}	2.33728355_{164}^{454}	0.9609829_{2908}^{3167}
$r=\frac{7}{5}$	9.60233951_{13}^{32}	6.65028944_{27}^{69}	4.2626259_{6687}^{7089}	2.39961234_{575}^{954}	1.00733021_{566}^{904}
$r=\frac{6}{5}$	9.72776225_{58}^{83}	6.78309776_{09}^{67}	4.3985526_{5862}^{6408}	2.52803757_{468}^{984}	1.10282400_{286}^{746}
$r=\frac{5}{5}$	10.0203943_{66}^{73}	7.0929597_{365}^{447}	4.7156820_{8616}^{9403}	2.82765067_{085}^{827}	1.3255954_{2591}^{3252}

Further we obtain

Theorem 8. *For fixed R the sum $\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}$ of D_r is strongly decreasing in r with $r \geq 1$.*

Proof. The monotonicity of the sum follows from the properties of the Fourier coefficients of $|f_r'(\phi)|$, cf. (3.6). For details see [13]. □

Remark 1. For similar conformal mappings more monotonicity results are given in [13].

As the last example we consider the conformal mapping $f_x(z) = \frac{z-x}{1-zx}$ with $x \in \mathbb{R}$ and $0 \leq x < 1$. For the first eigenvalue of the fixed membrane problem this issue has been considered in [2]. For $Rx \leq 1$ some examples are given on Fig. 1

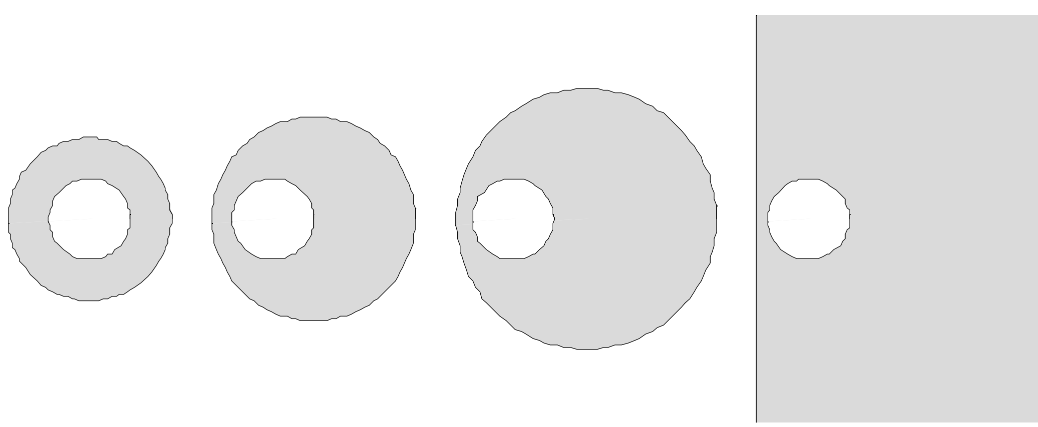

Fig. 1

for $x = 0, x = 1/4, x = 1/3, x = 1/2$ and $R = 2$. The case $Rx > 1$ gives completely different images. For $x = 2/3, x = 3/4$ and $R = 2$ the pictures are given on Fig. 2.

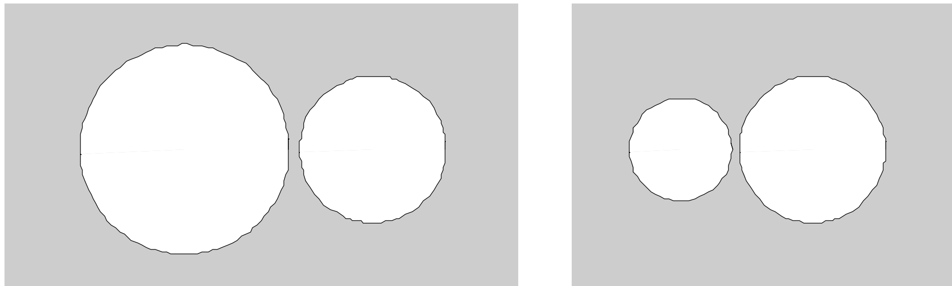

Fig. 2

Using the conformal mapping

$$\tilde{f}_x(z) = \frac{1}{R} \cdot \frac{1 - x^2 R^2}{1 - x^2} \cdot \frac{z - x}{1 - zx}$$

we obtain the numerical results summarized in the tables below.

