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RATE OF CONVERGENCE FOR THE BÉZIER VARIANT OF THE MKZD OPERATORS

VIJAY GUPTA AND HARUN KARSLI

Abstract. We estimate the rate of convergence of the Bézier variant of Durrmeyer type Meyer–König and Zeller operators for functions with derivatives of bounded variation defined on $[0,1]$.

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1. Introduction

For a function f defined on the interval [0,1], the Meyer–König and Zeller (MKZ) operators $\tilde{M}_n(f,x)$ [13] are defined as

$$\tilde{M}_n(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x)\, f\left(\frac{k}{n+k}\right), \tag{1}$$

where $m_{n,k}(x) = \binom{n+k-1}{k} x^k(1-x)^n$. To approximate Lebesgue integrable functions on the interval $[0,1]$, Guo [6] introduced the integrated MKZ operators

$$\hat{M}_n(f;x) = \sum_{k=0}^{\infty} \hat{m}_{n,k}(x) \int_{I_k} f(t)\, dt \tag{2}$$

where $I_k = [\frac{k}{n+k}, \frac{k+1}{n+k+1}]$ and $\hat{m}_{n,k}(x) = (n+1)\binom{n+k+1}{k} x^k(1-x)^n$. For the rate of convergence of some integral modifications of the MKZ operator we refer the reader to [5], [7] and [11]. In [15], Zeng defined, for each $\alpha \geq 1$, Bézier variants of the MKZ operators (1) and (2) by

$$\tilde{M}_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} Q_{n,k}^{\alpha}(x) f\left(\frac{k}{n+k}\right) \tag{3}$$

and

$$\hat{M}_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} \left(\frac{Q_{n,k}^{\alpha}(x)}{\int_{I_k} dt}\right) \int_{I_k} f(t)\, dt\,, \tag{4}$$

where $Q_{n,k}^{(\alpha)}(x) = (J_{n,k}(x))^{\alpha} - (J_{n,k+1}(x))^{\alpha}$ and $J_{n,k}(x) = \sum_{j=k}^{\infty} m_{n,j}(x)$ be the Bézier basis functions, which were introduced by P. Bézier [1]. In particular when $\alpha = 1$, the operators (3) and (4) reduce to the operators (1) and (2), respectively.

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Very recently for $f \in L_1[0,1]$ and $\alpha \geq 1$, Gupta [8] introduced a Bézier variant of the different Durrmeyer type MKZ operators (MKZD) by

$$M_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} Q_{n,k}^{\alpha}(x) \int_0^1 b_{n,k}(t)\, f(t)dt, \quad x \in [0,1], \tag{5}$$

where $b_{n,k}(t) = n \begin{pmatrix} n+k \\ k \end{pmatrix} t^k(1-t)^{n-1}$. Gupta [8] investigated the rate of convergence of the operators (5), for functions of bounded variation on $[0,1]$ (see also [9]).

The aim of this paper is to extend the study on the operators (5) for functions having derivatives of bounded variation on $[0,1]$. Here we establish the rate of convergence of operators $M_{n,\alpha}$ for functions with derivatives of bounded variation defined on $[0,1]$. Several researchers have studied on MKZ operators and its different variants. We also mention the work on similar type of operators due to Bojanic and Cheng (see [2], [3]) who estimated the rate of convergence with derivatives of bounded variation for Bernstein and Hermite–Fejer polynomials by using different methods. Some of the important papers on this topic are due to Bojanic and Khan [4], Pych-Taberska [14], and Gupta et al. [10], [12].

Let $DBV[0,1]$ denotes the class of real valued differentiable functions defined on $[0,1]$, whose derivatives are of bounded variation on $[0,1]$, which can be written as

$$f(x) = f(0) + \int_0^x \Psi(t)dt, \quad x \in [0,1],$$

where $\Psi \in BV[0,1]$. In this sense it is justified to call this class of functions with derivatives of bounded variation and will be denoted as equivalently

$$DBV[0,1] = \{f : f' \in BV[0,1]\}.$$

The main result of this paper is the following assertion.

Theorem. *Let $\alpha \geq 1$ and f be a function with derivatives of bounded variation on $[0,1]$. If f' has a discontinuity of the first kind in $x \in (0,1)$, then for each $\lambda > 2$ and $\varepsilon > 0$, there is an integer $N(x,\lambda)$ such that for all $n \geq N(x,\lambda)$ we have*

$$\begin{aligned}|M_{n,\alpha}(f;x) - f(x)| \leq{}& \frac{\alpha}{\alpha+1}\,|f'(x+) - f'(x-)| \sqrt{\frac{\alpha\lambda(x+\varepsilon)(1-x)^2}{n}} \\ &+ \frac{\alpha\lambda(x+\varepsilon)(1-x)}{n-1}\frac{1}{\alpha+1}\,|f'(x+) + \alpha f'(x-)| \\ &+ \frac{1}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}} (f'_x) + \frac{\alpha\lambda(x+\varepsilon)+x}{nx} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (f'_x),\end{aligned} \tag{6}$$

where

$$f_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq 1, \\ 0, & t = x, \\ f(t) - f(x-), & 0 \leq t < x \end{cases} \tag{7}$$

and $\bigvee_a^b (f'_x)$ *is the total variation of* f'_x *on* $[a, b]$.

