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OSCILLATION CRITERIA FOR A CLASS OF FUNCTIONAL PARABOLIC EQUATIONS

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Abstract. Oscillations of parabolic equations with functional arguments are studied, and sufficient conditions are derived for all solutions of certain boundary value problems to be oscillatory in a cylindrical domain. Our approach is to reduce the multi-dimensional problems to one-dimensional problems for functional differential inequalities.

1. Introduction

We are concerned with the oscillatory behavior of solutions of the parabolic equation with functional arguments

$$(1)\qquad \frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{\ell} h_i(t)u(x,\rho_i(t))\right) -a(t)\Delta u(x,t)-\sum_{i=1}^{k} b_i(t)\Delta u(x,\tau_i(t)) -c(x,t,(z_i[u](x,t))_{i=1}^{M}) = f(x,t), \quad (x,t)\in\Omega\equiv G\times(0,\infty),$$

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where G is a bounded domain of $\mathbf{R}^n$ with piecewise smooth boundary ∂G. It is assumed that

$$z_i[u](x,t)=\begin{cases} u(x,\sigma_i(t)) & (i=1,2,...,m), \\ \max_{s\in B_i(t)} u(x,s) & (i=m+1,m+2,...,m_1), \\ \displaystyle\sum_{j=1}^{N_i}\int_G K_{ij}(x,t,y)\omega_{ij}(u(y,\sigma_{ij}(t)))dy & (i=m_1+1,m_1+2,...,M), \end{cases}$$

where $B_i(t)$ $(i=m+1,m+2,...,m_1)$ are closed bounded sets of $[0,\infty)$ with the property that $\lim_{t\to\infty}\min_{s\in B_i(t)} s=\infty$, $\sigma_{ij}(t)\in C([0,\infty);\mathbf{R})$ $(i=m_1+1,m_1+2,...,M;\ j=1,2,...,N_i)$, $\lim_{t\to\infty}\sigma_{ij}(t)=\infty$, $K_{ij}(x,t,y)\in C(\overline{\Omega}\times\overline{G};[0,\infty))$, and $\omega_{ij}(s)\in C(\mathbf{R};\mathbf{R})$ are odd functions such that $\omega_{ij}(s)\geq 0$ for $s>0$.

We note that $z_i[u](x,t)\geq 0$ $(i=1,2,...,M)$ in $G\times[T,\infty)$ for some $T>0$ if u is eventually positive in $G\times(0,\infty)$.

We assume that :

$$\begin{aligned} &h_i(t)\in C^1([0,\infty);[0,\infty)) \quad (i=1,2,...,\ell); \\ &a(t)\in C([0,\infty);[0,\infty)); \\ &b_i(t)\in C([0,\infty);[0,\infty)) \quad (i=1,2,...,k); \\ &\rho_i(t)\in C^1([0,\infty);\mathbf{R}),\ \lim_{t\to\infty}\rho_i(t)=\infty \quad (i=1,2,...,\ell); \\ &\tau_i(t)\in C([0,\infty);\mathbf{R}),\ \lim_{t\to\infty}\tau_i(t)=\infty \quad (i=1,2,...,k); \\ &\sigma_i(t)\in C([0,\infty);\mathbf{R}),\ \lim_{t\to\infty}\sigma_i(t)=\infty \quad (i=1,2,...,m); \\ &f(x,t)\in C(\overline{\Omega};\mathbf{R}). \end{aligned}$$

Moreover, we assume that :

$$\begin{aligned} &c(x,t,(\xi_i)_{i=1}^M)\in C(\overline{\Omega}\times\mathbf{R}^M;\mathbf{R}), \\ &c(x,t,(\xi_i)_{i=1}^M)\geq\sum_{i=1}^m p_i(t)\varphi_i(\xi_i) \quad \text{for} \quad (x,t,(\xi_i)_{i=1}^M)\in\Omega\times[0,\infty)^M, \\ &c(x,t,(-\xi_i)_{i=1}^M)\leq-\sum_{i=1}^m p_i(t)\varphi_i(\xi_i) \quad \text{for} \quad (x,t,(\xi_i)_{i=1}^M)\in\Omega\times[0,\infty)^M, \end{aligned}$$

where $[0,\infty)^j=[0,\infty)\times[0,\infty)^{j-1}$ $(j=1,2,...,M)$, $p_i(t)\in C([0,\infty);[0,\infty))$, $\varphi_i(s)\in C([0,\infty);[0,\infty))$, and $\varphi_i(s)$ are convex in $(0,\infty)$ $(i=1,2,...,m)$.

The boundary conditions to be considered are the following :

(B_1) $u=\psi$ on $\partial G\times(0,\infty)$,

(B_2) $\dfrac{\partial u}{\partial\nu}-\mu u=\tilde{\psi}$ on $\partial G\times(0,\infty)$,

where $\psi,\ \tilde{\psi}\in C(\partial G\times(0,\infty);\mathbf{R})$, $\mu\in C(\partial G\times(0,\infty);[0,\infty))$ and ν denotes the unit exterior normal vector to ∂G.

