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COMPLEX GEOMETRY OF THE UNIVERSAL TEICHMÜLLER SPACE. II

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Dedicated to the memory of I. N. Vekua
on the occasion of his 100th birth anniversary

Abstract. We give an alternate and simpler proof of the important theorem stating that all invariant distances on the universal Teichmüller space **T** coincide, and solve for **T** the problem of Kra on isometric embeddings of a disk into Teichmüller spaces.

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Let me start with my personal warm reminiscences of scientist Ilia N. Vekua.

As a graduate student of the Institute of Mathematics of the Siberian Branch of the USSR (now Russian) Academy of Sciences at Novosibirsk, I attended in 1961–1963 a graduate course and two research seminars of Ilia Vekua at Novosibirsk State University. At that time, he was Rector of the university. His course was devoted to generalized analytic functions, a new theory established and developed by Vekua, Bers *et al.* The seminars dealt with partial differential equations, complex analysis, geometry, boundary value problems, etc.

At the end of 1962, Vekua brought to a seminar meeting numerous new mathematical books by some leading American, French and German mathematicians on Riemann surfaces, complex manifolds, the modern theory of differential equations on manifolds, geometry, and topology, which had been recently translated into Russian. He then made the following comment to the participants of his seminar:

"What we have been doing until now can be regarded only as walking or at most riding a bicycle along narrow tracks, whereas the best foreign mathematicians are rushing by car on wide roads and they have opened new vistas in mathematics. We must diligently study their works, starting with these books. Especially, this concerns our young researchers."

This statement had a strong impact on me and influenced my research in a significant way. Later I also had several very fruitful personal discussions with I. N. Vekua on these and other fields and methods in mathematics. These discussions were very instructive in that they indicated me how a world renowned

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scientist conveys ideas to a student only beginning his research. Subsequently, I successfully applied his methods and ideas. Later he also played an important role in my life.

In my opinion, I. N. Vekua was not only one of the greatest mathematicians and an outstanding representative of his native country Georgia and the world of science and culture, but also a wonderful person.

This paper concerns both geometric complex analysis and differential geometry, two closely related fields to which the contributions of I. N. Vekua were fundamental.

1. Main Theorem

The aim of this paper is twofold. First, we give an alternate and much simpler proof of the important theorem that all contractible invariant distances on the universal Teichmüller space $\mathbf{T}$ coincide. This result established in [13] has many applications. Here we apply this result to solve for $\mathbf{T}$ the problem of Kra on isometric embeddings of a disk into Teichmüller spaces.

Denote the Carathéodory, Kobayashi and Teichmüller metrics on $\mathbf{T}$ by $c_{\mathbf{T}}$, $d_{\mathbf{T}}$ and $\tau_{\mathbf{T}}$, respectively. It is elementary that $c_{\mathbf{T}} \le d_{\mathbf{T}} \le \tau_{\mathbf{T}}$.

Theorem 1.1. *All contractible invariant distances on the universal Teichmüller space* $\mathbf{T}$ *coincide with its Teichmüller distance, and for any two points* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{T}(X)$ *we have the equality*

$$c_{\mathbf{T}}(\mathbf{x}_1, \mathbf{x}_2) = d_{\mathbf{T}}(\mathbf{x}_1, \mathbf{x}_2) = \tau_{\mathbf{T}}(\mathbf{x}_1, \mathbf{x}_2) = \inf \operatorname{hyp}(h^{-1}(\mathbf{x}_1), (h^{-1}(\mathbf{x}_2)), \qquad (1.1)$$

where hyp *denotes the hyperbolic Poincaré metric on the unit disk of curvature* -4 *and the infimum is taken over all holomorphic maps* $h: \Delta \to \mathbf{T}(X)$.

The corresponding infinitesimal metrics coincide with the canonical Finsler structure $F_{\mathbf{T}}(\mathbf{x}, v)$ *on the tangent bundle* $\mathcal{T}(\mathbf{T})$ *of this space*

A basic tool in the proof is the Grunsky inequalities technique. Another essential ingredient is based on extremal properties of Finsler metrics of constant negative holomorphic curvature.

In the last section, we apply this theorem to solving Kra's problem mentioned above.

2. Preliminaries

We briefly present here certain underlying results needed for the proof of Theorem 1.1. The exposition is adapted to our special case.

2.1. **Basic Finsler metrics on Teichmüller spaces.** The universal Teichmüller space $\mathbf{T}$ is the space of quasisymmetric homeomorphisms of the unit circle $S^1 = \partial\Delta$ factorized by Möbius maps. The canonical complex Banach structure on $\mathbf{T}$ is defined by factorization of the ball of Beltrami coefficients

$$\operatorname{Belt}(\Delta)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|\Delta^* = 0, \ \|\mu\| < 1\},$$

letting $\mu_1, \mu_2 \in \operatorname{Belt}(\Delta)_1$ be equivalent if the corresponding quasiconformal maps w^{μ_1}, w^{μ_2} (solutions of the Beltrami equation $\partial_{\bar{z}} w = \mu \partial_z w$ with $\mu = \mu_1, \mu_2$)

coincide on S^1 (hence, on $\overline{\Delta^*}$) and passing to Schwarzian derivatives S_{f^μ}. The defining projection $\phi_{\mathbf{T}} : \mu \to S_{w^\mu}$ is a holomorphic map from $L_\infty(\Delta)$ to $\mathbf{B}$. The equivalence class of a map w^μ will be denoted by $[w^\mu]$. An appropriate normalization of w^μ will be indicated in subsection 2.3.

