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ON THE CONVERGENCE OF THE METHOD OF LINES FOR QUASI–NONLINEAR FUNCTIONAL EVOLUTIONS IN BANACH SPACES

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Abstract. This paper is concerned with the existence of global limit solutions for the quasi–nonlinear functional evolution problem

$$\begin{aligned} &x' \in A(t, x_t)x + G(t, x_t, L_t x), t \in [0, T], \\ &x_0 = \phi, \end{aligned} \qquad \text{(FDE,}\phi\text{)}$$

where $A(t, \psi_1)$ and $G(t, \psi_1, L_t\psi_2)$ are defined, with respect to ψ_1, on a subspace of the space $PC([-r, 0], X)$ of all piecewise continuous functions $f : [-r, 0] \to X$. An appropriate subspace of $PC([-r, t], X)$ is the domain of definition of the nonlinear operators L_t, $t \in [0, T]$. The operators $A(t, \psi)x$ are w-dissipative and Lipschitz — like in (t, ψ) which are more general conditions than those of Karsatos–Liu. The operators G and L_t are Lipschitzian mappings on their respective domains. Moreover, we investigate the uniqueness and strong solution for such problem.

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1. Introduction and preliminaries

Dyson–Bressan [1] have established the existence and uniqueness of integral solution for the problem (FDE,ϕ) with $A \equiv A(t, x_t)$, $G \equiv G(t, x_t)$.

The method of lines for the problem (FDE,ϕ) with $A \equiv -A_t$ and $G \equiv G(t, x_t)$ was developed for space X with X^* uniformly convex by Kartsatos–Parrott [5, 6].

Recently, Kartsatos–Liu [4] have developed method of line for (FDE,ϕ) with m-accretive operators $-A(t, \psi)x$.

Our object is construct an approximate solution for the problem

$$\begin{aligned} &x' \in A(t, x_t)x + G(t, x_t, L_t x), \quad t \in [0, T], \\ &x_0 = \phi, \end{aligned} \tag{FDE,ϕ}$$

where $A(t, \psi_1)$ and $G(t, \psi_1, L_t\psi_2)$ are defined, with respect to ψ_1, on a subspace of the space $PC([-r, 0], X)$ of all piecewise continuous functions $f : [-r, 0] \to X$. An appropriate subspace of $PC([-r, t], X)$ is the domain of definition of the nonlinear operators L_t, $t \in [0, T]$. The operators $(A(t, \psi)x - wI)$ are m-dissipative in x and Lipschitz like in (t, ψ). The operators G and L_t are Lipschitzian mappings on their respective domains.

In this paper, we constitutes a approximation scheme for (FDE,ϕ) without fixed the functional term of $A(t, x_t)$ by assumption of $A(t, \psi)$ which is more general than the condition of Kartsatos–Liu [4].

In what follows, X stands for a real Banach space with dual space X^* and normalized duality mapping J.

We recall that for $x, y \in X$,

$$\langle y, x\rangle_+ = \lim_{h\to 0^+} \frac{\|x + hy\| - \|x\|}{h}, \qquad \langle y, x\rangle_- = \lim_{h\to 0^-} \frac{\|x + hy\| - \|x\|}{h}.$$

For some properties of $\langle\cdot,\cdot\rangle_\pm$, we refer the reader to Kobayashi [7] and Pavel [8].

An operator $A : D(A) \subset A \to 2^X$ is called "dissipative" if for every $x, y \in D(A)$, there exists $j \in J(x-y)$ such that $\langle u-v, j\rangle \le 0$ for all $u \in Ax$, $v \in Ay$. A dissipative operator A is "m-dissipative" if $R(I - \lambda A) = X$ for all $\lambda \in (0, \infty)$. Also, A is said to be accretive if $-A$ is dissipative.

We denote by PC the space of all piecewise continuous functions $f : [-r, 0] \to \overline{B_{\overline{r}}(0)}$ associated with the supremum norm, where $B_{\overline{r}}(0)$ is the ball of X with radius $\overline{r}$ and center 0.

We consider the following assumptions:

(A1) There exists $w \in \mathbb{R}$ such that for each $(t, \psi) \in [0, T] \times PC$, $A(t, \psi) - wI$ is m-dissipative for $0 \le \lambda \le \lambda_0 = 1/\max(0, w)$.

(A2) There exists a continuous increasing function $r_0 : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned}\langle y_1 - y_2,\ x_1 - x_2\rangle_- \leq & w\|x_1 - x_2\| + r_0(\|\psi_1\|_{PC}, \|\psi_2\|_{PC}, \|x_2\|) \\ & \cdot [|t_1 - t_2|(1 + \|y_2\|) + |\psi_1 - \psi_2\|_{PC}]\end{aligned}$$

for all $y_i \in A(t_i, \psi_i)x_i$, $i = 1, 2$, $(t_i, \psi_i) \in [0, T] \times PC$.