2. Auxiliary Results

In this section we give certain results, which are necessary to prove our main theorem.

Lemma 1 ([9]). *For* $s \in \mathbb{N}^0$ *(the set of nonnegative integers), if we define*

$$M_{n,1}((t-x)^s; x) = \sum_{k=0}^{\infty} m_{n,k}(x) \int_0^1 b_{n,k}(t)\,(t-x)^s dt,$$

then

$$|M_{n,1}((t-x); x)| = \frac{x(1-x)}{n-1}$$

and

$$M_{n,1}((t-x)^2; x) \leq \frac{4x}{n-1} + \frac{2(1-x)^2}{(n-1)(n-2)}.$$

In particular, given $\lambda > 2$ *and* $\varepsilon > 0$*, there is an integer* $N(x, \lambda)$ *such that for all* $n \geq N(x, \lambda)$ *and* $x \in [0, 1]$,

$$|M_{n,1}((t-x); x)| \leq \frac{\lambda(x+\varepsilon)(1-x)}{n-1} \tag{8}$$

and

$$M_{n,1}((t-x)^2; x) \leq \frac{\lambda(x+\varepsilon)(1-x)^2}{n}. \tag{9}$$

Remark 1 ([15]). For all $n, k \in N$, there holds $Q^{\alpha}_{n,k}(x) \leq \alpha m_{n,k}(x)$, $x \in [0, 1]$.

Define

$$K_{n,\alpha}(x, t) = \sum_{k=0}^{\infty} Q^{\alpha}_{n,k}(x) b_{n,k}(t)$$

and

$$\lambda_{n,\alpha}(x, t) = \int_0^t K_{n,\alpha}(x, u) du.$$

Note that

$$\lambda_{n,\alpha}(x, 1) = \int_0^1 K_{n,\alpha}(x, u) du = 1. \tag{10}$$

Lemma 2 ([9]). *For each $\lambda > 2$ and $\varepsilon > 0$, there is an integer $N(x,\lambda)$ such that, for all $n \geq N(x,\lambda)$ and $x \in (0,1)$,*

$$\lambda_{n,\alpha}(x,y) \leq \alpha \frac{\lambda(x+\varepsilon)(1-x)^2}{n(x-y)^2}, \quad 0 \leq y < x, \tag{11}$$

$$1 - \lambda_{n,\alpha}(x,z) \leq \alpha \frac{\lambda(x+\varepsilon)(1-x)^2}{n(z-x)^2}, \quad x < z \leq 1. \tag{12}$$

Remark 2. From Cauchy–Schwarz–Bunyakowsky inequality, we get from (9)

$$M_{n,\alpha}(|t-x|\,;x) \leq \left(M_{n,\alpha}((t-x)^2;x)\right)^{\frac{1}{2}} \leq \sqrt{\alpha \frac{\lambda(x+\varepsilon)(1-x)^2}{n}}\,. \tag{13}$$

3. Proof of The Main Result

Now, we can prove the theorem.

Proof of Theorem. According to (5) and equation (10), we can write the difference between $M_{n,\alpha}(f;x)$ and $f(x)$ as follows:

$$\begin{aligned} M_{n,\alpha}(f;x) - f(x) &= \sum_{k=0}^{\infty} Q_{n,k}^{\alpha}(x) \int_0^1 b_{n,k}(t)\, f(t) dt - f(x) \\ &= \int_0^1 [f(t) - f(x)] K_{n,\alpha}(x,t)\, dt. \end{aligned} \tag{14}$$

Since $f(t) \in DBV[0,1]$, we can rewrite equation (14) as

$$\begin{aligned} M_{n,\alpha}(f;x) - f(x) &= \int_0^x [f(t) - f(x)] K_{n,\alpha}(x,t)\, dt + \int_x^1 [f(t) - f(x)] K_{n,\alpha}(x,t)\, dt \\ &= -\int_0^x \left[\int_t^x f'(u)\, du\right] K_{n,\alpha}(x,t)\, dt + \int_x^1 \left[\int_x^t f'(u)\, du\right] K_{n,\alpha}(x,t) dt \\ &= -I_1(x) + I_2(x), \end{aligned}$$

where

$$I_1(x) := \int_0^x \left[\int_t^x f'(u)\, du\right] K_{n,\alpha}(x,t)\, dt \tag{15}$$

and

$$I_2(x) := \int_x^1 \left[\int_x^t f'(u)\, du\right] K_{n,\alpha}(x,t) dt. \tag{16}$$