Remark 2. For the monotonicity results we refer to [13].

Next, we consider the eigenvalue problem

$$\begin{aligned} \Delta u + \mu u &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad A \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad B, \end{aligned} \tag{3.8}$$

with the same notation as in (3.3).

	$R = \frac{10}{5}$	$R = \frac{9}{5}$	$R = \frac{8}{5}$	$R = \frac{7}{5}$	$R = \frac{6}{5}$
$x = 0$	2.36381326615^{4}_{3}	2.00448789384^{4}_{3}	1.6028191827^{1}_{0}	1.147446770^{20}_{19}	0.62175793040^{1}_{0}
$x = \frac{1}{20}$	2.41035584456^{4}_{3}	2.03662400383^{4}_{3}	1.61894066967^{7}_{6}	1.15867812288^{8}_{7}	0.62624267544^{4}_{3}
$x = \frac{2}{20}$	2.55540437222^{3}_{2}	2.13609555048^{8}_{7}	1.66853260815^{5}_{4}	1.19303364432^{3}_{2}	0.63989241578^{8}_{7}
$x = \frac{3}{20}$	2.816837574^{60}_{59}	2.31283493394^{4}_{3}	1.7555144293^{2}_{1}	1.25259648^{500}_{499}	0.6633152902^{2}_{1}
$x = \frac{4}{20}$	3.23080147568^{8}_{7}	2.58622446309^{9}_{8}	1.88729690847^{7}_{6}	1.341196023^{60}_{59}	0.69760150661^{1}_{0}
$x = \frac{5}{20}$	3.865611398^{60}_{59}	2.99086332188^{8}_{7}	2.07646518969^{9}_{8}	1.465056502^{90}_{89}	0.74445588766^{6}_{5}
$x = \frac{6}{20}$	4.85549541504^{4}_{3}	3.58913146693^{3}_{2}	2.34412421458^{8}_{7}	1.633988769^{30}_{29}	0.80642520756^{6}_{5}
$x = \frac{7}{20}$	6.49389038493^{3}_{2}	4.500708245^{20}_{19}	2.726793293^{80}_{79}	1.86359380634^{4}_{3}	0.887281315^{10}_{09}
$x = \frac{8}{20}$	9.56012203187^{7}_{6}	5.98239191535^{5}_{4}	3.2918481420^{1}_{0}	2.179527155^{60}_{59}	0.99267888345^{5}_{4}
$x = \frac{9}{20}$	17.1249737439^{9}_{5}	8.69917470034^{4}_{3}	4.17680402975^{5}_{4}	2.62637677107^{7}_{6}	1.13133002201^{1}_{0}

	$R = \frac{10}{5}$	$R = \frac{9}{5}$	$R = \frac{8}{5}$	$R = \frac{7}{5}$	$R = \frac{6}{5}$
$x = \frac{11}{20}$	$18.414302^{30122}_{29867}$	$---$	$---$	$---$	$---$
$x = \frac{12}{20}$	11.252634634^{1}_{0}	$19.049081^{73109}_{58105}$	$---$	$---$	$---$
$x = \frac{13}{20}$	8.37125245587^{5}_{4}	11.074325382^{5}_{4}	20.77800058844^{3}_{2}	$---$	$---$
$x = \frac{14}{20}$	6.81440424446^{1}_{0}	8.05243376636^{8}_{7}	9.80597048852^{2}_{1}	$---$	$---$
$x = \frac{15}{20}$	5.84471637348^{8}_{7}	6.45832690207^{7}_{6}	6.72800966975^{5}_{4}	17.66161067237^{7}_{6}	$---$
$x = \frac{16}{20}$	5.18664436325^{5}_{4}	5.47811481661^{1}_{0}	5.25999788324^{4}_{3}	9.07317730102^{2}_{1}	$---$
$x = \frac{17}{20}$	4.71340970124^{4}_{3}	4.81801134489^{9}_{8}	4.40375575608^{8}_{7}	6.25454574111^{1}_{0}	$21.^{647}_{548}$
$x = \frac{18}{20}$	4.35851482158^{8}_{7}	4.34561851804^{4}_{3}	3.84560832909^{9}_{8}	4.85117749988^{8}_{7}	7.436184^{481}_{223}
$x = \frac{19}{20}$	4.08374722138^{8}_{7}	3.99245705916^{6}_{5}	3.45482235713^{3}_{2}	4.01507474980^{0}_{8}	4.59253289867^{7}_{6}