Definition 1. By a *solution* of the boundary value problems (1), (B_i) ($i = 1, 2$), we mean a function $u(x,t) \in C^2(\overline{G} \times [t_{-1}, \infty); \mathbf{R}) \cap C^1(\overline{G} \times [\tilde{t}_{-1}, \infty); \mathbf{R}) \cap C(\overline{G} \times [\hat{t}_{-1}, \infty); \mathbf{R})$ which satisfies (1), (B_i) ($i = 1, 2$), where

$$\begin{aligned} t_{-1} &= \min\left\{0, \min_{1\le i\le k}\left\{\inf_{t\ge 0} \tau_i(t)\right\}\right\}, \\ \tilde{t}_{-1} &= \min\left\{0, \min_{1\le i\le \ell}\left\{\inf_{t\ge 0} \rho_i(t)\right\}\right\}, \\ \hat{t}_{-1} &= \min\left\{0, \min_{1\le i\le m}\left\{\inf_{t\ge 0} \sigma_i(t)\right\}, \min_{\substack{m_1+1\le i\le M \\ 1\le j\le N_i}}\left\{\inf_{t\ge 0} \sigma_{ij}(t)\right\}\right\}. \end{aligned}$$

Definition 2. A solution u of the boundary value problems (1), (B_i) ($i = 1, 2$) is said to be *oscillatory* in Ω if u has a zero in $G \times (t, \infty)$ for any $t > 0$.

In 1983, Bykov and Kultaev [2] have studied the oscillations of functional parabolic equations including the special case of (1). Oscillation theory for functional parabolic equations has been extensively developed in recent years by several authors; see, for example, [4–6, 8–10]. However, most of the papers except [2, 9] pertain to the parabolic equations (1) with $-c(x,t,(z_i[u](x,t))_{i=1}^M)$ replaced by $c(x,t,(z_i[u](x,t))_{i=1}^M)$. We mention in particular the paper [1] which deals with impulsive nonlinear parabolic equations.

The purpose of this paper is to derive sufficient conditions for every solution u of the boundary value problems (1), (B_i) ($i = 1, 2$) to be oscillatory in Ω. In Section 2 we reduce the multi-dimensional problems to one-dimensional oscillation problems. Section 3 is devoted to the nonexistence of eventually positive solutions of the associated functional differential inequalities. In Section 4 we combine the results of Sections 2 and 3 to obtain various oscillation results for the functional parabolic equation (1).

2. Reduction to functional differential inequalities

In this section we show that the boundary value problems (1), (B_i) ($i = 1, 2$) can be reduced to one-dimensional oscillation problems.

It is known that the smallest eigenvalue λ_1 of the eigenvalue problem

$$\begin{aligned} -\Delta w &= \lambda w \quad \text{in} \quad G, \\ w &= 0 \quad \text{on} \quad \partial G \end{aligned}$$

is positive and the corresponding eigenfunction $\Phi(x)$ may be chosen so that $\Phi(x) > 0$ in G (see Courant and Hilbert [3]).

We use the notation :

$$F(t) = \int_G f(x,t)\Phi(x)dx \cdot \left(\int_G \Phi(x)dx\right)^{-1},$$

$$\Psi(t) = \int_{\partial G} \psi \frac{\partial \Phi}{\partial \nu}(x)dS \cdot \left(\int_G \Phi(x)dx\right)^{-1},$$

$$\tilde{F}(t) = \frac{1}{|G|}\int_G f(x,t)dx,$$

$$\tilde{\Psi}(t) = \frac{1}{|G|}\int_{\partial G} \tilde{\psi}\, dS,$$

where $|G| = \int_G dx$.

We define the function spaces $\mathcal{B}_\gamma(\Omega)$ and $\tilde{\mathcal{B}}_\Gamma$ by

$$\mathcal{B}_\gamma(\Omega) = \left\{u(x,t) \in C^2(\overline{\Omega};\mathbf{R}); |u(x,t)| \le \gamma(x,t) \text{ on } \overline{\Omega}\right\},$$

$$\tilde{\mathcal{B}}_\Gamma = \left\{y(t) \in C^1([T_y,\infty);\mathbf{R}); |y(t)| \le \Gamma(t) \text{ on } [T_y,\infty)\right\},$$

where T_y is a positive constant depending on $y(t)$, $\gamma(x,t)$ is a positive continuous function on $\overline{\Omega}$, and

$$\Gamma(t) = \int_G \gamma(x,t)\Phi(x)dx \cdot \left(\int_G \Phi(x)dx\right)^{-1}.$$

