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# SOME COMBINATORIAL PROPERTIES OF FINITE LINE-SYSTEMS IN THE EUCLIDEAN PLANE

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**Abstract.** We consider finite systems of straight lines in the Euclidean plane $\mathbf{R}^2$ with some of their combinatorial characteristics. Euler's formula is applied for obtaining results of combinatorial type for such systems. In particular, a lower estimate for the number of two-sided and three-sided domains determined by a given finite line-system in $\mathbf{R}^2$ is presented and it is shown that this estimate is precise in a certain sense.

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Let $\mathcal{L} = \{l_i : i \in I\}$ be a finite family of pairwise distinct straight lines in $\mathbf{R}^2$. Evidently, this family produces the finite point-system in $\mathbf{R}^2$ whose elements are common points of the above-mentioned lines. This associated point-system is empty if and only if all lines from $\mathcal{L} = \{l_i : i \in I\}$ are parallel. In our further consideration we shall avoid this trivial case. In other words, we shall assume that there are at least two distinct lines from $\mathcal{L}$ which have a common point.

If $\text{card}(I)$ is fixed, then the question naturally arises how many combinatorial types of mutual positions for $\{l_i : i \in I\}$ are possible. Another interesting question: how can we describe the combinatorial type of a mutual position of $\{l_i : i \in I\}$ in terms of the associated point-system? Many analogous questions can be posed for finite line-systems in $\mathbf{R}^2$. As a rule, they are simple to formulate but quite often turn out rather difficult. Some of those questions are of certain interest for combinatorial and discrete geometry (see [2]; cf. also [3] and [4]).

Obviously, any finite line-system $\mathcal{L} = \{l_i : i \in I\}$ yields a decomposition of $\mathbf{R}^2$ into polygonal domains (some of them are necessarily unbounded). Let us assume that $\text{card}(I)$ is fixed and denote $\text{card}(I) = m$.

For our further purposes, it is also convenient to introduce the following notation:

$V(m)$ = the total number of vertices of the obtained polygonal domains (equivalently, $V(m)$ is the number of elements of the point-system produced by a given line-system $\{l_i : i \in I\}$).

$E(m)$ = the total number of sides (edges) of the obtained domains (note that among the sides of some of these domains there are rays, so they are necessarily unbounded);

$F(m)$ = the total number of the obtained domains.

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Also, we denote by $F_k(m)$ the number of those domains from this decomposition, which have exactly $k$ sides, where $k = 2, 3, \dots$. In addition, we denote by $V_k(m)$ the number of those vertices which belong to exactly $k$ sides (edges), where $k = 4, 6, 8, \dots$.

Clearly, each domain with exactly two sides is unbounded. Three-sided domains may also be unbounded as well as bounded (in the latter case they are triangular domains or, simply, triangles).

**Theorem 1.** *For $F_2(m)$ and $F_3(m)$, the inequality $2m+4 \leq 2F_2(m)+F_3(m)$ holds true.*

*Proof.* Starting with Euler's formula (see, e.g., [1] or [4]), one can easily deduce that

$$F(m) + V(m) = E(m) + 1.$$

Also, it is not difficult to check the validity of the following relations:

$$F(m) = F_2(m) + F_3(m) + F_4(m) + \cdots ,$$

$$V(m) = V_4(m) + V_6(m) + V_8(m) + \cdots .$$

$$2m + 4V_4(m) + 6V_6(m) + 8V_8(m) + \cdots = 2E(m),$$

$$2F_2(m) + 3F_3(m) + 4F_4(m) + \cdots = 2E(m).$$

Consequently, we have

$$2F_2(m) + 3F_3(m) + 4(F(m) - F_2(m) - F_3(m)) \leq 2E(m),$$

$$4F(m) - 2E(m) \leq 2F_2(m) + F_3(m),$$

whence it follows that

$$4(E(m) + 1 - V(m)) - 2E(m) = 4 + 2E(m) - 4V(m) \leq 2F_2(m) + F_3(m),$$

$$(4 + 2m) + 2V_6(m) + 4V_8(m) + \cdots \leq 2F_2(m) + F_3(m).$$

Since $2V_6(m) + 4V_8(m) + \cdots \geq 0$, we finally obtain $4 + 2m \leq 2F_2(m) + F_3(m)$. This completes the proof of the statement. □