From (7), for any $f(t) \in DBV[0,1]$, we decompose $f'(t)$ into four parts as

$$f'(t) = \frac{1}{\alpha+1}\left(f'(x+) + \alpha f'(x-)\right) + f'_x(t) + \left(\frac{f'(x+) - f'(x-)}{2}\right) \times \left(\operatorname{sgn}(t-x) + \frac{\alpha-1}{\alpha+1}\right) + \delta_x(t)\left(f'(x) - \frac{f'(x+) + f'(x-)}{2}\right), \tag{17}$$

where

$$\delta_x(t) = \begin{cases} 1, & x = t, \\ 0, & x \neq t. \end{cases} \tag{18}$$

If we use (17) in (15) and (16), we have the following expressions.

$$I_1(x) = \int_0^x \left[\int_t^x \left\{\frac{1}{\alpha+1}\left(f'(x+) + \alpha f'(x-)\right) + f'_x(u) + \left(\frac{f'(x+) - f'(x-)}{2}\right)\left(\operatorname{sgn}(u-x) + \frac{\alpha-1}{\alpha+1}\right) + \delta_x(t)\left(f'(x) - \frac{f'(x+) + f'(x-)}{2}\right)\right\} du\right] K_{n,\alpha}(x,t)\, dt$$

and

$$I_2(x) = \int_x^1 \left[\int_x^t \left\{\frac{1}{\alpha+1}\left(f'(x+) + \alpha f'(x-)\right) + f'_x(u) + \left(\frac{f'(x+) - f'(x-)}{2}\right)\left(\operatorname{sgn}(u-x) + \frac{\alpha-1}{\alpha+1}\right) + \delta_x(t)\left[f'(x) - \frac{f'(x+) + f'(x-)}{2}\right]\right\} du\right] K_{n,\alpha}(x,t)\, dt.$$

Firstly, we evaluate $I_1(x)$.

By (18), it is obvious that $\int_x^t \delta_x(u)\, du = 0$. We have

$$I_1(x) = \frac{1}{\alpha+1}\left[f'(x+) + \alpha f'(x-)\right] \int_0^x (x-t) K_{n,\alpha}(x,t)\, dt + \frac{f'(x+) - f'(x-)}{2} \int_0^x \left[-1 + \frac{\alpha-1}{\alpha+1}\right](x-t) K_{n,\alpha}(x,t)\, dt + \int_0^x \left[\int_t^x f'_x(u)\, du\right] K_{n,\alpha}(x,t)\, dt. \tag{19}$$

Using similar method, for evaluating $I_2(x)$, we find that

$$I_2(x) = \frac{1}{\alpha+1}\left[f'(x+) + \alpha f'(x-)\right] \int_x^1 (t-x) K_{n,\alpha}(x,t)dt$$
$$+ \frac{f'(x+) - f'(x-)}{2} \int_x^1 \left[1 + \frac{\alpha-1}{\alpha+1}\right](t-x) K_{n,\alpha}(x,t)dt$$
$$+ \int_x^1 \left[\int_x^t f'_x(u)\, du\right] K_{n,\alpha}(x,t)\, dt. \tag{20}$$

Since $\alpha \geq 1$, from equations (19) and (20), we obtain an estimate for the difference between $M_{n,\alpha}(f;x)$ and $f(x)$ as follows;

$$|M_{n,\alpha}(f;x) - f(x)| \leq \frac{1}{\alpha+1}\left|f'(x+) + \alpha f'(x-)\right| \left|\int_0^1 (t-x) K_{n,\alpha}(x,t)dt\right|$$
$$+ \frac{\alpha}{\alpha+1}\left|f'(x+) - f'(x-)\right| \left|\int_0^1 |t-x|\, K_{n,\alpha}(x,t)\, dt\right|$$
$$+ \left| - \int_0^x \left[\int_t^x f'_x(u)\, du\right] K_{n,\alpha}(x,t)\, dt\right|$$
$$+ \left|\int_x^1 \left[\int_x^t f'_x(u)\, du\right] K_{n,\alpha}(x,t)\, dt\right|. \tag{21}$$