Theorem 1. If the functional differential inequalities

$$(2_\pm) \qquad \frac{d}{dt}\left(y(t) + \sum_{i=1}^{\ell} h_i(t)y(\rho_i(t))\right) + \lambda_1 a(t)y(t) + \lambda_1 \sum_{i=1}^{k} b_i(t)y(\tau_i(t)) - \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \ge \pm G(t)$$

have no eventually positive solutions of class $\tilde{\mathcal{B}}_\Gamma$, then every solution $u \in \mathcal{B}_\gamma(\Omega)$ of the boundary value problem *(1), (B_1)* is oscillatory in Ω, where

$$G(t) = F(t) - a(t)\Psi(t) - \sum_{i=1}^{k} b_i(t)\Psi(\tau_i(t)).$$

Proof. Suppose to the contrary that there exists a solution $u \in \mathcal{B}_\gamma(\Omega)$ of the problem (1), (B_1) which is nonoscillatory in Ω. First we assume that $u > 0$ in $G \times [t_0,\infty)$ for some $t_0 > 0$. The hypothesis implies that

$$c(x,t,(z_i[u](x,t))_{i=1}^{M}) \ge \sum_{i=1}^{m} p_i(t)\varphi_i(u(x,\sigma_i(t))) \quad \text{in} \quad G \times [t_1,\infty)$$

for some $t_1 \geq t_0$. Hence, from (1) we see that

$$(3)\qquad \frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{\ell} h_i(t)u(x,\rho_i(t))\right) -a(t)\Delta u(x,t)-\sum_{i=1}^{k} b_i(t)\Delta u(x,\tau_i(t)) -\sum_{i=1}^{m} p_i(t)\varphi_i(u(x,\sigma_i(t))) \geq f(x,t) \quad \text{in} \quad G\times[t_1,\infty).$$

Multiplying (3) by $\Phi(x)\cdot\left(\int_G \Phi(x)dx\right)^{-1}$ and then integrating over G yields

$$(4)\qquad \frac{d}{dt}\left(U(t)+\sum_{i=1}^{\ell} h_i(t)U(\rho_i(t))\right)-a(t)K_\Phi\int_G \Delta u(x,t)\Phi(x)dx -\sum_{i=1}^{k} b_i(t)K_\Phi\int_G \Delta u(x,\tau_i(t))\Phi(x)dx -\sum_{i=1}^{m} p_i(t)K_\Phi\int_G \varphi_i(u(x,\sigma_i(t)))\Phi(x)dx \geq F(t), \quad t\geq t_1,$$

where

$$K_\Phi=\left(\int_G \Phi(x)dx\right)^{-1},$$
$$U(t)=\int_G u(x,t)\Phi(x)dx\cdot\left(\int_G \Phi(x)dx\right)^{-1}.$$

It follows from Green's formula that

$$(5)\qquad \begin{aligned} K_\Phi\int_G \Delta u(x,t)\Phi(x)dx &= K_\Phi\int_{\partial G}\left[\frac{\partial u}{\partial\nu}(x,t)\Phi(x)-u(x,t)\frac{\partial\Phi}{\partial\nu}(x)\right]dS+K_\Phi\int_G u(x,t)\Delta\Phi(x)dx \\ &= -K_\Phi\int_{\partial G}\psi\,\frac{\partial\Phi}{\partial\nu}(x)dS-\lambda_1 K_\Phi\int_G u(x,t)\Phi(x)dx \\ &= -\Psi(t)-\lambda_1 U(t), \quad t\geq t_1. \end{aligned}$$

Analogously we have

$$(6)\qquad K_\Phi\int_G \Delta u(x,\tau_i(t))\Phi(x)dx=-\Psi(\tau_i(t))-\lambda_1 U(\tau_i(t)), \quad t\geq t_2$$

for some $t_2 \geq t_1$. Applying Jensen's inequality [7, p.160], we obtain

$$K_\Phi \int_G \varphi_i(u(x,\sigma_i(t)))\Phi(x)dx \geq \varphi_i\left(U(\sigma_i(t))\right), \quad t \geq t_2. \tag{7}$$

Combining (4)–(7) yields

$$\begin{aligned}&\frac{d}{dt}\left(U(t) + \sum_{i=1}^{\ell} h_i(t)U(\rho_i(t))\right) + \lambda_1 a(t)U(t)\\ &+\lambda_1 \sum_{i=1}^{k} b_i(t)U(\tau_i(t)) - \sum_{i=1}^{m} p_i(t)\varphi_i(U(\sigma_i(t))) \geq G(t), \quad t \geq t_2.\end{aligned}$$

It is easy to check that

$$|U(t)| \leq K_\Phi \int_G |u(x,t)|\Phi(x)dx \leq K_\Phi \int_G \gamma(x,t)\Phi(x)dx = \Gamma(t),$$

and therefore $U(t) \in \tilde{\mathcal{B}}_\Gamma$. Hence, (2_+) has an eventually positive solution $U(t)$ of class $\tilde{\mathcal{B}}_\Gamma$. This contradicts the hypothesis. If $u < 0$ in $G \times [t_0, \infty)$, it can be shown that

$$c(x,t,(z_i[u](x,t))_{i=1}^M) \leq -\sum_{i=1}^{m} p_i(t)\varphi_i(-u(x,\sigma_i(t))) \quad \text{in} \quad G \times [t_1, \infty)$$

for some $t_1 \geq t_0$. Letting $v \equiv -u$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left(v(x,t) + \sum_{i=1}^{\ell} h_i(t)v(x,\rho_i(t))\right)\\ &-a(t)\Delta v(x,t) - \sum_{i=1}^{k} b_i(t)\Delta v(x,\tau_i(t))\\ &-\sum_{i=1}^{m} p_i(t)\varphi_i(v(x,\sigma_i(t))) \geq -f(x,t) \quad \text{in} \quad G \times [t_1, \infty).\end{aligned}$$