(L1) $L_t : PC([-r, t], X) \to X$ for every $t \in [0, T]$. Moreover,

$$\|L_t\psi_1 - L_t\psi_2\| \leq a_1(t)\|\psi_1 - \psi_2\|_t \tag{L1,1}$$

and

$$\|L_t\psi - L_s\psi\| \leq r_1(\|\psi\|_T)|t - s|, \tag{L1,2}$$

for every $t, s \in [0, T]$ and every $\psi, \psi_1, \psi_2 \in PC([-r, T], X)$, where $a_1 : [0, T] \to \mathbb{R}^+$ is continuous, $r_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function and $\|\psi\|_t = \sup_{\theta\in[-r,t]} \|\psi(\theta)\|$.

(G1)

$$\|G(t, \psi_1, y_1) - G(t, \psi_2, y_2)\| \leq a_2(t)[\|\psi_1 - \psi_2\|_{PC} + \|y_1 - y_2\|] \tag{G1,1}$$

and

$$\|G(t, \psi, y) - G(s, \psi, y)\| \leq r_2(\|\psi\|_{PC}, \|y\|)|t - s|, \tag{G1,2}$$

for every $t, s \in [0, T]$, every $\psi, \psi_1, \psi_2 \in PC$ and every $y, y_1, y_2 \in B_{\overline{r}}(0)$, where $a_2 : [0, T] \to \mathbb{R}^+$ is continuous and $r_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing in both variables.

(ϕ1) $\phi \in PC$ is a given Lipschitz continuous function with Lipschitz constant C_ϕ and $\phi(0) \in D(A(0, \phi))$.

From (A1), the resolvents J_λ and Yosida approximants A_λ of A are defined by $J_\lambda y = (I - \lambda A)^{-1}y$ and $A_\lambda y = \lambda^{-1}(J_\lambda - I)y$, respectively. It is readily verified that

1. $A_\lambda y \in AJ_\lambda y$ for all $y \in X$,
2. $\|J_\lambda x - J_\lambda y\| \leq (1 - \lambda w)^{-1}\|x - y\|$ for all $x, y \in X$,
3. $\|A_\lambda u\| \leq (1 - \lambda w)^{-1} \inf\{\|y\| \mid y \in Au\}$ for all $u \in D(A)$, and so
$$\lim_{\lambda\to 0^+} \|A_\lambda u\| \leq \inf\{\|y\| \mid y \in Au\} \equiv |Au|.$$

Further properties of J_λ and A_λ can be found in Pavel [8]. We set $C_1 \equiv \max_{t\in[0,T]} a_1(t)$ and $C_2 \equiv \max_{t\in[0,T]} a_2(t)$ in (L1) and (G1).

From the Proposition 2.2 in [8], it can be seen that (A2) is equivalent to the condition

(A3) For all $y_i \in A(t_i,\psi_i)x_i$, $i=1,2$, $(t_i,\psi_i) \in [0,T]\times PC$, there is a function r_0 as in (A2) such that for $\lambda > 0$

$$\begin{aligned}(1-\lambda w)\|x_1-x_2\| \le &\|x_1-x_2-\lambda(y_1-y_2)\| \\ &+ \lambda r_0(\|\psi_1\|_{PC},\|\psi_2\|_{PC},\|x_2\|) \\ &\cdot [|t_1-t_2|(1+\|y_2\|)+\|\psi_1-\psi_2\|_{PC}],\end{aligned}$$

which implies

(A4)

$$\begin{aligned}(1-\lambda w)\|x-u\| \le &\|x-\lambda y-u\| + \lambda|A(s,\psi_2)u| \\ &+ \lambda r_0(\|\psi_1\|_{PC},\|\psi_2\|_{PC},\|u\|) \\ &\cdot [|t-s|(1+|A(s,\psi_2)u|)+\|\psi_1-\psi_2\|_{PC}]\end{aligned}$$

for all $\lambda > 0$, $u \in D(A(s,\psi_2))$.

Also, from [1], it will be seen that (A3) is equivalent to the condition (A5) which for $x_1 = x_2$ is condition (A2) in [3] and [4].

(A5) There exists a function $\overline{r_0} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ which is increasing and continuous such that for all $x_1, x_2 \in X$ and $0 < \lambda < \lambda_0$,

$$\begin{aligned}\|J_\lambda(t_1,\psi_1)x_1 - &J_\lambda(t_2,\psi_2)x_2\| \\ &\le \frac{1}{1-\lambda w}\|x_1-x_2\| + \lambda\overline{r_0}(\|\psi_1\|_{PC},\|\psi_2\|_{PC},\|x_2\|) \\ &\quad \cdot [|t_1-t_2|(1+\|A_\lambda(t_2,\psi_2)x_2\|)+\|\psi_1-\psi_2\|_{PC}].\end{aligned}$$

Therefore our condition (A1) and (A2) are more general than conditions (A1) and (A2) of Kartsatos–Liu [4].