On the other hand, since

$$\int_0^1 |t-x|\, K_{n,\alpha}(x,t)\, dt = M_{n,\alpha}(|t-x|\,;x) \tag{22}$$

and

$$\int_0^1 (t-x) K_{n,\alpha}(x,t)\, dt = M_{n,\alpha}(t-x;x), \tag{23}$$

then using (22) and (23) in (21), we obtain

$$|M_{n,\alpha}(f;x) - f(x)| \leq \frac{1}{\alpha+1}\left|f'(x+) + \alpha f'(x-)\right| \left|M_{n,\alpha}(t-x;x)\right|$$
$$+ \frac{\alpha}{\alpha+1}\left|f'(x+) - f'(x-)\right| \left|M_{n,\alpha}(|t-x|\,;x)\right|$$
$$+ \left| - \int_0^x \left[\int_t^x f'_x(u)\, du\right] K_{n,\alpha}(x,t)\, dt\right|$$

$$+\left|\int_x^1\left[\int_x^t f_x'(u)\,du\right]K_{n,\alpha}(x,t)\,dt\right|. \tag{24}$$

From the definition of $\lambda_{n,\alpha}(x,t)$, we write

$$\int_0^x\left[\int_t^x f_x'(u)\,du\right]K_{n,\alpha}(x,t)dt=\int_0^x\left[\int_t^x f_x'(u)\,du\right]\frac{\partial}{\partial t}\lambda_{n,\alpha}(x,t)\,dt. \tag{25}$$

Using integration by parts in the right-hand side of (25), we obtain

$$\int_0^x\left[\int_t^x f_x'(u)\,du\right]\frac{\partial}{\partial t}\lambda_{n,\alpha}(x,t)\,dt=\int_0^x f_x'(t)\,\lambda_{n,\alpha}(x,t)\,dt.$$

Thus

$$\left|-\int_0^x\left[\int_t^x f_x'(u)\,du\right]K_{n,\alpha}(x,t)\,dt\right|\le\int_0^x|f_x'(t)|\;\lambda_{n,\alpha}(x,t)\,dt$$

and

$$\left|-\int_0^x\left[\int_t^x f_x'(u)\,du\right]K_{n,\alpha}(x,t)dt\right|\le\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)|\;\lambda_{n,\alpha}(x,t)\,dt+\int_{x-\frac{x}{\sqrt{n}}}^{x}|f_x'(t)|\;\lambda_{n,\alpha}(x,t)\,dt.$$

Since $f_x'(x)=0$ and $\lambda_{n,\alpha}(x,t)\le 1$,

$$\int_{x-\frac{x}{\sqrt{n}}}^{x}|f_x'(t)|\,\lambda_{n,\alpha}(x,t)\,dt=\int_{x-\frac{x}{\sqrt{n}}}^{x}|f_x'(t)-f_x'(x)|\,\lambda_{n,\alpha}(x,t)\,dt\le\int_{x-\frac{x}{\sqrt{n}}}^{x}\bigvee_t^x(f_x')\,dt.$$

Besides from (11), we have

$$\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)|\;\lambda_{n,\alpha}(x,t)\,dt\le\frac{\alpha\lambda(x+\varepsilon)(1-x)^2}{n}\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)|\;\frac{dt}{(x-t)^2}\le\frac{\alpha\lambda(x+\varepsilon)(1-x)^2}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x(f_x')\frac{dt}{(x-t)^2}.$$

Make the change of variables $t = x - \frac{x}{u}$, then

$$\int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x (f_x') \frac{dt}{(x-t)^2} = \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x (f_x') \frac{(\frac{x}{u^2})\,du}{(-\frac{x}{u})^2} = \frac{1}{x} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x (f_x')\,du = \frac{1}{x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f_x')$$

and

$$\int_{x-\frac{x}{\sqrt{n}}}^x \bigvee_t^x (f_x')\,dt \le \bigvee_{x-\frac{x}{\sqrt{n}}}^x (f_x') \int_{x-\frac{x}{\sqrt{n}}}^x dt = \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (f_x').$$

Consequently

$$\left| -\int_0^x \left[\int_t^x f_x'(u)\,du \right] K_{n,\alpha}(x,t)\,dt \right| \le \frac{\alpha\lambda(x+\varepsilon)(1-x)^2}{nx} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f_x') + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (f_x'). \tag{26}$$

By the same way, from (12) we obtain

$$\left| \int_x^1 \left[\int_x^t f_x'(u)\,du \right] K_{n,\alpha}(x,t)\,dt \right| \le \frac{\alpha\lambda(x+\varepsilon)(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{1-x}{k}} (f_x') + \frac{1-x}{\sqrt{n}} \bigvee_x^{x+\frac{1-x}{\sqrt{n}}} (f_x'). \tag{27}$$

Combining (8), (13), (26) and (27) in (24), we get (6).

Thus the proof is completed. □

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Authors' addresses:

V. Gupta
School of Applied Sciences
Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110075
India
E-mail: vijaygupta2001@hotmail.com

H. Karsli
Ankara University
Faculty of Sciences
Department of Mathematics
06100 Tandogan-Ankara
Turkey
E-mail: karsli@science.ankara.edu.tr