Proceeding as in the case where $u > 0$, we are led to a contradiction. The proof is complete. □

Theorem 2. If the functional differential inequalities

$(8_\pm)$

$$\frac{d}{dt}\left(y(t) + \sum_{i=1}^{\ell} h_i(t)y(\rho_i(t))\right) - \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \geq \pm\tilde{G}(t)$$

have no eventually positive (bounded) solutions, then every (bounded) solution u of the boundary value problem (1), *(B_2)* is oscillatory in Ω, where

$$\tilde{G}(t) = \tilde{F}(t) + a(t)\tilde{\Psi}(t) + \sum_{i=1}^{k} b_i(t)\tilde{\Psi}(\tau_i(t)).$$

Proof. Assume on the contrary, that there exists a (bounded) solution u of the problem (1), (B_2) such that $u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Arguing as in the proof of Theorem 1, we observe that the inequality (3) holds for some $t_1 \geq t_0$. Dividing (3) by $|G|$ and then integrating over G yields

$$\begin{aligned}(9)\quad & \frac{d}{dt}\left(\tilde{U}(t) + \sum_{i=1}^{\ell} h_i(t)\tilde{U}(\rho_i(t))\right) - a(t)\frac{1}{|G|}\int_G \Delta u(x,t)dx \\ & -\sum_{i=1}^{k} b_i(t)\frac{1}{|G|}\int_G \Delta u(x,\tau_i(t))dx - \sum_{i=1}^{m} p_i(t)\frac{1}{|G|}\int_G \varphi_i(u(x,\sigma_i(t)))dx \\ & \geq \tilde{F}(t), \quad t \geq t_1.\end{aligned}$$

It follows from the divergence theorem that

$$\begin{aligned}(10)\qquad \frac{1}{|G|}\int_G \Delta u(x,t)dx &= \frac{1}{|G|}\int_{\partial G} \frac{\partial u}{\partial \nu}(x,t)dS \\ &= \frac{1}{|G|}\int_{\partial G} \left(\mu\, u(x,t) + \tilde{\psi}\right) dS \\ &\geq \tilde{\Psi}(t), \quad t \geq t_1.\end{aligned}$$

Analogously we obtain

$$(11)\qquad \frac{1}{|G|}\int_G \Delta u(x,\tau_i(t))dx \geq \tilde{\Psi}(\tau_i(t)), \quad t \geq t_2$$

for some $t_2 \geq t_1$. An application of Jensen's inequality shows that

$$(12)\qquad \frac{1}{|G|}\int_G \varphi_i(u(x,\sigma_i(t)))dx \geq \varphi_i\left(\tilde{U}(\sigma_i(t))\right), \quad t \geq t_2.$$

Combining (9)–(12) yields

$$\frac{d}{dt}\left(\tilde{U}(t) + \sum_{i=1}^{\ell} h_i(t)\tilde{U}(\rho_i(t))\right) - \sum_{i=1}^{m} p_i(t)\varphi_i(\tilde{U}(\sigma_i(t))) \geq \tilde{G}(t), \quad t \geq t_2,$$

which means that $\tilde{U}(t)$ is an eventually positive (bounded) solution of (8_+). This contradicts the hypothesis. The case where $u < 0$ can be treated similarly, and we are led to a contradiction. The proof is complete. □

3. Functional differential inequalities

In this section we investigate the nonexistence of eventually positive solutions of the functional differential inequality

(13)
$$\frac{d}{dt}\left(y(t)+\sum_{i=1}^{\ell}h_i(t)y(\rho_i(t))\right)-\sum_{i=1}^{m}p_i(t)\varphi_i(y(\sigma_i(t)))\geq G(t).$$

Theorem 3. Assume that $\sum_{i=1}^{\ell}h_i(t)\leq 1,\ \rho_i(t)\leq t\ (i=1,2,...,\ell)$, and $\varphi_j(s)$ is nondecreasing on $[0,\infty)$ for some $j\in\{1,2,...,m\}$. The inequality (13) has no eventually positive bounded solution if there is a function $\Theta(t)\in C^1((0,\infty);\mathbf{R})$ such that $\Theta(t)$ is bounded and oscillatory at $t=\infty$, $\Theta'(t)=G(t)$, and