2. Existence and convergence of the method of lines

In this section, we show the existence of a method of lines for the problem (FDE), Theorem 1, and then we show that this method converges uniformly to a "limit solution" of the problem (FDE,ϕ), Theorem 2. As the similar process of Kartsatos–Liu [4], we have the following theorem.

Theorem 1. Assume that conditions *(A1), (G1) and (L1)* hold and that $\phi \in PC$. Assume that for every pair of piecewise continuous functions $\psi \in PC$, $w : [-r,T] \to \overline{B_{\bar r}(0)}$, every $x \in D(A(t,\psi))$, with $\|x\| > \bar r$, $t \in [0,T]$, and every $u \in A(t,\psi)x$ there exists a functional $g \in J_x$ such that

$$\langle u + G(t,w_t,L_t(w)), g\rangle \le 0. \qquad (*)$$

Then there exists a method of lines $\{\bar{z}_n^n(t)\}$ on $\overline{B_{\bar{r}}(0)}$ for the problem *(FDE,ϕ)* such that

$$\bar{z}_j^n(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ z_1^n, & t \in (0, t_1^n] \\ \vdots & \\ z_{j-1}^n, & t \in (t_{j-2}^n, tj-1^n] \\ z_j^n, & t \in (t_{j-1}^n, T], n = 1, 2, \dots, \ j = 1, 2, \dots, n, \end{cases}$$

and

$$\frac{z_j^n - z_{j-1}^n}{h_n} \in A(t_j^n, (\bar{z}_{j-1}^n)_{t_{j-1}^n}) z_j^n + G(t_j^n, (\bar{z}_j^n)_{t_j^n}, L_{t_j^n}(\bar{z}_j^n)),$$

$$t_j^n = j\frac{T}{n} = jh_n \text{ on } [0, T], \qquad \text{where } \bar{z}_0^n = \begin{cases} \phi(t), & t \in [-r, 0] \\ \phi(0), & t \in (0, T]. \end{cases}$$

Proof. We know that the function $F_j(t_j^n)x$ is Lipschitz continuous with Lipschitz constants $C_2(1+C_1)$ on $\overline{B_{\bar{r}}(0)}$. We also observe that the mapping

$$x \to [(\frac{1}{h_n})I - A(t_j^n, (\bar{z}_{j-1}^n)_{t_{j-1}^n})]^{-1} x$$

is Lipschitz continuous with constant $h_n(1-h_n w)^{-1}$. Thus, $S : \overline{B_{\bar{r}}(0)} \to X$ is Lipschitz continuous with constant $C_2(1+C_1)(1-h_n w)^{-1} h_n$. We choose n so large that $C_2(1+C_1)(1-h_n w)^{-1} h_n < 1$, say $n \geq n_0$, and we show that S maps the ball $\overline{B_{\bar{r}}(0)}$ into itself. In fact, given $x \in \overline{B_{\bar{r}}(0)}$, let $u = Sx$. Then, for some $v \in A(t_j^n, (\bar{z}_{j-1}^n)_{t_{j-1}^n})u$, we have

$$\left(\frac{1}{h_n}\right) u - v - F_j(t_j^n)x - \frac{z_{j-1}^n}{h_n} = 0.$$

We proceed by induction. We assume that the vector z_{j-1}^n has already been obtained and that it belongs to the ball $\overline{B_{\bar{r}}(0)}$. We already know that this is true for the point z_0^n. Assuming that $\|u\| > \bar{r}$ and picking an appropriate $g \in Ju$, we apply $(*)$ to obtain

$$\begin{aligned} 0 &= -\langle v + F_j(t_j^n)x, g\rangle + (\frac{1}{h_n})\langle u - z_{j-1}^n, g\rangle \\ &\geq (\frac{1}{h_n})(\|u\|^2 - \|z_{j-1}^n\|\|u\|) \qquad (**) \\ &\geq (\frac{1}{h_n})(\|u\| - \bar{r})\|u\| > 0. \end{aligned}$$

This is a contradiction. Here we have used the fact that $\|(f_j(x))_{t_j^n}\|_\infty \le \bar{r}$. By Banach contraction principle, S has a unique fixed point in $\overline{B_{\bar{r}}(0)}$. This fixed point is the next point z_j^n in the constraction of the method of lines. $\square$

Boundary conditions like $(**)$ have already been applied to ellipic–type problems, involving maximal monotone and $m-$accretive in [2] and [3]. From the proof of Theorem 1, we deduce that $z_j^n \in \overline{B_{\bar{r}}(0)} \cap D(A(t_j^n,(\bar{z}_{j-1}^n)_{t_{j-1}^n}))$ for every $n = 1,2,\dots,\ j = 1,2,\dots,n$.

Lemma 1. The double sequence $\{\dfrac{z_j^n - z_{j-1}^n}{h_n}\}$, $n = 1,2,\dots,\ j = 1,2,\dots,n$, is bounded.