*Remark* 1. In a certain sense, the inequality $2m + 4 \leq 2F_2(m) + F_3(m)$ is exact. Indeed, in Fig. 1 we have $m - 1$ parallel lines and one more line not parallel to them and, hence, intersecting all of them. In that case, we obviously have

$$F_2(m) = 4,\ F_3(m) = 2(m - 2),\ 2F_2(m) + F_3(m) = 2m + 4,$$

so the above-mentioned inequality reduces to the equality. Moreover, the argument used in the proof of Theorem 1 yields that the relation $2F_2(m)+F_3(m) = 2m + 4$ is satisfied if and only if

$$F_5(m) = F_6(m) = \cdots = 0,\ V_6(m) = V_8(m) = \cdots = 0.$$

The same Fig. 1 shows us that, for every natural number $m \geq 2$, there exist systems in the plane containing exactly $m$ lines, for which $F_2(m) = 4$. Consequently, the value $F_2(m)$ can be bounded from above for arbitrarily large $m$. Furthermore, for any natural number $m \geq 3$, it is not difficult to point out a system consisting of $m$ lines in the plane and such that $F_2(m) = 3$ (see Fig. 2).

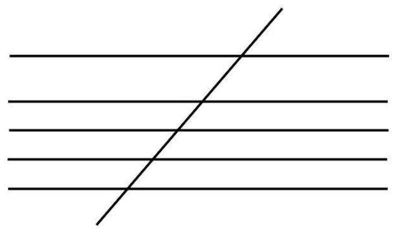

Fig. 1

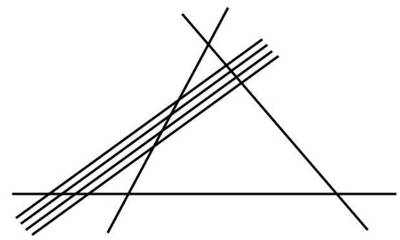

Fig. 2

On the other hand, we can formulate the following statement.

**Theorem 2.** *The inequality $F_2(m) \geq 3$ is valid for any natural $m \geq 2$.*

*Proof.* The above inequality is trivial if all points associated with a line-system $\mathcal{L} = \{l_i : i \in I\}$ are collinear. Indeed, in that case we come either to a family of lines passing through a point or to a family of pairwise parallel lines all of which intersect one more line not parallel to them (see again Fig. 1). In both cases, we have $F_2(m) \geq 4 > 3$. Suppose now that the point-system associated with $\{l_i : i \in I\}$ is not collinear and consider its convex hull $T$. Evidently, $T$ is a convex polygon whose all vertices belong to this point-system. It can easily be observed that each vertex of $T$ is simultaneously a vertex of some two-sided domain determined by $\{l_i : i \in I\}$. Also, any two distinct vertices of $T$ correspond to distinct two-sided domains. Since the number of vertices of a nondegenerate convex polygon $T$ is greater than or equal to 3, we at once obtain the required inequality $F_2(m) \geq 3$. □

*Remark* 2. For an arbitrary line-system in $\mathbf{R}^2$ consisting of $m \geq 2$ elements, the inequality $F_2(m) \leq 2m$ holds true. To see this fact, it suffices to observe that, for any line from our system, there are at most four two-sided domains which have a common ray with this line. Thus we come to the estimates

$$3 \leq F_2(m) \leq 2m,$$

which are precise. Indeed, Fig. 2 shows that the relation $F_2(m) = 3$ can be valid for arbitrarily large numbers $m$ and the relation $F_2(m) = 2m$ holds true for any system consisting of $m$ lines in $\mathbf{R}^2$ passing through a point.