(14)
$$\int_{s_0}^{\infty}p_j(s)\varphi_j\left(\left[\left(1-\sum_{i=1}^{\ell}h_i(\sigma_j(s))\right)[\Theta(\sigma_j(s))]_- + \Theta(\sigma_j(s))-\sum_{i=1}^{\ell}h_i(\sigma_j(s))\Theta(\rho_i(\sigma_j(s)))\right]_+\right)ds=\infty$$

for some $s_0>0$, where

$$[\Theta(t)]_{\pm}=\max\{\pm\Theta(t),\ 0\}.$$

Proof. Assume on the contrary, that there exists an eventually positive bounded solution $y(t)$ of (13) such that $y(t)>0$ on $[t_0,\infty)$ for some $t_0>0$. Then, $y(\rho_i(t))>0\ (i=1,2,...,\ell)$, $y(\sigma_i(t))>0\ (i=1,2,...,m)$ on $[t_1,\infty)$ for some $t_1\geq t_0$. Letting

$$z(t)=y(t)+\sum_{i=1}^{\ell}h_i(t)y(\rho_i(t))-\Theta(t),$$

we see that

$$z'(t)\geq\sum_{i=1}^{m}p_i(t)\varphi_i(y(\sigma_i(t)))\geq 0,\quad t\geq t_1,$$

and therefore $z(t)$ is nondecreasing for $t\geq t_1$. Hence, we find that either $z(t)>0$ or $z(t)\leq 0$ on $[t_2,\infty)$ for some $t_2\geq t_1$. If $z(t)\leq 0$ on $[t_2,\infty)$, then

(15)
$$y(t)+\sum_{i=1}^{\ell}h_i(t)y(\rho_i(t))\leq\Theta(t),\quad t\geq t_2.$$

The left hand side of (15) is positive, but the right hand side of (15) is oscillatory at $t=\infty$. This is a contradiction. Hence, we conclude that

$z(t) > 0$ on $[t_2, \infty)$. Since $z(t) + \Theta(t) > 0$ on $[t_2, \infty)$, we find that $z(t) > -\Theta(t)$ on $[t_2, \infty)$, and therefore

$$z(t) \geq [\Theta(t)]_- \quad \text{for} \quad t \geq t_2. \tag{16}$$

In view of the fact that $y(t) \leq z(t) + \Theta(t)$ and $z(t)$ is nondecreasing, we obtain

$$\begin{aligned} y(t) &= z(t) - \sum_{i=1}^{\ell} h_i(t) y(\rho_i(t)) + \Theta(t) \\ &\geq z(t) - \sum_{i=1}^{\ell} h_i(t) \left(z(\rho_i(t)) + \Theta(\rho_i(t)) \right) + \Theta(t) \\ &\geq \left(1 - \sum_{i=1}^{\ell} h_i(t) \right) z(t) + \Theta(t) - \sum_{i=1}^{\ell} h_i(t) \Theta(\rho_i(t)), \quad t \geq t_2. \end{aligned} \tag{17}$$

Combining (16) with (17) yields

$$y(t) \geq \left(1 - \sum_{i=1}^{\ell} h_i(t) \right) [\Theta(t)]_- + \Theta(t) - \sum_{i=1}^{\ell} h_i(t) \Theta(\rho_i(t)), \quad t \geq t_2.$$

Since $y(t) > 0$ for $t \geq t_2$, we observe that

$$y(\sigma_j(t)) \geq \left[\left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(t)) \right) [\Theta(\sigma_j(t))]_- + \Theta(\sigma_j(t)) - \sum_{i=1}^{\ell} h_i(\sigma_j(t)) \Theta(\rho_i(\sigma_j(t))) \right]_+$$

on $[t_3, \infty)$ for some $t_3 \geq t_2$. Hence, we obtain

$$\begin{aligned} z'(t) &\geq \sum_{i=1}^{m} p_i(t) \varphi_i(y(\sigma_i(t))) \\ &\geq p_j(t) \varphi_j(y(\sigma_j(t))) \\ &\geq p_j(t) \varphi_j \left(\left[\left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(t)) \right) [\Theta(\sigma_j(t))]_- \right. \right. \\ &\qquad \left. \left. + \Theta(\sigma_j(t)) - \sum_{i=1}^{\ell} h_i(\sigma_j(t)) \Theta(\rho_i(\sigma_j(t))) \right]_+ \right), \quad t \geq t_3. \end{aligned} \tag{18}$$

Integrating (18) over $[t_3, t]$ yields

$$(19)\quad z(t) - z(t_3) \geq \int_{t_3}^{t} p_j(s)\varphi_j\left(\left[\left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(s))\right)[\Theta(\sigma_j(s))]_- + \Theta(\sigma_j(s)) - \sum_{i=1}^{\ell} h_i(\sigma_j(s))\Theta(\rho_i(\sigma_j(s)))\right]_+\right)ds, \quad t \geq t_3.$$

The left hand side of (19) is bounded from above, but the right hand side of (19) tends to infinity as $t \to \infty$. This is a contradiction and the proof is complete. □

Next we consider the functional differential inequality

$$(20)\qquad y'(t) - p(t)y(\sigma(t)) \geq q(t), \quad t \geq T,$$

where T is some positive number, $p(t) \in C([T,\infty);[0,\infty))$, $q(t) \in C([T,\infty);\mathbf{R})$ and $\sigma(t) \in C([T,\infty);\mathbf{R})$ for which $\lim_{t\to\infty}\sigma(t) = \infty$, $\sigma(t) \geq t$ and $\sigma(t)$ is nondecreasing on $[T,\infty)$.