An intrinsic complete metric on the space $\mathbf{T}$ is the **Teichmüller metric** defined by

$$\tau_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi_{\mathbf{T}}(\nu)) = \frac{1}{2} \inf\big\{\log K\big(w^{\mu_*} \circ \big(w^{\nu_*}\big)^{-1}\big) : \mu_* \in \phi_{\mathbf{T}}(\mu),\ \nu_* \in \phi_{\mathbf{T}}(\nu)\big\}. \quad (2.1)$$

It is generated by the **Finsler structure** on the tangent bundle $\mathcal{T}(\mathbf{T}) = \mathbf{T} \times \mathbf{B}$ of $\mathbf{T}$ defined by

$$\begin{aligned} F_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi'_{\mathbf{T}}(\mu)\nu) = \inf\big\{\big\|\nu_*(1 - |\mu|^2)^{-1}\big\|_\infty : \phi'_{\mathbf{T}}(\mu)\nu_* = \phi'_{\mathbf{T}}(\mu)\nu; \\ \mu \in \mathrm{Belt}(\Delta)_1;\ \nu, \nu_* \in L_\infty(\mathbb{C})\big\}. \end{aligned} \quad (2.2)$$

The space $\mathbf{T}$ as a complex Banach manifold has also the contractible invariant metrics. Two of these (the largest and the smallest one) are of special interest. They are called the Kobayashi and the Carathéodory metric, respectively, and are defined as follows.

The **Kobayashi metric** $d_{\mathbf{T}}$ on $\mathbf{T}$ is the largest pseudometric d on $\mathbf{T}$ contracted by holomorphic maps $h :\ \Delta \to \mathbf{T}$ so that for any two points $\psi_1,\ \psi_2 \in \mathbf{T}$ we have

$$d_{\mathbf{T}}(\psi_1, \psi_2) \le \inf\{d_\Delta(0, t) :\ h(0) = \psi_1,\ h(t) = \psi_2\},$$

where d_Δ is the **hyperbolic Poincaré metric** on Δ of Gaussian curvature -4, with the differential form

$$ds = \lambda_{\mathrm{hyp}}(z)|dz| := |dz|/(1 - |z|^2). \quad (2.3)$$

The **Carathéodory** distance between ψ_1 and ψ_2 is

$$c_{\mathbf{T}}(\psi_1, \psi_2) = \sup \mathrm{hyp}(\widetilde{h}(\psi_1), \widetilde{h}(\psi_2)),$$

where the supremum is taken over all holomorphic maps $\widetilde{h} :\ \mathbf{T} \to \Delta$.

The corresponding differential (infinitesimal) forms of the Kobayashi and Carathéodory metrics are defined for the points $(\psi, v) \in \mathcal{T}(\mathbf{T})$, respectively, by

$$\begin{aligned} \mathcal{K}_{\mathbf{T}}(\psi, v) &= \inf\{1/r :\ r > 0,\ h \in \mathrm{Hol}(\Delta_r, \mathbf{T}), h(0) = \psi, dh(0) = v\}, \\ \mathcal{C}_{\mathbf{T}}(\psi, v) &= \sup\{|df(\psi)v| :\ f \in \mathrm{Hol}(\mathbf{T}, \Delta), f(\psi) = 0\}, \end{aligned} \quad (2.4)$$

where $\mathrm{Hol}(X, Y)$ denotes the collection of holomorphic maps of a complex manifold X into Y and Δ_r is the disk $\{|z| < r\}$.

Due to the fundamental Gardiner–Royden theorem, the Kobayashi metric on Teichmüller spaces is equal to the Teichmüller metric (cf. [4], [6], [8], [24]). Its strengthened version obtained in [14] for the universal Teichmüller space by applying the Grunsky inequalities states more:

Proposition 2.1. *The differential Kobayashi metric $\mathcal{K}_{\mathbf{T}}(\varphi, v)$ on the tangent bundle $\mathcal{T}(\mathbf{T})$ of the universal Teichmüller space $\mathbf{T}$ is logarithmically plurisubharmonic in $\varphi \in \mathbf{T}$, equals the canonical Finsler structure $F_{\mathbf{T}}(\varphi, v)$ on $\mathcal{T}(\mathbf{T})$*

generating the Teichmüller metric of $\mathbf{T}$ *and has the constant sectional holomorphic curvature* $\kappa_{\mathcal{K}}(\varphi, v) = -4$ *on* $\mathcal{T}(\mathbf{T})$.

Recall that the **sectional holomorphic curvature** of an upper semicontinuos Finsler metric on $\mathbf{T}$ equals the supremum of the Gaussian curvatures

$$\kappa_\lambda(t) = -\frac{\mathbf{\Delta} \log \lambda(t)}{\lambda(t)^2} \tag{2.5}$$

over an appropriate collection of holomorphic maps from the disk into X for a given tangent direction at the image. For an arbitrary upper semicontinuous Finsler metric $ds = \lambda(t)|dt|$ on a plane domain, the curvature (2.5) is defined using the **generalized Laplacian**

$$\mathbf{\Delta}\lambda(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \Big\{ \frac{1}{2\pi} \int\limits_0^{2\pi} \lambda(t + re^{i\theta}) d\theta - \lambda(t) \Big\}$$

(for $-\infty \leq \lambda(t) < \infty$). Similarly to C^2 functions, for which $\mathbf{\Delta}$ coincides with the usual Laplacian, λ is subharmonic on its domain if and only if $\mathbf{\Delta}\lambda(t) \geq 0$, and at the points t_0 of local maxima of λ with $\lambda(t_0) > -\infty$, we have $\mathbf{\Delta}\lambda(t_0) \leq 0$.

As is well known, the holomorphic curvature of the Kobayashi metric $\mathcal{K}(x, v)$ of any complete hyperbolic manifold X satisfies $\kappa_{\mathcal{K}_X} \geq -4$ at all points (x, v) of the tangent bundle $\mathcal{T}(X)$ of X, and for the Carathéodory metric $\mathcal{C}_X$ we have $\kappa_{\mathcal{C}}(x, v) \leq -4$ (provided X is complete C-hyperbolic). For details and general properties of invariant metrics, we refer to [3], [11] (see also [1], [14]).