Proof. From $(\phi 1)$ and (A4),

$$
\begin{aligned}
&(1-wh_n)\|z_1^n - z_0^n\| \\
&\quad \le \|z_1^n - z_0^n - h_n(\frac{z_1^n - z_0^n}{h_n} - G(t_1^n,(\bar{z}_1^n)_{t_1^n}, L_{t_1^n}(\bar{z}_1^n))) - z_0^n\| \\
&\qquad + h_n|A(0,\phi)\phi(0)| + h_n r_0(\|(\bar{z}_0^n)_{t_0^n}\|_{PC}, \|\phi\|_{PC}, \|z_0^n\|) \\
&\qquad \cdot [|t_1^n - 0|(1+|A(0,\phi)\phi(0)|) + \|(\bar{z}_0^n)_{t_0^n} - \phi\|_{PC}] \\
&\quad = h_n\|G(t_1^n,(\bar{z}_1^n)_{t_1^n}, L_{t_1^n}(\bar{z}_1^n))\| + h_n|A(0,\phi)\phi(0)| \\
&\qquad + h_n r_0(\|\phi\|_{PC}, \|\phi\|_{PC}, \|\phi(0)\|) \cdot [h_n(1+|A(0,\phi)\phi(0)|)] \\
&\quad \le h_n\|G(t_1^n,(\bar{z}_1^n)_{t_1^n}, L_{t_1^n}(\bar{z}_1^n)) - G(t_1^n,\bar{0},0)\| \\
&\qquad + h_n\|G(t_1^n,\bar{0},0)\| + h_n T r_0(\bar{r},\bar{r},\bar{r})(1+|A(0,\phi)\phi(0)|) \\
&\quad \le C_2 h_n[\|(\bar{z}_1^n)_{t_1^n}\|_{PC} + \|L_{t_1^n}(\bar{z}_1^n)\|] \\
&\qquad + h_n C_3 + h_n T r_0(\bar{r},\bar{r},\bar{r})(1+|A(0,\phi)\phi(0)|),
\end{aligned}
$$

where $C_3 = \max_{t\in[0,T]}\|G(t,\bar{0},0)\|$ and $\bar{0}$ denotes the zero function in $PC([-r,0],X)$. Thus, we have

$$
\begin{aligned}
&(1-wh_n)\|z_1^n - z_0^n\| \\
&\qquad \le h_n[C_2(\bar{r}+C_4) + C_3 + T r_0(\bar{r},\bar{r},\bar{r})(1+|A(0,\phi)\phi(0)|)] \\
&\qquad \equiv [C_5 + T r_0(\bar{r},\bar{r},\bar{r})(1+|A(0,\phi)\phi(0)|)]h_n \equiv C_6 h_n,
\end{aligned}
$$

where

$$
\begin{aligned}
\|L_t(\psi)\| &\le \|L_t(\psi) - L_t(0)\| + \|L_t(0)\| \\
&\le C_1\|\psi\|_T + \|L_t(0) - L_0(0)\| + \|L_0(0)\| \\
&\le C_1\bar{r} + r_1(0)|t-0| + \|L_0(0)\| \equiv C_4,
\end{aligned}
$$

and $C_5 \equiv C_2(\bar{r}+C_4) + C_3$. For $j = 2,3,\dots,n$, we get from (A3) that

$(1-wh_n)\|z_j^n - z_{j-1}^n\|$

$$\le \|z^n_{j-1}-z^n_{j-2}\|+h_n\|G(t^n_j,(\bar z^n_j)_{t^n_j},L_{t^n_j}(\bar z^n_j))-G(t^n_{j-1},(\bar z^n_{j-1})_{t^n_{j-1}},L_{t^n_{j-1}}(\bar z^n_{j-1}))\|$$
$$+ h_n r_0(\bar r,\bar r,\bar r)[h_n(1+\frac{\|z^n_{j-1}-z^n_{j-2}\|}{h_n}+\|G(t^n_{j-1},(\bar z^n_{j-1})_{t^n_{j-1}},L_{t^n_{j-1}}(\bar z^n_{j-1}))\|)$$
$$+\|(\bar z^n_{j-1})_{t^n_{j-1}}-(\bar z^n_{j-2})_{t^n_{j-2}}\|_{PC}].$$

We observe that from (G1)

$$\|G(t^n_j,(\bar z^n_j)_{t^n_j},L_{t^n_j}(\bar z^n_j))-G(t^n_{j-1},(\bar z^n_{j-1})_{t^n_{j-1}},L_{t^n_{j-1}}(\bar z^n_{j-1}))\|$$
$$\le r_2(\bar r,C_4)h_n+C_2[\max_{1\le k\le j}\|z^n_k-z^n_{k-1}\|+C_\phi h_n+C_1\|z^n_j-z^n_{j-1}\|+r_1(\bar r)h_n].$$