Lemma 1. The inequality (20) has no eventually positive solution if there exists a sequence $\{t_n\}$ such that:

$$\lim_{n\to\infty} t_n = \infty,$$
$$\int_{t_n}^{\sigma(t_n)} p(s)ds \geq 1,$$
$$\int_{t_n}^{\sigma(t_n)} q(s)ds + \int_{t_n}^{\sigma(t_n)} p(s)\left(\int_{\sigma(t_n)}^{\sigma(s)} q(\xi)d\xi\right)ds \geq 0.$$

Proof. Suppose that there exists a solution $y(t)$ of (20) for which $y(t) > 0$ on $[T_0,\infty)$ for some $T_0 > T$. Integrating (20) over $[t,\sigma(t)]$, we obtain

$$(21)\qquad y(\sigma(t)) - y(t) - \int_{t}^{\sigma(t)} p(s)y(\sigma(s))ds \geq \int_{t}^{\sigma(t)} q(s)ds, \quad t \geq T_0.$$

Since

$$(22)\qquad y'(t) \geq q(t) \quad \text{for} \quad t \geq T_0,$$

an integration of (22) over $[\sigma(t),\sigma(s)]$ yields

$$y(\sigma(s)) - y(\sigma(t)) \geq \int_{\sigma(t)}^{\sigma(s)} q(\xi)d\xi \quad \text{for} \quad s \geq t,$$

and therefore

$$y(\sigma(s)) \geq y(\sigma(t)) + \int_{\sigma(t)}^{\sigma(s)} q(\xi) d\xi \quad \text{for} \quad s \geq t. \tag{23}$$

Combining (21) with (23), we obtain

$$y(\sigma(t)) - y(t) - \int_t^{\sigma(t)} p(s) \left(y(\sigma(t)) + \int_{\sigma(t)}^{\sigma(s)} q(\xi) d\xi \right) ds \geq \int_t^{\sigma(t)} q(s) ds, \ t \geq T_0,$$

or equivalently

$$\begin{aligned} & -y(t) - y(\sigma(t)) \left(\int_t^{\sigma(t)} p(s) ds - 1 \right) \\ \geq & \int_t^{\sigma(t)} q(s) ds + \int_t^{\sigma(t)} p(s) \left(\int_{\sigma(t)}^{\sigma(s)} q(\xi) d\xi \right) ds, \quad t \geq T_0. \end{aligned} \tag{24}$$

It is easy to see that $t_n \geq T_0$ $(n \geq N)$ for some positive integer N. We easily see that the left hand side of (24) with $t = t_n$ $(n \geq N)$ is negative, whereas the right hand side of (24) with $t = t_n$ $(n \geq N)$ is nonnegative. This is a contradiction and the proof is complete. □

Theorem 4. Assume that $\sum_{i=1}^{\ell} h_i(t) \leq 1$, $\rho_i(t) \leq t$ $(i = 1, 2, ..., \ell)$, and $\varphi_j(s) \geq \beta s$ in $(0, \infty)$ for some $\beta > 0$ and some $j \in \{1, 2, ..., m\}$. Moreover, assume that $\sigma_j(t) \geq t$ and $\sigma_j(t)$ is nondecreasing in $(0, \infty)$, and that there is a function $\Theta(t) \in C^1((0, \infty); \mathbf{R})$ such that $\Theta(t)$ is oscillatory at $t = \infty$ and $\Theta'(t) = G(t)$. The inequality (13) has no eventually positive solution if there exists a sequence $\{t_n\}$ for which

$$\lim_{n \to \infty} t_n = \infty, \tag{25}$$

$$\int_{t_n}^{\sigma_j(t_n)} p_j(s) \left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(s)) \right) ds \geq \frac{1}{\beta}, \tag{26}$$

$$\begin{aligned} & \int_{t_n}^{\sigma_j(t_n)} Q(s) ds + \\ & \beta \int_{t_n}^{\sigma_j(t_n)} p_j(s) \left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(s)) \right) \left(\int_{\sigma_j(t_n)}^{\sigma_j(s)} Q(\xi) d\xi \right) ds \geq 0, \end{aligned} \tag{27}$$

where

$$Q(t) = \beta p_j(t) \left(\Theta(\sigma_j(t)) - \sum_{i=1}^{\ell} h_i(\sigma_j(t)) \Theta(\rho_i(\sigma_j(t))) \right).$$

Proof. Let $y(t)$ be a solution of (13) such that $y(t) > 0$ on $[t_0, \infty)$ for some $t_0 > 0$. Proceeding as in the proof of Theorem 3, we see that (17) holds, and hence we obtain

$$(28)\quad y(\sigma_j(t)) \ \geq\ \left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(t))\right) z(\sigma_j(t)) + \Theta(\sigma_j(t)) - \sum_{i=1}^{\ell} h_i(\sigma_j(t))\Theta(\rho_i(\sigma_j(t))), \quad t \geq t_3$$

for some $t_3 \geq t_2$. Then it can be shown that

$$(29)\quad \begin{aligned} z'(t) &\geq \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \\ &\geq p_j(t)\varphi_j(y(\sigma_j(t))) \\ &\geq \beta p_j(t) y(\sigma_j(t)), \quad t \geq t_3. \end{aligned}$$