2.2. **Frame maps and Strebel points.** Let $f_0 := f^{\mu_0}$ be an extremal representative of its class $[f_0]$ with dilatation

$$k(f_0) = \|\mu_0\|_\infty = \inf\{k(f^\mu) : f^\mu|S^1 = f_0|S^1\},$$

and assume that in this class there exists a quasiconformal map f_1 whose Beltrami coefficient μ_{f_1} satisfies the inequality

$$\operatorname{ess\,sup}_{A_r} |\mu_{f_1}(z)| < k(f_0)$$

in some annulus $A_r := \{z : r < |z| < 1\}$. Then f_1 is called the **frame map** for the class $[f_0]$, and the corresponding point in the space $\mathbf{T}$ is called the **Strebel point**.

We use the following important properties of Srebel points adapted to our case.

Proposition 2.2.

(i) *If a class* $[f]$ *has a frame map, then the extremal map* f_0 *in this class is unique and either conformal or a Teichmüller map with Beltrami coefficient* $\mu_0 = k|\psi_0|/\psi_0$ *on* Δ*, defined by an integrable holomorphic quadratic differential* ψ *on* Δ *and a constant* $k \in (0, 1)$ [25].

(ii) *The set of Strebel points is open and dense in* $\mathbf{T}$ [8].

The assertion (i) holds, for example, for asymptotically conformal (hence for all smooth) curves $f(S^1)$.

We shall use the following construction employed in [8]. Suppose f_0 with Beltrami coefficient μ_0 is extremal in its class. Fix a number ϵ between 0 and 1, and take an increasing sequence $\{r_n\}_1^\infty$ with $0 < r_n < 1$ approaching 1. Put

$$\mu_n(z) = \begin{cases} \mu_0(z) & \text{if } |z| < r_n, \\ (1-\epsilon)\mu_0(z) & \text{otherwise} \end{cases} \tag{2.6}$$

and let f_n be a quasiconformal map with Beltrami coefficient μ_n. Then, for sufficiently large n, f_n is a frame map for its class, and the dilatation k_n of the extremal map in the class of f_n approaches $k_0 = k(f_0)$.

Similar results hold also for arbitrary Riemann surfaces (cf. [5], [8]).

2.3. **Grunsky inequalities.** The classical Grunsky theorem states that a holomorphic function

$$f(z) = z + \text{const} + O(z^{-1})$$

in a neighborhood U_0 of $z = \infty$ can be extended to a univalent holomorphic function on the disk

$$\Delta^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : \ |z| > 1\}$$

if and only if its Grunsky coefficients α_{mn} satisfy the inequalities

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le 1, \tag{2.7}$$

where α_{mn} are generated by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\Delta^*)^2,$$

$\mathbf{x} = (x_n)$ runs over the unit sphere $S(l^2)$ of the Hilbert space l^2 with $\|\mathbf{x}\|^2 = \sum_1^\infty |x_n|^2$, and the principal branch of the logarithmic function is chosen (cf. [9]). The quantity

$$\varkappa(f) := \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| : \mathbf{x} = (x_n) \in S(l^2) \Big\} \tag{2.8}$$

is called the **Grunsky norm** of f.

Let Σ denote the collection of all univalent holomorphic functions

$$f(z) = z + b_0 + b_1 z^{-1} + \cdots : \Delta^* \to \widehat{\mathbb{C}} \setminus \{0\},$$

and let $\Sigma(k)$ be its subset of functions with k-quasiconformal extensions to the unit disk $\Delta = \{|z| < 1\}$ so that $f(0) = 0$. Put $\Sigma^0 = \bigcup_k \Sigma(k)$.

This collection is closely related to **universal Teichmüller space T** modelled as a bounded domain in the Banach space **B** of holomorphic functions in Δ^* with norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{\Delta^*}(|z|^2 - 1)^2|\varphi(z)|.$$

All $\varphi \in \mathbf{B}$ can be regarded as the **Schwarzian derivatives**

$$S_f = (f''/f')' - (f''/f')^2/2$$

of locally univalent holomorphic functions in Δ^*. The points of $\mathbf{T}$ represent the functions $f \in \Sigma^0$ whose minimal dilatation

$$k(f) := \inf\{k(w^\mu) = \|\mu\|_\infty : \ w^\mu|\partial\Delta^* = f\}$$

determines the Teichmüller metric on $\mathbf{T}$; here $\|\mu\|_\infty = \operatorname{ess\,sup}_{\mathbb{C}} |\mu(z)|$.

Grunsky's theorem was strengthened for the functions with quasiconformal extensions by many authors, resulting in the following sharp equalities for $f \in \Sigma^0$ ([17], [20]; see also [23], [18], [26]):

$$\varkappa(f) \le k(f) \le \frac{3}{2\sqrt{2}}\varkappa(f) = 1.08\dots\varkappa(f). \tag{2.9}$$

The second inequality in (2.9) is important in applications of Grunsky inequalities for example to Fredholm eigenvalues. The first inequality in (2.9) is important in applications to Teichmüller spaces.

A point is that for a generic function $f \in \Sigma^0$, we have in (2.9) the strict inequality $\varkappa(f) < k(f)$ (cf. [12], [21], [19]). The functions with $\varkappa(f) = k(f)$ play a crucial role in applications of the Grunsky inequality technique and are characterized as follows.

Denote by $A_1(\Delta)$ the subspace of $L_1(\Delta)$ formed by holomorphic functions in Δ, and consider the set

$$A_1^2 = \{\psi \in A_1(\Delta) : \ \psi = \omega^2, \ \omega \text{ holomorphic}\}$$

which consists of the integrable holomorphic functions on Δ having only zeros of even order. Put

$$\langle \mu, \psi \rangle_\Delta = \iint\limits_D \mu(z)\psi(z)dxdy, \quad \mu \in L_\infty(\Delta), \ \psi \in L_1(\Delta) \ \ (z = x + iy).$$

Proposition 2.3 ([12], [16]). *The equality $\varkappa(f) = k(f)$ holds if and only if the function f is the restriction to Δ^* of a quasiconformal self-map w^{μ_0} of $\widehat{\mathbb{C}}$ with Beltrami coefficient μ_0 satisfying the condition*

$$\sup |\langle \mu_0, \psi \rangle_\Delta| = \|\mu_0\|_\infty, \tag{2.10}$$

where the supremum is taken over holomorphic functions $\psi \in A_1^2(\Delta)$ with $\|\psi\|_{A_1(\Delta)} = 1$.