Here we used the fact that

$$\begin{aligned}\|(\bar z^n_j)_{t^n_j}-(\bar z^n_{j-1})_{t^n_{j-1}}\|_{PC}&\le \max_{1\le k\le j}\|z^n_k-z^n_{k-1}\|+C_\phi h_n,\\ \|L_{t^n_j}(\bar z^n_j)-L_{t^n_{j-1}}(\bar z^n_{j-1})\|&\le C_1\|\bar z^n_j-\bar z^n_{j-1}\|_{t^n_{j-1}}+r_1(\bar r)h_n\\ &\le C_1\|z^n_j-z^n_{j-1}\|+r_1(\bar r)h_n.\end{aligned}$$

Therefore, we have

$$\begin{aligned}(1-wh_n)\frac{\|z^n_j-z^n_{j-1}\|}{h_n}\le&\frac{\|z^n_{j-1}-z^n_{j-2}\|}{h_n}+h_n[r_2(\bar r,C_4)+C_2C_\phi+C_2r_1(\bar r)\\ &+Tr_0(\bar r,\bar r,\bar r)(1+C_5+C_\phi)]\\ &+h_n\max_{1\le k\le j}\frac{\|z^n_k-z^n_{k-1}\|}{h_n}(C_2+C_1C_2+2r_0(\bar r,\bar r,\bar r)).\end{aligned}$$

Letting $C_7\equiv r_2(\bar r,C_4)+C_2(C_\phi+r_1(\bar r))+r_0(\bar r,\bar r,\bar r)(1+C_5+C_\phi)$ and $p=1-h_n(w+C_2+C_1C_2+2r_0(\bar r,\bar r,\bar r))$ and assuming that n is large enough, we have $p\in(0,1)$ and

$$\frac{p}{h_n}\max_{1\le k\le j}\|z^n_k-z^n_{k-1}\|\le\frac{1}{h_n}\max_{1\le k\le j-1}\|z^n_k-z^n_{k-1}\|+C_7h_n.$$

This implies

$$\begin{aligned}\frac{p}{h_n}\max_{1\le k\le j}\|z^n_k-z^n_{k-1}\|&\le C_7h_n\sum_{s=0}^{j-2}\frac{1}{p^s}+\frac{1}{p^{(j-2)}h_n}\|z^n_1-z^n_0\|\\ &\le C_7h_n\sum_{s=0}^{j-1}\frac{1}{ps}+\frac{C_6}{p^{j-1}}\end{aligned}$$

which yields, for $j=2,3,\dots,n$

$$\begin{aligned}\frac{1}{h_n}\max_{1\le k\le j}\|z^n_k-z^n_{k-1}\|&\le C_7h_n\sum_{s=1}^{j}\frac{1}{p^s}+\frac{1}{p^j}C_6\\ &\le C_7h_n\sum_{s=1}^{n}\frac{1}{p^n}+\frac{1}{p^j}C_6\le\frac{C_7T+C_6}{p^n}.\end{aligned}$$

Since

$$p^n = \left[1 - \frac{T(w + C_2 + C_1C_2 + 2r_0(\bar{r}, \bar{r}, \bar{r}))}{n}\right]^n \to e^{-[w+C_2+C_1C_2+2(\bar{r},\bar{r},\bar{r})]T},$$

as $n \to \infty$, we have the desired conclusion. □

The reader should keep in mind that the approximation index n is large enough so that the proof of the Lemma 1 can go through. We set

$$C_8 \equiv \sup_{1 \le n \le \infty} \max_{1 \le k \le n} \frac{\|z_k^n - z_{k-1}^n\|}{h_n}.$$

The next lemma establishes a Lipschitz–like condition for the functions $\bar{z}_n^n(t)$.

Lemma 2. Let $u_n(t) \equiv \bar{z}_n^n(t)$, $t \in [-r, T]$, $n = 1, 2, \dots$. Then there exists a constant C_{10} such that

$$\|u_n(t) - u_n(s)\| \le C_{10}(|t - s| + h_n), \quad \text{for all} \quad t, s \in [-r, T],\ n = 1, 2, \dots .$$

Proof. We define the "Rothe functions" $z^n(t)$ as follows:

$$z^n(t) \equiv \begin{cases} \phi(t), & t \in [-r, 0], \\ z_{j-1}^n + (t - t_{j-1}^n)\dfrac{(z_j^n - z_{j-1}^n)}{h_n}, & t \in [t_{j-1}^n, t_j^n], \end{cases}$$

for $n = 1, 2, \dots$, $j = 1, 2, \dots, n$. It is easy to see that the sequence $\{z^n(t)\}$ is Lipschitz continuous on $[-r, T]$ with Lipschitz constant $C_9 \equiv \max\{C_8, C_\phi\}$ and we have $\|u_n(t) - z^n(t)\| \le C_9 h_n$ for all $t \in [0, T]$. By the Lipschitz continuity of $z^n(t)$, we have

$$\begin{aligned} \|u_n(t) - u_n(s)\| &\le \|u_n(t) - z^n(t)\| + \|z^n(t) - z^n(s)\| + \|z^n(s) - u_n(s)\| \\ &\le 2C_9(|t - s| + h_n) = C_{10}(|t - s| + h_n), \end{aligned}$$