Combining (28) with (29), we observe that $z(t)$ is a positive solution of

$$(30)\quad z'(t) - \beta p_j(t) \left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(t))\right) z(\sigma_j(t)) \geq Q(t)$$

for $t \geq t_3$. However, Lemma 1 implies that (30) has no eventually positive solution. This is a contradiction and the proof is complete. □

4. Functional parabolic equations

Combining the results in Sections 2 and 3, we can derive various oscillation theorems for the boundary value problems (1), (B_i) $(i = 1, 2)$.

Lemma 2. If $(2_\pm)$ have eventually positive solutions $y_r(t) \in \tilde{\mathcal{B}}_\Gamma$ $(r = 1, 2)$, respectively, then $y_r(t)$ are eventually positive solutions of the differential inequalities

$$(31)\quad \frac{d}{dt}\left(y(t) + \sum_{i=1}^{\ell} h_i(t) y(\rho_i(t))\right) - \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \geq G_r(t),$$

where

$$G_r(t) = (-1)^{r-1} G(t) - \lambda_1 a(t)\Gamma(t) - \lambda_1 \sum_{i=1}^{k} b_i(t)\Gamma(\tau_i(t)) \quad (r = 1, 2).$$

Proof. Since

$$\lambda_1 a(t) y_r(t) \le \lambda_1 a(t)\Gamma(t), \quad \lambda_1 \sum_{i=1}^{k} b_i(t) y_r(\tau_i(t)) \le \lambda_1 \sum_{i=1}^{k} b_i(t)\Gamma(\tau_i(t)),$$

we easily see that $y_r(t)$ are eventually positive solutions of (31). □

Theorem 5. Assume that $\sum_{i=1}^{\ell} h_i(t) \le 1$, $\rho_i(t) \le t$ $(i = 1, 2, ..., \ell)$, $\varphi_j(s)$ is nondecreasing on $[0, \infty)$ for some $j \in \{1, 2, ..., m\}$. Every solution $u \in \mathcal{B}_K$ (K is a positive constant) of the boundary value problem *(1), (B_1)* is oscillatory in Ω if there is a function $\Theta_r(t) \in C^1((0, \infty); \mathbf{R})$ $(r = 1, 2)$ such that $\Theta_r(t)$ is bounded and oscillatory at $t = \infty$, $\Theta_r'(t) = G_r(t)$ with $\Gamma(t) \equiv K$, and that

$$\int_{s_0}^{\infty} p_j(s)\varphi_j\left(\left[\left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(s))\right)[\Theta_r(\sigma_j(s))]_- + \Theta_r(\sigma_j(s)) - \sum_{i=1}^{\ell} h_i(\sigma_j(s))\Theta_r(\rho_i(\sigma_j(s)))\right]_+\right) ds = \infty$$

for some $s_0 > 0$.

Proof. It follows from Theorem 3 that (31) have no eventually positive bounded solutions, and hence Lemma 2 with $\gamma(x, t) = \Gamma(t) = K$ implies that ($2_\pm$) have no eventually positive solutions $y(t) \in \tilde{\mathcal{B}}_K$. The conclusion follows from Theorem 1. □

Theorem 6. Assume that $\sum_{i=1}^{\ell} h_i(t) \le 1$, $\rho_i(t) \le t$ $(i = 1, 2, ..., \ell)$, and $\varphi_j(s) \ge \beta s$ in $(0, \infty)$ for some $\beta > 0$ and some $j \in \{1, 2, ..., m\}$. Moreover, assume that $\sigma_j(t) \ge t$ and $\sigma_j(t)$ is nondecreasing in $(0, \infty)$, and that there is a function $\Theta_r(t) \in C^1((0, \infty); \mathbf{R})$ $(r = 1, 2)$ such that $\Theta_r(t)$ is oscillatory at $t = \infty$ and $\Theta_r'(t) = G_r(t)$. Every solution $u \in \mathcal{B}_\gamma(\Omega)$ of the boundary value problem *(1), (B_1)* is oscillatory in Ω if there exists a sequence $\{t_{r,n}\}$ $(r = 1, 2)$ for which (25) – (27) with $t_n = t_{r,n}$ and $Q(t)$ replaced by

$$Q_r(t) = \beta p_j(t)\left(\Theta_r(\sigma_j(t)) - \sum_{i=1}^{\ell} h_i(\sigma_j(t))\Theta_r(\rho_i(\sigma_j(t)))\right)$$

hold.