If, in addition, the class $[f]$ contains a frame map (is a Strebel point), then μ_0 is of the form

$$\mu_0(z) = \|\mu_0\|_\infty|\psi_0(z)|/\psi_0(z) \quad \text{with} \ \ \psi_0 \in A_1^2 \ \text{in} \ \Delta. \tag{2.11}$$

For analytic curves $f(S^1)$, the equality (2.11) was obtained by a different method in [21].

As is shown in [12], the elements of A_1^2 are represented in the form

$$\psi(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn}\; x_m x_n z^{m+n} \tag{2.12}$$

with $\mathbf{x} = (x_n) \in S(l^2)$.

Geometrically, the equality (2.10) means that on the holomorphic disk

$$\Delta(\mu_0) = \{\phi_\infty(t\mu_0/\|\mu_0\|) : t \in \Delta\} \subset \mathbf{T},$$

where $\phi_{\mathbf{T}}$ is the canonical quotient map determining $\mathbf{T}$, the Carathéodory and Teichmüller metrics of $\mathbf{T}$ are equal. Together with (2.8) and (2.12), this equality allows us to derive from the collection of holomorphic maps

$$h_{\mathbf{x}}(\varphi) = \sum_{m,n=1}^{\infty} \sqrt{mn}\; \alpha_{mn}(\varphi) x_m x_n :\ \mathbf{T} \to \Delta \tag{2.13}$$

a sequence which is maximizing for the Carathéodory distance $c_{\mathbf{T}}(\psi, \mathbf{0})$, $\psi \in \Delta(\mu_0)$.

3. Two Auxiliary Transforms

3.1. We shall need an extension of Proposition 2.3 to quadratic differentials with zeros of odd order obtained as follows.

Fix $a \in \Delta \setminus \{0\}$ and compose the functions $f \in \Sigma^0$ with the fractional linear map

$$\gamma_a(z) = \frac{\overline{a}}{a}\,\frac{z+a}{1+\overline{a}z},$$

preserving both disks Δ and Δ^*. Then

$$f \circ \gamma_a(z) = f\Big(\frac{1/z + 1/a}{1 + 1/(\overline{a}z)}\Big) = f\Big(\frac{1}{a}\Big) + \Big(1 - \frac{1}{|a|^2}\Big) f'\Big(\frac{1}{a}\Big)\frac{1}{z} + \cdots .$$

We define the transform

$$\begin{aligned} \mathcal{L}_a :\ f \mapsto f_a(z) &= \frac{(1 - 1/|a|^2) f'(1/a)}{f \circ \gamma_a(z) - f(1/a)} + \frac{1}{2}\Big[\Big(1 - \frac{1}{|a|^2}\Big)\frac{f''(1/a)}{f'(1/a)} - \frac{2}{a}\Big] + \cdots \\ &= z + \frac{b_{1,a}}{z} + \cdots , \end{aligned} \tag{3.1}$$

which preserves the classes $\Sigma(k)$ and Σ^0. Both maps $f \circ \gamma_a$ and f_a have the same Beltrami coefficient

$$\gamma_a^*(\mu) = (\mu \circ \gamma_a)\overline{\gamma_a'}/\gamma_a'.$$

Put

$$\log \frac{f_a(z) - f_a(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn}^a z^{-m} \zeta^{-n},$$

choosing a single-valued branch of logarithm similar to above, and let $\varkappa_a(f) = \varkappa(f_a)$.

We shall also use the transform

$$\mathcal{M}: \ f(z) \mapsto f_2(z) = \sqrt{f(z^2)} = z + \frac{b_0^1}{2z^3} + \cdots , \qquad (3.2)$$

whose images are the odd functions from Σ^0, and write

$$f_{a,2}(z) = \sqrt{f_a(z^2)}, \quad \varkappa_{a,2}(f) = \varkappa(f_{a,2}).$$

The squaring map

$$\mathcal{M}_0: \ z \mapsto z^2$$

is a branched holomorphic covering map $\widetilde{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widetilde{\mathbb{C}}$ is the two-sheeted sphere ramified over the points 0 and ∞. The transform (3.2) is well-defined on the maps $f^\mu \in \Sigma^0$.

Each $\mathcal{M} f^\mu$ is a fiber-wise map of $\widetilde{\mathbb{C}}$, which means that this map is compatible with the projection $\mathcal{M}_0$. Its restriction to one sheet of (choosing the branch of the square root accordingly to (3.3)) gives an injective odd quasiconformal map of $\widehat{\mathbb{C}}$.

3.2. Note that a holomorphic map $g : \Delta \to \Delta$ transforms the Beltrami and integrable forms on Δ, respectively, by

$$\mu \mapsto g^*(\mu) := (\mu \circ g)\overline{g'}/g', \quad \psi \mapsto g^*(\psi) := (\psi \circ g)(g')^2; \qquad (3.3)$$

hence

$$g^*(\mu) g^*(\psi) = (\mu\psi) \circ g |g'|^2.$$

In particular, let a quadratic differential $\psi = \psi(\zeta) d\zeta^2 \in A_1(\Delta)$ have zero of odd order at the origin, i.e.,

$$\varphi(\zeta) = \sum_{2m-1}^{\infty} d_j \zeta^j \ \text{ with } \ d_{2m-1} \neq 0.$$

Lifting ψ to $\widetilde{\mathbb{C}}$, one gets the differential

$$\widetilde{\psi}(z) = \mathcal{M}_0^* \psi = 4(d_{2m-1} z^{4m-2} + d_{2m} z^{4m} + \cdots) dz^2 \quad (\zeta = z^2)$$

which has at $z = 0$ zero of even order. If ψ has other zeros $a_1, a_2, \ldots$ in $\Delta \setminus \{0\}$, then $\widetilde{\psi}$ has zeros of the same order at the points $\pm\sqrt{a_j}$ on $\widetilde{\mathbb{C}}$. A choice of sign determines the corresponding single-valued branch of the root as well as a sheet of $\widetilde{\mathbb{C}}$.