Now, the relations

$$2F_2(m) + 3(F(m) - F_2(m)) \leq 2E(m),$$

$$F_2(m) \geq 3F(m) - 2E(m) = 3F(m) - 2(F(m) + V(m) - 1) = F(m) - 2(V(m) - 1),$$

are obviously valid, whence it follows (in view of Remark 2) that

$$F(m) - 2(V(m) - 1) \leq 2m, \ F(m) \leq 2(V(m) - 1) + 2m.$$

Moreover, it is not difficult to show that the equality $F_2(m) = 2m$ is true if and only if all lines of our system pass through a point. At the same time, we may write

$$F_2(m) + F_3(m) \leq F(m) \leq 2(V(m) - 1) + 2m,$$

so we have the inequalities

$$2m + 4 \le 2F_2(m) + F_3(m), \;\; F_2(m) + F_3(m) \le 2(V(m) - 1) + 2m$$

and, as said above, both of them are precise.

**Theorem 3.** *If a system $\mathcal{L} = \{l_i : i \in I\}$ in $\mathbf{R}^2$ contains exactly $m$ straight lines, which are not parallel to each other and do not pass through a point, then $F_3(m) \ge 2m/3$. Consequently, we always have $\lim_{m\to\infty} F_3(m) = +\infty$.*

*Proof.* Here we need a slightly more delicate argument. Suppose first that the given line-system $\mathcal{L}$ can be represented in the form $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$, where:

(a) $\mathcal{L}' \cap \mathcal{L}'' = \emptyset$;

(b) all lines from $\mathcal{L}'$ are parallel to each other;

(c) all lines from $\mathcal{L}''$ pass through a point $x$;

(d) the point $x$ lies on the boundary of $conv(\cup\mathcal{L}')$.

This situation is illustrated by Fig. 3.

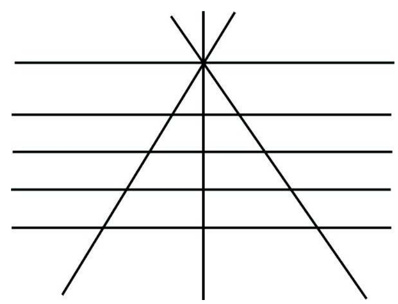

Fig. 3

Denoting $k_1 = \mathrm{card}(\mathcal{L}')$ and $k_2 = \mathrm{card}(\mathcal{L}'')$ and taking into account that $m = k_1 + k_2$, we have

$$F_3(m) = 2(k_1 - 1) + 2(k_2 - 1) = 2m - 4 \ge 2m/3.$$

Suppose now the our line-system $\mathcal{L}$ does not admit a partition $\{\mathcal{L}', \mathcal{L}''\}$ with the above-mentioned properties. Then it can be verified that, for each line $l_i$ from $\mathcal{L}$, there exist at least two distinct three-sided domains $A_i$ and $B_i$ lying in the two half-planes determined by $l_i$ and such that $A_i \cap l_i$ (respectively, $B_i \cap l_i$) is a side of $A_i$ (respectively, a side of $B_i$). This circumstance readily implies the required estimate $F_3(m) \ge 2m/3$. □

*Remark* 3. Having the inequality $F_3(m) \ge 2m/3$, we cannot assert, in general, that there are sufficiently many triangular domains (i.e. triangles) in the decomposition of the plane produced by $\mathcal{L} = \{l_i : i \in I\}$. Indeed, take an arbitrary angle in $\mathbf{R}^2$ and intersect its both sides by many parallel lines (each of them is assumed not be passing through the vertex of the angle). In this way we obtain a certain finite line-system in the plane, which yields only one triangular domain. Notice, in this context, that if a finite line-system $\mathcal{L}$ in the plane contains three distinct lines in a general position, then $\mathcal{L}$ produces at least one triangular domain (which can be shown by easy induction on $\mathrm{card}(I)$). The last circumstance is closely connected with the question of the rigidity (in the natural sense) of a given finite line-system $\mathcal{L}$ in the Euclidean plane. It is not difficult to demonstrate that $\mathcal{L}$ is rigid if and only if the following two relations hold:

(i) $\mathcal{L}$ contains at least three distinct lines in a general position;

(ii) any line from $\mathcal{L}$ contains at least two points of the associated point-system.