for $t, s \in [-r, T]$, where $C_{10} \equiv 2C_9$. This completes the proof. □

Lemma 3. Let $\{x_j^n\}_{j=0}^n$ and $\{y_k^m\}_{k=0}^m$ be as in Theorem 1. We have

$$\frac{x_j^n - x_{j-1}^n}{h_n} \in A(s_j^n, (\bar{x}_j^n)_{s_{j-1}^n})x_j^n + G(s_j^n, (\bar{x}_j^n)_{s_j^n}, L_{s_j^n}(\bar{x}_j^n)),$$

where $h_n = s_j^n - s_{j-1}^n = T/n$ and $x_0^n = \phi(0)$. Also

$$\frac{y_k^m - y_{k-1}^m}{h_m} \in A(t_k^m, (\bar{y}_k^m)_{t_{k-1}^m})y_k^m + G(t_k^m, (\bar{y}_k^m)_{t_k^m}, L_{t_k^m}(\bar{y}_k^m)),$$

where $h_m = t^m_k - t^m_{k-1} = T/m$ and $y^m_0 = \hat{\phi}(0)$. Then

$$\begin{aligned}\hat{\gamma}_{j,k}\|x^n_j - y^m_k\| \le & \frac{h_n}{h_n+h_m}\hat{\gamma}_{j,k-1}\|x^n_j - y^m_{k-1}\| + \frac{h_m}{h_n+h_m}\hat{\gamma}_{j-1,k}\|x^n_{j-1} - y^m_k\| \\ & + \frac{h_nh_m}{h_n+h_m}\{C_{11}|t^n_j - t^m_k| + r_0(\bar{r},\bar{r},\bar{r})\|(\bar{x}^n_{j-1})_{s^n_{j-1}} - (\bar{y}^m_{k-1})_{t^m_{k-1}}\|_{pc} \\ & + \langle x^n_j - y^m_k, G(s^n_j, (\bar{x}^n_j)_{t^n_j}, L_{s^n_j}(\bar{x}^n_j)) - G(t^m_k, (\bar{y}^m_k)_{t^m_k}, L_{t^m_k}(\bar{y}^m_k))\rangle_+\}\end{aligned}$$

for $0 \le p \le j \le n$, $0 \le q \le k \le m$, where $C_{11} = (1 + C_8 + C_5)r_0(\bar{r},\bar{r},\bar{r})$ and $\hat{\gamma}_{j,k} = (1 - wh_n)^{j-p}(1 - wh_m)^{k-q}$.

Proof. We choose $\lambda \in (0,1)$ and let $\sigma = h_nh_m/(h_n + h_m)$. Then

$$\begin{aligned} & x^n_j - y^m_k - \sigma\lambda[(\frac{x^n_j - x^n_{j-1}}{h_n} - G(t^n_j, (\bar{x}^n_j)_{s^n_j}, L_{s^n_j}(\bar{x}^n_j)) \\ & \quad - (\frac{y^m_k - y^m_{k-1}}{h_m} - G(t^m_k, (\bar{y}^m_k)_{t^m_k}, L_{t^m_k}(\bar{y}^m_k)))] \\ & = (1-\lambda)(x^n_j - y^m_k) + \frac{h_n\lambda}{h_n+h_m}(x^n_j - y^m_{k-1}) + \frac{h_m\lambda}{h_n+h_m}(x^n_{j-1} - y^m_k) \\ & \quad + \sigma\lambda[G(s^n_j, (\bar{x}^n_j)_{s^n_j}, L_{s^n_j}(\bar{x}^n_j)) - G(t^m_k, (\bar{y}^m_k)_{t^m_k}, L_{t^m_k}(\bar{y}^m_k))]\end{aligned}$$

which implies, by (A3)

$$\begin{aligned} & (1 - w\sigma\lambda)(1-\lambda)\|x^n_j - y^m_k\| + (1 - w\sigma\lambda)\lambda\|x^n_j - y^m_k\| \\ & = (1 - w\sigma\lambda)\|x^n_j - y^m_k\| \\ & \le \|(1-\lambda)(x^n_j - y^m_k) + \sigma\lambda[G(s^n_j, (\bar{x}^n_j)_{s^n_j}, L_{s^n_j}(\bar{x}^n_j)) - G(t^m_k, (\bar{y}^m_k)_{t^m_k}, L_{t^m_k}(\bar{y}^m_k))]\| \\ & \quad + \frac{h_m\lambda}{h_n+h_m}\|x^n_{j-1} - y^m_k\| + \frac{h_n\lambda}{h_n+h_m}\|x^n_j - y^m_{k-1}\| + \sigma\lambda r_0(\|(\bar{x}^n_j)_{s^n_{j-1}}\|_{PC}, \\ & \quad \|(\bar{y}^m_k)_{t^m_{k-1}}\|_{PC}, \|y^m_k\|) \cdot [|s^n_j - t^m_k|(1 + C_8 + C_5) + \|(\bar{x}^n_{j-1})_{s^n_{j-1}} - (\bar{y}^m_{k-1})_{t^m_{k-1}}\|_{PC}].\end{aligned}$$