3.3. Together with Proposition 2.3, this yields

Proposition 3.1. *Equality $\varkappa_{a,2}(f) = k(f)$ for $f \in \Sigma^0$ holds if and only if the function f is the restriction to Δ^* of a quasiconformal map w^{μ_0} of $\widehat{\mathbb{C}}$ whose Beltrami coefficient $\mu_0(z)$ in Δ satisfies the relation*

$$\|\mu_0(z)\|_\infty = \|\gamma_a^* \mu_0(z^2)\|_\infty = 2 \sup \Big| \iint\limits_{\Delta} \gamma_a^* \mu_0(z^2) \ \gamma_a^* \psi(z^2) \ |z|^2 dx dy \Big|, \qquad (3.4)$$

where the supremum is taken over the set of functions $\psi \in A_1(\Delta \setminus \{a\})$ *such that* $\gamma_a^* \psi(z^2) z^2 \in A_1^2(\Delta)$ *and*

$$\|\gamma_a^* \psi(z^2) z^2\|_{A_1(\Delta)} = \frac{1}{2} \ \|\psi\|_{A_1(\Delta \setminus \{a\})} = \frac{1}{2}\,. \tag{3.5}$$

This proposition involves a non-injective change of variables by integration. A general rule says that for the Sobolev continuous maps $g = (g_1, \dots , g_n) \in W^1_{n,\mathrm{loc}}(G)$ of a bounded domain $G \subset \mathbb{R}^n$ with the boundary of zero $(n-1)$-Lebesgue measure, we have the equality

$$\int\limits_G u \circ g(x) |J_g(x)| dx = \int\limits_{\mathbb{R}^n} u(y) N(y, g, G) dy, \tag{3.6}$$

where $J_g(x)$ is the Jacobian of g at the points $x \in G$, and $N(y, g, G)$ denotes the multiplicity of the map g at y determined as the (finite or infinite) number of points of the set $g^{-1}\{y\} \cap G$ (see, e.g., [7]).

The actions of maps $\mathcal{L}_a^*$ and $\mathcal{M}^*$ on elements $\mu \in \mathrm{Belt}(\Delta)_1$ by (3.3) are holomorphic self-maps of this ball which descend to holomorphic self-maps of $\mathbf{T}$.

4. Proof of Theorem 1.1

In view of the density of Strebel points in $\mathbf{T}$, it suffices to establish the equality of metrics on Teichmüller extremal disks

$$\Delta(\mu_0^*) = \{\phi_{\mathbf{T}}(t\mu_0^*) : \ t \in \Delta\}, \quad \mu_0^* = |\psi_0|/\psi_0, \quad \text{with} \ \ \psi_0 \in A_1(\Delta) \setminus \{\mathbf{0}\}, \tag{4.1}$$

and Proposition 2.3 allows us to consider only the disks (4.1) corresponding to differentials ψ_0 having zeros of odd order in Δ. The proof will be accomplished in three stages.

Step 1. Quadratic differentials with a finite number of zeros of odd order. We will use here the notation $f^0 = f^{t\mu_0^*}$ for any $t \in \Delta$.

Let ψ_0 have in Δ a single zero a of odd order; then the map $f^0_{a,2} := \mathcal{L}_a \circ \mathcal{M} f^0$ satisfies

$$\varkappa(f_{a,2}) := \varkappa_{a,2}(f^0) = k(f^0).$$

Applying (2.7), (2.12) and Proposition 3.1, one gets for an appropriate sequence $\mathbf{x}^{(p)} = (x_n^{(p)}) \in S(l^2)$, the relations $|h_{\mathbf{x}^{(p)}}(\varphi)| < 1$ for all $\varphi \in \mathbf{T}$, while

$$\lim_{p\to\infty} |dh_{\mathbf{x}^{(p)}}(\mathbf{0})(tS_{f^0_{a,2}})| = |\langle t(\mathcal{M} \circ \mathcal{L}_a)^* \mu_0^*, (\mathcal{M} \circ \mathcal{L}_a)^* \psi_0 \rangle_\Delta| = |t|. \tag{4.2}$$

This equality means that

$$\mathcal{C}_{\mathbf{T}}(\mathbf{0}, \phi'_{\mathbf{T}}(\mu_0^*)) = F_{\mathbf{T}}(\mathbf{0}, \phi'_{\mathbf{T}}(\mu_0^*)) = |t|. \tag{4.3}$$

By Schwarz's lemma, we obtain from here the equality of metrics $c_{\mathbf{T}}$ and $\tau_{\mathbf{T}}$ on the disk (4.1).

Assume now that equality (4.3) holds for any quadratic differential ψ with $n = m \geq 1$ distinct zeros of odd order and consider a differential ψ_0 whose zeros of odd order in Δ are the distinct points

$$a_1, a_2, \dots, \ a_m, a_{m+1}.$$

Applying again the transforms (3.2) and (3.3), we get the extremal map $f^0_{a,2} := \mathcal{M} \circ \mathcal{L}_{a_{m+1}} f^0$ whose Beltrami coefficient

$$\mu_{f^0_{a,2}} = t|\psi_{0,m+1}|/\psi_{0,m+1}$$

is defined by quadratic differential

$$\psi_{0,m+1} = (\mathcal{M} \circ \mathcal{L}_{a_{m+1}})^* \psi_0. \tag{4.4}$$

The differential (4.4) has, on the covering surface $\widetilde{\mathbb{C}}$, at most $2m$ distinct zeros of odd order, located at the points

$$\pm\sqrt{\mathcal{M} \circ \mathcal{L}_{a_{m+1}}(a_j)}, \quad j = 1, \dots, m.$$

Hence each sheet of $\widetilde{\mathbb{C}}$ contains at most m zeros of $\psi_{0,m+1}$.