Starting with relations (i) and (ii), one can infer that a finite line-system $\mathcal{L}$ in the Euclidean plane is non-rigid if and only if at least one of the following two assertions is true:

(1) $\mathcal{L}$ admits a partition $\{\mathcal{L}', \mathcal{L}''\}$, where all lines from $\mathcal{L}'$ are parallel to each other and so are all lines from $\mathcal{L}''$;

(2) $\mathcal{L}$ admits a partition $\{\mathcal{L}', \mathcal{L}''\}$, where all lines from $\mathcal{L}'$ are parallel to each other and all lines from $\mathcal{L}''$ pass through a point which belongs to $\cup\mathcal{L}'$ (see, e.g., Fig. 3).

*Remark* 4. For any natural number $m \geq 5$, one can construct a system $\mathcal{L}$ consisting of $m$ lines in $\mathbf{R}^2$, no three of which have a common point, and such that no unbounded three-sided domain is generated by $\mathcal{L}$. For this purpose, it suffices to consider all those lines which carry the sides of a regular convex polygon with $m$ vertices.

*Remark* 5. Theorem 3 also enables us to prove the following statement which generalizes a result presented in [3]. Namely, suppose that a line-system $\mathcal{L}$ is given in the plane, whose elements are not parallel to each other and do not pass through a point. Then either this line-system is infinite or there exists a circumference tangent to exactly three lines from $\mathcal{L}$. Moreover, if $\mathrm{card}(\mathcal{L}) = m$, then there are at least $2m/3$ circumferences such that each of them is tangent to exactly three lines from $\mathcal{L}$.

The statement mentioned in Remark 5 can be regarded as a certain analog of the well-known Sylvester theorem on collinear points (see, for instance, [2], [5]–[7]). This theorem states that if a given finite point-system $P$ on the plane is such that the straight line determined by any two distinct points from $P$ contains at least three points of $P$, then $P$ itself is contained in a straight line (i.e. $P$ is collinear). A dual version, in the sense of projective geometry, of the Sylvester theorem reads as follows: if a given finite line-system $\mathcal{L}$ on the plane is such that no point of the plane belongs to exactly two lines from $\mathcal{L}$, then either all lines from $\mathcal{L}$ are parallel or all of them have a common point.

A slightly weaker form of the latter statement can be formulated in the following manner:

(*) Let $\mathcal{L}$ be a finite family of lines on the (projective) plane having the property that no point (finite or infinite) of the plane belongs to exactly two lines from $\mathcal{L}$. Then all lines from $\mathcal{L}$ have a common point (finite or infinite).

A similar fact can be stated for the two-dimensional unit sphere $S_2$ (here the role of lines is played by diametral sections of this sphere, i.e. by its great circumferences):

(**) Suppose that a finite system $\mathcal{C}$ of great circumferences of $S_2$ is given such that no point of $S_2$ belongs to exactly two circumferences from $\mathcal{C}$. Then all circumferences from $\mathcal{C}$ contain some pair of antipodal points of $S_2$.

Note that (**) is implied by one purely combinatorial consequence of Euler's formula. Indeed, an easy argument based on Euler's formula shows that, for any finite system of great circumferences of $S_2$, which do not contain a common pair of antipodal points of $S_2$, the inequality

$$3V_3 + 2V_4 + V_5 \geq 12$$

holds true, where $V_k$ denotes the number of all those vertices of the associated graph on $S_2$, which belong to exactly $k$ edges (arcs). The proof of the above-mentioned inequality can be found, e.g., in [4]. But it is obvious that this inequality cannot be valid for the given system $\mathcal{C}$ because in our situation $V_k = 0$ for all $k \leq 5$. Therefore either $\mathcal{C}$ is one-element or $\cap\mathcal{C}$ coincides with some set consisting of precisely two antipodal points of $S_2$ (cf. [4]).

The assertion (*) follows from (**) if we use a standard projective trick which enables us to replace great circumferences of $S_2$ by straight lines in $\mathbf{R}^2$. We thus conclude that the Sylvester theorem is, in fact, of purely combinatorial nature.

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