Multiplying $(1 - wh_n)^{j-p}(1 - wh_m)^{k-q} \equiv \hat{\gamma}_{j,k}$ in the above inequality and letting $\xi = \lambda/(1-\lambda)$. Then we have

$$\begin{aligned} & (1 - w\sigma\lambda)\hat{\gamma}_{j,k}\{\frac{1-\lambda}{\lambda}\|x^n_j - y^m_k\| + \frac{\lambda}{\lambda}\|x^n_j - y^m_k\|\} \\ & \le \frac{h_n}{h_n+h_m}\|x^n_j - y^m_{k-1}\|\hat{\gamma}_{j,k-1} + \frac{h_m}{h_n+h_m}\|x^n_{j-1} - y^m_k\|\hat{\gamma}_{j-1,k} \\ & \quad + \hat{\gamma}_{j,k}C_{11}\sigma|s^n_j - t^m_k| + \hat{\gamma}_{j,k}\sigma r_0(\bar{r},\bar{r},\bar{r})\|(\bar{x}^n_{j-1})_{s^n_{j-1}} - (\bar{y}^m_{k-1})_{t^m_{k-1}}\|_{pc} \\ & \quad + \hat{\gamma}_{j,k}\frac{1}{\xi}\|(x^n_j - y^m_k) + \xi\sigma[G(s^n_j, (\bar{x}^n_j)_{s^n_j}, L_{s^n_j}(\bar{x}^n_j)) - G(t^m_k, (\bar{y}^m_k)_{t^m_k}, L_{t^m_k}(\bar{y}^m_k))]\|.\end{aligned}$$

Rearranging the above inequality,

$$\begin{aligned}\hat{\gamma}_{j,k}\|x_j^n - y_k^m\| \leq \frac{h_n}{h_n+h_m}\|x_j^n - y_{k-1}^m\|\hat{\gamma}_{j,k-1} + \frac{h_m}{h_n+h_m}\|x_{j-1}^n - y_k^m\|\hat{\gamma}_{j-1,k} \\ + \hat{\gamma}_{j,k}C_{11}\sigma|s_j^n - t_k^m| + \hat{\gamma}_{j,k}\sigma r_0(\bar{r},\bar{r},\bar{r})\|(\bar{x}_j^n)_{s_{j-1}^n} - (\bar{y}_k^m)_{t_{k-1}^m}\|_{pc} \\ + \hat{\gamma}_{j,k}\{\frac{1}{\xi}\|(x_j^n - y_k^m) + \xi\sigma[G(s_j^n,(\bar{x}_j^n)_{s_j^n}, L_{s_j^n}(\bar{x}_j^n)) \\ - G(t_k^m,(\bar{y}_k^m)_{t_k^m}, L_{t_k^m}(\bar{y}_k^m))]\|\} + \hat{\gamma}_{j,k}(w\sigma\lambda - 1)\frac{1}{\xi}\|x_j^n - y_k^m\|.\end{aligned}$$

Therefore

$$\begin{aligned}&\hat{\gamma}_{j,k}\|x_j^n - y_k^m\|(1-w\sigma) \\ &\quad\leq \frac{h_n}{h_n+h_m}\|x_j^n - y_{k-1}^m\|\hat{\gamma}_{j,k-1} + \frac{h_m}{h_n+h_m}\|x_{j-1}^n - y_k^m\|\hat{\gamma}_{j-1,k} \\ &\quad\quad + C_{11}\sigma|s_j^n - t_k^m|\hat{\gamma}_{j,k} + \sigma r_0(\bar{r},\bar{r},\bar{r})\hat{\gamma}_{j,k}\|(\bar{x}_j^n)_{s_{j-1}^n} - (\bar{y}_k^m)_{t_{k-1}^m}\|_{pc} \\ &\quad\quad + \hat{\gamma}_{j,k}\langle x_j^n - y_k^m, \sigma(G(s_j^n,(\bar{x}_j^n)_{s_j^n}, L_{s_j^n}(\bar{x}_j^n)) - G(t_k^m,(\bar{y}_k^m)_{t_k^m}, L_{t_k^m}(\bar{y}_k^m)))\rangle_\xi.\end{aligned}$$

We note that $(1-w\sigma)^{-1}\hat{\gamma}_{j,k} < 1$ since $0 < \hat{\gamma}_{j,k} < \max\{1-wh_n, 1-wh_m\} \leq 1-w\sigma < 1$. Dividing $(1-w\sigma)$ and letting $\lambda \to 0^+$ in the above inequlity, then we obtain the desired result. □