By assumption, there exists a sequence of holomorphic maps $h_p : \mathbf{T} \to \Delta$ with

$$\lim_{p\to\infty} |dh_p(\mathbf{0})[\phi'_{\mathbf{T}}(\mu^*_{f^0_{a,2}})]| = \mathcal{C}_{\mathbf{T}}(\mathbf{0}, \phi'_{\mathbf{T}}(\mu^*_{f^0_{a,2}})) = F_{\mathbf{T}}(\mathbf{0}, \phi'_{\mathbf{T}}(\mu^*_{f^0_{a,2}})) = 1. \tag{4.5}$$

Then the functions

$$h_p \circ (\mathcal{M} \circ \mathcal{L}_{a_{m+1}})^*, \quad p = 1, 2, \dots ,$$

form a maximizing sequence for the Carathéodory metric on the original disk (4.1), and the equalities (4.5) force equalizing this and Teichmüller metrics on $\Delta(\mu^*_0)$.

Applying induction, one obtains the desired equality of metrics on all disks (4.1) defined by quadratic differentials ψ_0 with a finite number of zeros of odd order.

Step 2. Asymptotically conformal maps. We begin to consider the extremal disks (4.1) corresponding to quadratic differentials ψ_0 having in Δ infinitely many zeros of odd order.

Assume first that the restriction of maps f^μ from a given class $[f_*]$ to S^1 is **asymptotically conformal**; in other words, for any pair of points a, b on $L = f(S^1)$, we have

$$\max_{z\in L(a,b)} \frac{|a - z| + |z - b|}{|a - b|} \to 1 \quad \text{as} \quad |a - b| \to 0,$$

where $L(a, b)$ denotes the subarc of L with the endpoints a, b of smaller diameter.

Any such class $[f_*]$ is a Strebel point (cf. [25]), hence it contains a unique Teichmüller map f_0 with Beltrami coefficient

$$\mu_* = k_*|\psi_*|/\psi_*, \quad 0 < k_* < 1, \quad \psi_* \in A_1(\Delta) \setminus \{\mathbf{0}\}. \tag{4.6}$$

Assume that ψ_* has in Δ an infinite set of zeros of odd order (otherwise, there is nothing to prove). Take a sequence of numbers $r_n < 1$ approaching 1 and put

$$\psi_n(z) := \psi_*(r_n z), \quad \mu_n(z) = k_*|\psi_n(z)|/\psi_n(z),$$

extending all μ_n by zero to Δ^*.

The asymptotic conformality of the curve $f_0(S^1) = f_*(S^1)$ implies

$$\lim_{n\to\infty} \|S_{f_0} - S_{f^{\mu_n}}\|_{\mathbf{B}} = 0. \tag{4.7}$$

Indeed, for any asymptotically conformal quasicircle $L = f(S^1)$ we have (cf. [2])

$$\sigma_f(r) := \sup_{1<|z|<r} (|z|^2 - 1)^2 |S_f(z)| \to 0 \quad \text{as} \quad r \to 1.$$

Hence, for $r > 1$ and all n,

$$\begin{aligned} \sup_{|z|>1} (|z|^2 - 1)^2 |S_{f_0}(z) - S_{f^{\mu_n}}(z)| &\le \sup_{|z|>r} (|z|^2 - 1)^2 |S_{f_0}(z) - S_{f^{\mu_n}}(z)| \\ &\quad + \sup_{1<|z|<r} (|z|^2 - 1)^2 |S_{f_0}(z)| + \sup_{1<|z|<r} (|z|^2 - 1)^2 |S_{f^{\mu_n}}(z)| \\ &\le \sup_{|z|>r} (|z|^2 - 1)^2 |S_{f_0}(z) - S_{f^{\mu_n}}(z)| + \sigma_f(r) + 6(r^2 - 1)^2. \end{aligned} \tag{4.8}$$

Given a small $\varepsilon > 0$, we fix $r > 1$ so that the second and the third terms on the right-hand side of (4.8) do not exceed each $\varepsilon/3$, and thereafter find n_0 such that for all $n \ge n_0$, the first term becomes less than $\varepsilon/3$. This proves (4.7).

We turn back to the proof of our theorem. Every ψ_n has in Δ only a finite number of zeros of odd order, thus by Step 1,

$$c_{\mathbf{T}}(S_{f_n}, \mathbf{0}) = \tau_{\mathbf{T}}(S_{f_n}, \mathbf{0}), \quad n = 1, 2, \dots ,$$

and, by continuity, we have the desired equality

$$c_{\mathbf{T}}(S_{f_0}, \mathbf{0}) = \tau_{\mathbf{T}}(S_{f_0}, \mathbf{0}) \tag{4.9}$$

for all points of the disk (4.1) determined by ψ_*.

Step 3. General quadratic differentials with infinitely many zeros of odd order. Let $\varphi^* \in \mathbf{T}$ be a Strebel point, and its class $[f^*]$ contain a Teichmüller extremal map with a Beltrami coefficient of the form (4.6) (again extended by 0 to Δ^*) defined by the holomorphic quadratic differential ψ^* having in Δ infinitely many zeros $a_1, a_2, \dots$ of odd order.