In the case of an annulus $A(1,R)$, with the Dirichlet conditions on the outer circle $A^{(0)}$ of radius R and the Neumann conditions on the inner circle $B^{(0)}$ of radius 1 the eigenfunctions have the form

$$u^{(0)}_{m,n} = (A_{m,n} J_n(k_{m,n} \cdot r) + B_{m,n} Y_n(k_{m,n} \cdot r)) \begin{cases} \cos n\phi \\ \sin n\phi \end{cases},$$

where J_n and Y_n Bessel functions of first and second kind, respectively, see [18], [15]. The constants $A_{m,n}$, $B_{m,n}$ and $k_{m,n}$ are determined by the boundary conditions and the normalizing condition $\int_A u^{(0)^2}_{m,n}\, dA = 1$. The eigenvalues are given by $\sqrt{\mu^{(0)}_{m,n}} = k_{m,n}$.

We write in the sequel $\mu^{(0)}_j$ and also $u^{(0)}_j$ for the eigenvalues and the eigenfunctions of the annulus $A(1,R)$, where the eigenvalues are numbering in increasing order.

Let f map the annulus onto Ω, and let $R(z,\zeta)$ be the Robin function in $A(1,R)$. In the same way as before we obtain for the eigenvalues of Ω

Theorem 9.

$$\sum_{j=1}^{n} \frac{1}{\mu_j} \geq \sum_{j=1}^{n} \int\limits_{\Omega} \int\limits_{\Omega} R(z,\zeta) |f'(z)|^2 |f'(\zeta)|^2 v_j(z) v_j(\zeta)\, dA_z\, dA_\zeta$$

with the conditions $\int_\Omega |f'|^2 v_i v_j\, dA = \delta_{ij}$ *and* $v_j \in L_2(A(1,R))$.

Again we define $b_{j,k} = \int_{A(1,R)} u^{(0)}_k v_j |f'|^2\, dA$ and $d_{i,j} = \int_{A(1,R)} |f'|^2 u^{(0)}_i u^{(0)}_j\, dA$. It follows as above

$$\sum_{j=1}^{n} \frac{1}{\mu_j} \geq \sum_{j=1}^{n} \frac{d_{j,j}}{\mu^{(0)}_j},$$

where $mu_j^{(0)}$ is $mu_{m,n}^{(0)}$ for a suitable choice of numbering. Now we have to calculate $d_{j,j}$, see [14]. For f, we have

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_{-1} z^{-1} + a_{-2} z^{-2} + a_{-3} z^{-3} + \cdots$$

for $1 < r_0 < R$. Therefore it follows with $z = re^{i\phi}$ and $1 < r_0 < R$ that

$$\int\limits_{|z|=r_0} f'(z)\, d\phi = 2\pi a_1 \tag{3.9}$$

and, by using Schwarz's inequality, we obtain

$$2\pi \int\limits_{|z|=r_0} |f'(z)|^2\, d\phi \geq \left(\int\limits_{|z|=r_0} |f'(z)|\, d\phi \right)^2 \geq \left| \int\limits_{|z|=r_0} f'(z)\, d\phi \right|^2 = (2\pi |a_1|)^2. \tag{3.10}$$

Let $u_j^{(0)}$ be an eigenfunction of radial type. So we have with (3.9) and (3.10) and the normalizing condition $\int_{A(1,R)} u_j^{(0)^2}\, dA = 1$

$$\begin{aligned} d_{j,j} &= \int\limits_{A(1,R)} |f'|^2 u_j^{(0)^2}\, dA = \int\limits_1^R r \cdot |f'|^2 u_j^{(0)^2}\, dr\, d\phi \geq 2\pi |a_1|^2 \int\limits_1^R r \cdot u_j^{(0)^2}\, dr \\ &= |a_1|^2 \int\limits_A u_j^{(0)^2}\, dA = |a_1|^2. \end{aligned} \tag{3.11}$$