Proof. Theorem 4 implies that (31) have no eventually positive solutions. Hence, it follows from Lemma 2 that ($2_\pm$) have no eventually positive solutions $y(t) \in \tilde{\mathcal{B}}_\Gamma$. The conclusion follows from Theorem 1. □

Theorem 7. Assume that $\sum_{i=1}^{\ell} h_i(t) \le 1$, $\rho_i(t) \le t$ $(i = 1, 2, ..., \ell)$, $\varphi_j(s)$ is nondecreasing on $[0, \infty)$ for some $j \in \{1, 2, ..., m\}$. Every bounded solution u of the boundary value problem *(1), (B_2)* is oscillatory in Ω if there is a function $\Theta(t) \in C^1((0,\infty); \mathbf{R})$ such that $\Theta(t)$ is bounded and oscillatory at $t = \infty$, $\Theta'(t) = \tilde{G}(t)$, and that

$$\int_{s_0}^{\infty} p_j(s)\varphi_j\left(\left[\left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(s))\right)[\Theta(\sigma_j(s))]_{\mp} \pm \left(\Theta(\sigma_j(s)) - \sum_{i=1}^{\ell} h_i(\sigma_j(s))\Theta(\rho_i(\sigma_j(s)))\right)\right]_{+}\right)ds = \infty$$

for some $s_0 > 0$.

Proof. Combining Theorem 2 with Theorem 3, we are led to the conclusion. □

Theorem 8. Assume that $\sum_{i=1}^{\ell} h_i(t) \le 1$, $\rho_i(t) \le t$ $(i = 1, 2, ..., \ell)$, and $\varphi_j(s) \ge \beta s$ in $(0, \infty)$ for some $\beta > 0$ and some $j \in \{1, 2, ..., m\}$. Moreover, assume that $\sigma_j(t) \ge t$ and $\sigma_j(t)$ is nondecreasing in $(0, \infty)$, and that there is a function $\Theta(t) \in C^1((0,\infty); \mathbf{R})$ such that $\Theta(t)$ is oscillatory at $t = \infty$ and $\Theta'(t) = \tilde{G}(t)$. Every solution u of the boundary value problem (1), *(B_2)* is oscillatory in Ω if there exists a sequence $\{t_n\}$ satisfying *(25), (26)* and

$$\int_{t_n}^{\sigma_j(t_n)} Q(s)ds + \beta \int_{t_n}^{\sigma_j(t_n)} p_j(s)\left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(s))\right)\left(\int_{\sigma_j(t_n)}^{\sigma_j(s)} Q(\xi)d\xi\right)ds = 0.$$

Proof. The conclusion follows by combining Theorem 2 with Theorem 4. □

We conclude with an example which illustrates Theorem 7.

Example. We consider the problem

$$\begin{aligned}(32)\qquad & \frac{\partial}{\partial t}\left(u(x,t) + (1/2)u(x,t+\pi)\right) \\ & -u_{xx}(x,t) - u_{xx}(x,t+\pi) - u(x,t+(\pi/2)) \\ & = -(\cos x + 1)\cos t, \quad (x,t) \in (0,\pi)\times(0,\infty),\end{aligned}$$

$$(33)\qquad -u_x(0,t) = u_x(\pi,t) = 0, \quad t > 0.$$

Here $n = 1$, $G = (0, \pi)$, $\Omega = (0, \pi) \times (0, \infty)$, $\ell = k = m = M = 1$, $h_1(t) = 1/2$, $\rho_1(t) = t + \pi$, $a(t) = 1$, $b_1(t) = 1$, $\tau_1(t) = t + \pi$, $p_1(t) = 1$,

$\sigma_1(t) = t + (\pi/2)$, $\varphi_1(s) = s$, $\mu \equiv 0$, $\tilde{\psi} \equiv 0$ and $f(x,t) = -(\cos x + 1)\cos t$. It is easily seen that $\tilde{\Psi}(t) \equiv 0$ and

$$\tilde{G}(t) = \tilde{F}(t) = \frac{1}{\pi}\int_0^{\pi} f(x,t)dx = -\cos t.$$

Choosing $\Theta(t) = -\sin t$, we see that $\Theta(t) \in C^1((0,\infty);\mathbf{R})$, $\Theta'(t) = \tilde{G}(t)$, $\Theta(t)$ is bounded and oscillatory at $t = \infty$. It is easy to check that $\Theta(\sigma_1(s)) = -\cos s$, $\Theta(\rho_1(\sigma_1(s))) = \cos s$, and that

$$\begin{aligned} & \int_{s_0}^{\infty} \left[\left(1 - \frac{1}{2}\right) [-\cos s]_{\mp} \pm \left(-\cos s - \frac{1}{2}\cos s\right) \right]_+ ds \\ = \;& \frac{1}{2}\int_{s_0}^{\infty} \left[[-\cos s]_{\mp} \pm (-3\cos s) \right]_+ ds \\ = \;& \frac{1}{2}\int_{s_0}^{\infty} [\mp 3\cos s]_+ \, ds = \infty. \end{aligned}$$

Hence, it follows from Theorem 7 that every bounded solution u of the problem (32), (33) is oscillatory in $(0,\pi) \times (0,\infty)$. One such solution is $u = 2(\cos x + 1)\sin t$.

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