We complexify the construction involving the coefficients (2.6). Fix $r < 1$ close to 1 and define a family of Beltrami coefficients $\mu_t = \mu(\cdot, t)$ depending on a complex parameter t by

$$\mu(z,t) = \begin{cases} \mu_*(z), & |z| < r, \\ \left(1 - \frac{1}{1+t}\right)\mu_*(z), & r < |z| < 1, \end{cases}$$

and $\mu(z,t) = 0$ for $|z| > 1$. The admissible values of t are those for which $|\mu(z,t)| < 1$. This holds when t ranges over the disk

$$\Omega_a = \{t' \in \mathbb{C} : \ |t' + a| > R(a)\}$$
$$\text{with} \ \ a = a(k^*) = 1/[1 - (k^*)^2] > 1, \quad R(a) = a(a-1). \tag{4.10}$$

Two coefficients μ_t are of a special interest for us, namely, $\mu_\infty = \mu_*$ and μ_0 obtained by extension of $\mu_*|_{|z|<r}$ by zero to the disk $\{|z| > r\}$. We preserve for this μ_0 the notation from the previous step.

The map $f_0 := f^{\mu_0}$ is asymptotically conformal; thus, by Step 2, we have for its Schwarzian derivative $S_{f_0} \in \mathbf{T}$ both the equality (4.9) and its infinitesimal version.

Note also that the image $\phi_{\mathbf{T}}(\Omega_a)$ in $\mathbf{T}$ is a nondegenerate holomorphic disk, hence a simply connected Riemann surface; it contains the points $\varphi_* = S_{f_*^\mu}$ and $\varphi_0 = S_{f_0}$.

Indeed, for any complex linear functional $l(\varphi):\ \mathbf{B} \to \mathbb{C}$ with $l(\varphi_*) \neq l(\varphi_0)$, the composed map $l \circ \phi_{\mathbf{T}}$ is holomorphic on the disk (4.10), thus $l \circ \phi_{\mathbf{T}}(\Omega_a) = \omega_a$ is an open connected plane domain. Therefore, $\widetilde{\Omega}_a = l^{-1}(\omega_a)$ must also be open and connected (and then, by Zhuravlev's theorem, it is simply connected).

Let us compare the infinitesimal Carathéodory and Kobayashi metrics $\mathcal{C}_{\mathbf{T}}$ and $\mathcal{K}_{\mathbf{T}}$ on this disk $\widetilde{\Omega}_a$, applying Proposition 2.1 and Minda's maximum principle:

Lemma 4.1 ([22]). *If a function $u:\ \Omega \to [-\infty, +\infty)$ is upper semicontinuous in a domain $\Omega \subset \mathbb{C}$ and its generalized Laplacian satisfies the inequality $\mathbf{\Delta} u(z) \geq Ku(z)$ with some positive constant K at any point $z \in \Omega$, where $u(z) > -\infty$, then if $\limsup_{z\to\zeta} u(z) \leq 0$ for all $\zeta \in \partial\Omega$, then either $u(z) < 0$ for all $z \in \Omega$ or else $u(z) = 0$ for all $z \in \Omega$.*

The restrictions of $\mathcal{C}_{\mathbf{T}}$ and $\mathcal{K}_{\mathbf{T}}$ to $\widetilde{\Omega}_a$ are conformal subharmonic Finsler metrics $ds = \lambda_{\mathcal{C}}(t)|dt|$ and $ds = \lambda_{\mathcal{K}}(t)|dt|$ on this disk with generalized Gaussian curvatures $\kappa(\lambda_{\mathcal{C}}) \leq -4$ and $\kappa(\lambda_{\mathcal{K}}) \equiv -4$.

Denote the complex parameter on $\widetilde{\Omega}_a$ again by t. For a sufficiently small neighborhood U_0 of the origin $t = 0$, we put

$$M = \{\sup \lambda_{\mathcal{K}}(t) : t \in U_0\};$$

then in this neighborhood, $\lambda_{\mathcal{K}}(t) + \lambda_{\mathcal{C}}(t) \leq 2M$. Taking $u = \log \frac{\lambda_{\mathcal{C}}}{\lambda_{\mathcal{K}}}$, we get for $t \in U_0$,

$$\mathbf{\Delta} u(t) = \log \lambda_{\mathcal{C}}(t) - \lambda_{\mathcal{K}}(t) = 4[\lambda_{\mathcal{C}}(t)^2 - \lambda_{\mathcal{K}}(t)^2] \geq 8M[\lambda_{\mathcal{C}}(t) - \lambda_{\mathcal{K}}(t)]$$

(cf. [22], [15]). The elementary estimate

$$M \log(t/s) \geq t - s \quad \text{for} \quad 0 < s \leq t < M$$

(with equality only for $t = s$) implies that

$$M \log \frac{\lambda_{\mathcal{C}}(t)}{\lambda_{\mathcal{K}}(t)} \geq \lambda_{\mathcal{C}}(t) - \lambda_{\mathcal{K}}(t),$$

and hence, $\mathbf{\Delta} u(t) \geq 4M^2 u(t)$.

Applying Lemma 4.1, we obtain that in view of the equality (4.9) both metrics $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{K}}$ must be equal in U_0 and then in the entire disk $\widetilde{\Omega}_a$, in particular, at the point $\varphi^* = S_{f^{\mu^*}}$, which is equivalent to desired equality

$$c_{\mathbf{T}}(\mathbf{0}, \varphi^*) = d_{\mathbf{T}}(\mathbf{0}, \varphi^*) = \tau_{\mathbf{T}}(\mathbf{0}, \varphi^*).$$

We have established that the Carathéodory and Kobayashi distances $c_{\mathbf{T}}(\varphi, \mathbf{0})$ and $d_{\mathbf{T}}(\varphi, \mathbf{0})$ coincide for any point $\varphi \in \mathbf{T}$. Since the universal Teichmüller space is a homogeneous Banach domain, we get the equality of these distances between two arbitrary points $\varphi_1,\ \varphi_2$ in $\mathbf{T}$, which completes the proof.