Now let $u_j^{(0)}$ be a nonradial eigenfunction and $u_{j+1}^{(0)}$ be the eigenfunction to the same eigenvalue. So $u_j^{(0)^2} + u_{j+1}^{(0)^2}$ is radial. In the same way we conclude that

$$d_{j,j} + d_{j+1,j+1} = |a_1|^2 \int\limits_{A(1,R)} \left(u_j^{(0)^2} + u_{j+1}^{(0)^2} \right) dA \geq 2|a_1|^2. \tag{3.12}$$

(3.11) and (3.12) imply

Theorem 10. *For the eigenvalues of problem* (3.8), *in* Ω *we have*

$$\sum_{j=1}^n \frac{1}{\mu_j} \geq a_1^2 \sum_{j=1}^n \frac{1}{\mu_j^{(0)}}$$

for any n. *The equality holds if* A *is an annulus.*

Obviously, this theorem implies

Corollary 2. *Let* Φ *be a convex and increasing function, for any* n *we have*

$$\sum_{j=1}^n \Phi\Big(\frac{1}{\mu_j}\Big) \geq \sum_{j=1}^n \Phi\Big(|a_1^2| \frac{1}{\mu_j^{(0)}}\Big).$$

Remark 3. Theorem 10 and Corollary 2 were first proved by Laugesen and Morpugo [16].

For completeness we mention here a formula given in [3] which is closely related to Theorem 10.

Theorem 11. *Let f be a conformal mapping of the annulus $A(1,R)$ onto the domain Ω satisfying $\int_{A(1,R)} |f'(z)|^2\, dA_z < \infty$, then we have*

$$\int_{A(1,R)} \int_{A(1,R)} R^2(z,\zeta)|f'(z)|^2|f'(\zeta)|^2\, dA_z\, dA_\zeta = \sum_{j=1}^{\infty} \frac{1}{\mu_j^2}.$$

3.2. **Simply connected domain.** Now let Ω be a simply connected domain and consider problem (3.3) for it.

In the case of the semicircle with $A^{(0)} = \{z:\, -1 \le z \le 1\}$ we have $\lambda_n^{(0)} = n$ and $u_n^{(0)} = \sqrt{\frac{2}{\pi}}\, r^n \sin n\phi$. For further results in this context we refer to [7],[8]. If $R(z,\zeta)$ is the Robin function of the semicircle, we follow [3] and obtain

Theorem 12. *Let f be a conformal mapping of the semicircle onto the domain Ω with $f:\, A^{(0)} \to A$ and $f:\, B^{(0)} \to B$. For the eigenvalues of problem (3.3) for a simply connected domain Ω we have*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = \int_{B^{(0)}} \int_{B^{(0)}} R(z,\zeta)|f'(z)||f'(\zeta)|\, ds_z\, ds_\zeta.$$

By using Theorem 12 we obtain a formula only in terms of the Fourier coefficients of the mapping function.

Theorem 13. *For the eigenvalues of problem (3.3) for a simply connected domain Ω it follows that*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = \frac{\pi^2}{6} \cdot \frac{\alpha_0^2}{4} - \frac{\alpha_0}{2} \sum_{n=1}^{\infty} \frac{\alpha_{2n}}{n^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{\alpha_{n+m}^2}{n \cdot m}$$
$$- \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \frac{\alpha_{n+m}\alpha_{n-m}}{n \cdot m} + \frac{1}{2} \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \frac{\alpha_{n-m}^2}{n \cdot m}$$

with $(z \in B^{(0)})$

$$|f'(z)| \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos n\phi + \beta_n \sin n\phi).$$

This follows using the Fourier series and the Robin function of the semidisk on $B^{(0)}$

$$R(z,\zeta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\phi \sin n\theta}{n} = \frac{1}{2\pi} \ln \left| \frac{\overline{z} - \zeta}{z - \zeta} \right|$$

with $z = e^{i\phi}$ and $\zeta = e^{i\theta}$.

Applying Theorem 13 to the Joukowski mapping $f(z) = z + 1/z$ we calculate for the eigenvalues of the half-plane

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = 4.220796_{66}^{72}\,.$$

Further, for the eigenvalues of the semicircle with switched boundary components we obtain

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = 0.5687_{53}^{67}$$

using the mapping function $f(z) = \frac{z-i}{zi-1}, i^2 = -1$.

Remark 4. For more similar results see [13].

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