5. Problem of Kra

At the Conference on Complex Analysis and Dynamical Systems at Nahariya (Israel, January 2006), Irwin Kra posed the following

Problem. *Is any isometric imbedding of the disk into a finite dimensional Teichmüller space* $\mathbf{T}(\Gamma)$ *holomorphic or antiholomorphic*?

Theorem 1.1 allows us to answer this question affirmatively for the universal Teichmüller space. This case differs from the finite dimensional one, because for any two points of $\mathbf{T}$ there are infinitely many geodesic disks passing through these points.

Theorem 5.1. *Every isometric imbedding of the disk into the universal Teichmüller space* $\mathbf{T}$ *is either holomorphic or antiholomorphic.*

Proof. We apply the following consequence of Theorem 1.1.

Corollary 5.2. *The Teichmüller distance* $\tau_{\mathbf{T}}(\varphi, \psi)$ *is logarithmically plurisubharmonic in each of its variables. Moreover, the pluricomplex Green function of the space* $\mathbf{T}$ *equals*

$$g_{\mathbf{T}}(\varphi, \psi) = \log \tanh \tau_{\mathbf{T}}(\varphi, \psi) = \log k(\varphi, \psi), \tag{5.1}$$

where $k(\varphi, \psi)$ *denotes the extremal dilatation of quasiconformal maps determining the Teichmüller distance between the points* φ *and* ψ *in* $\mathbf{T}$*.*

Recall that the **pluricomplex Green function** of a Banach manifold X is

$$g_X(x, y) = \sup u_{\mathbf{y}}(\mathbf{x}) \quad (\mathbf{x}, \mathbf{y} \in X),$$

where the supremum is taken over plurisubharmonic functions $u_{\mathbf{y}}(\mathbf{x}) : X \to [-\infty, 0)$ such that

$$u_{\mathbf{y}}(\mathbf{x}) = \log \|\mathbf{x} - \mathbf{y}\| + O(1)$$

in a neighborhood of the point y (the pole of g_X); here $\|\cdot\|$ is the norm on the Banach space modeling X, and the remainder term $O(1)$ is bounded from above (cf. [3], [10], [14]). This function is related to Carathéodory and Kobayashi distances by

$$\log \tanh c_X(\mathbf{x}, \mathbf{y}) \le g_D(\mathbf{x}, \mathbf{y}) \le \log \tanh d_D(\mathbf{x}, \mathbf{y}). \tag{5.2}$$

Let now F be a given isometric embedding of the unit disk Δ into $\mathbf{T}$ so that the Teichmüller distance on $F(\Delta)$ equals the hyperbolic distance on Δ. Composing F with biholomorphic automorphisms of Δ and $\mathbf{T}$, one can take F so that $F(0) = \mathbf{0}$. We can also assume that F is orientation preserving, otherwise we precede it by reflection $t \mapsto \bar{t}$. In each case, the image $F(\Delta)$ is a topological disk in $\mathbf{T}$.

Corollary 5.2 and the equalities (5.1) imply that, for a fixed $\varphi_0 = F(t_0) \in F(\Delta)$,

$$g_{\mathbf{T}}(\varphi_0, \varphi) = g_\Delta(t_0, t) = \log[(t - t_0)/(1 - \overline{t_0} t)], \tag{5.3}$$

and that the norm $\|F(t)\|_\infty$ is a logarithmically subharmonic function on Δ.

The following lemma is a straightforward extension of Schwarz's lemma to logarithmically subharmonic functions (cf., e.g., [3], [13]).

Lemma 5.3. *Let a function* $u(z): \Delta \to [0,1)$ *be logarithmically subharmonic in the disk* Δ *and such that the ratio* $u(z)/|z|^m$ *is bounded in a neighborhood of the origin for some* $m \geq 1$. *Then*

$$u(z) \leq |z|^m \quad \text{for all} \;\; z \in \Delta \tag{5.4}$$

and

$$\limsup_{|z|\to 0} \frac{u(z)}{|z|^m} \leq 1. \tag{5.5}$$

Equality in (5.4), *even for one* $z_0 \neq 0$, *or in* (5.5), *can hold only for* $u(z) = |z|^m$.

Using this lemma for $m = 1$ and the equalities (5.3), one concludes that there exist the real tangent vectors $\frac{d}{dt}F(te^{i\theta})$ to $F(\Delta)$ in $\mathbf{T}$. Since F is an isometry, the Teichmüller norm of derivatives along the tangent vectors at the origin equals

$$\|F'(0)\|_\infty = \lim_{t\to 0} \frac{\|F(te^{i\theta})\|_\infty}{t} = 1 \quad (0 \leq \theta \leq 2\pi). \tag{5.6}$$

Since $\mathbf{T}$ is complete C-hyperbolic, the family $\mathcal{H} := \{h \in \mathrm{Hol}(\mathbf{T}, \Delta);\; h(\mathbf{0}) = 0\}$ is normal on $\mathbf{T}$ (with respect to convergence in the Carathéodory metric). Together with Theorem 1.1, this provides the existence of $h_0 \in \mathcal{H}$ with $|h_0'(\mathbf{0})| = 1$ and such that for all complex tangent vectors v to $\mathbf{T}$ at the origin we have the equality

$$\mathcal{C}_{\mathbf{T}}(\mathbf{0}, v) = |dh_0(\mathbf{0})v|. \tag{5.7}$$

We compose h_0 with F and take the function $\log|h_0 \circ F|$. Then the relations (5.1), (5.6) and (5.7) imply that this function must coincide on Δ with $g_{\mathbf{T}}(\mathbf{0}, \varphi) \circ F$, and by (5.3) these both functions are equal to $\log|t|$ on Δ. Hence

$$h_0 \circ F(t) \equiv t \quad \text{on} \quad \Delta.$$

The last identity implies that there exists the inverse function h_0^{-1} of the trace of h_0 on the disk $F(\Delta)$ which is injective and holomorphic on Δ and takes the same values as the given embedding F. This proves the theorem. □

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