Lemma 4. Let m and n be positive integers and let $\{z_j^n\}_{j=0}^n$, $\{z_k^m\}_{k=0}^m$ be constructed as in Theorem 1, for *(FDE,ϕ)*, which described in Lemma 3. Let $u_n(t) \equiv \bar{z}_n^n(t)$ and $u_m(t) = \bar{z}_m^m(t)$. Then there exist constants C_{12}, C_{15} and a positive sequence $\{\varepsilon_{n,m}\}$ with $\lim_{m,n\to\infty}\varepsilon_{n,m} = 0$ such that

$$\begin{aligned}&(1-wh_n)^j(1-wh_m)^k\|z_j^n - z_k^m\| \\ &\qquad\qquad \leq C_{15}D_{j,k} + E_j + jh_n(C_{15}D_{j,k} + \varepsilon_{n,m})),\end{aligned} \tag{1}$$

for $j = 0,1,\ldots,n$ and $k = 0,1,\ldots,m$, where the sequences $\{D_{j,k}\}$ and $\{E_j\}$ are defined by

$$D_{j,k} = \{(t_j^n - t_k^m)^2 + h_n t_j^n + h_m t_k^m\}^{1/2}$$

and

$$E_j = C_{12}\sum_{i=1}^{j}[\sup_{t\in[-r,t_i^n]}\|u_n(t) - u_m(t)\| \cdot h_n],$$

respectively.

Proof. By Lemma 3, we have

$$
\begin{aligned}
(1-wh_n)^j(1-wh_m)^k\|z_j^n - z_k^m\| \\
\le & \frac{h_n}{h_n+h_m}(1-wh_n)^j(1-wh_m)^{k-1}\|z_j^n - z_{k-1}^m\| \\
& + \frac{hm}{h_n+hm}(1-wh_n)^{j-1}(1-whm)^k\|z_{j-1}^n - z_k^m\| \\
& + \frac{h_nh_m}{h_n+h_m}\{C_{11}|t_j^n - t_k^m| + r_0(\bar r,\bar r,\bar r)\|(\bar z_{j-1}^n)_{t_{j-1}^n} - (\bar z_{k-1}^m)_{t_{k-1}^m}\|_{PC} \\
& + \langle z_j^n - z_k^m, G(t_j^n,(\bar z_j^n)_{t_j^n}, L_{t_j^n}(\bar z_j^n)) - G(t_k^m,(\bar z_k^m)_{t_k^m}, L_{t_k^m}(\bar z_k^m))\rangle_+\},
\end{aligned}
$$

for $1 \le j \le n$, $1 \le k \le m$. By (G1,1), (G1,2) and Lemma 2, we have

$$
\begin{aligned}
\|G(t_j^n,&(\bar z_j^n)_{t_j^n}, L_{t_j^n}(\bar z_j^n)) - G(t_k^m,(\bar z_k^m)_{t_k^m}, L_{t_k^m}(\bar z_k^m))\| \\
&\le \|G(t_j^n,(\bar z_j^n)_{t_j^n}, L_{t_j^n}(\bar z_j^n)) - G(t_j^n,(\bar z_k^m)_{t_k^m}, L_{t_k^m}(\bar z_k^m)\| \\
&\quad + \|G(t_j^n,(\bar z_k^m)_{t_k^m}, L_{t_k^m}(\bar z_k^m) - G(t_k^m,(\bar z_k^m)_{t_k^m}, L_{t_k^m}(\bar z_k^m)\| \\
&\le C_2[\|(\bar z_j^n)_{t_j^n} - (\bar z_k^m)_{t_k^m}\|_{pc} + \|L_{t_j^n}(\bar z_j^n) - L_{t_k^m}(\bar z_k^m)\| + r_2(\bar r, C_4)|t_j^n - t_k^m| \\
&\le C_2[\sup_{t\in[-r,t_j^n]}\|u_n(t) - u_m(t)\| + C_{10}(|t_j^n - t_k^m| + h_m) \\
&\quad + \|L_{t_j^n}(\bar z_j^n) - L_{t_k^m}(\bar z_k^m)\|] + r_2(\bar r, C_4)|t_j^n - t_k^m|,
\end{aligned}
$$

where

$$
\begin{aligned}
\|(\bar z_j^n)_{t_j^n} - (\bar z_k^m)_{t_k^m}\|_{PC} &\le \|(u_n)_{t_j^n} - (u_m)_{t_j^n}\|_{PC} + \|(u_m)_{t_j^n} - (u_m)_{t_k^m}\|_{PC} \\
&\le \sup_{t\in[-r,t_j^n]}\|u_n(t) - u_m(t)\| + C_{10}(|t_j^n - t_k^m| + h_m).
\end{aligned}
$$

We observe that for $0 \le t_k^m \le t_j^n$,

$$
\begin{aligned}
\|L_{t_j^n}(\bar z_j^n) - L_{t_k^m}(\bar z_k^m)\| &\le \|L_{t_j^n}(\bar z_j^n) - L_{t_k^m}(\bar z_j^n)\| + \|L_{t_k^m}(\bar z_j^n) - L_{t_k^m}(\bar z_k^m)\| \\
&\le r_1(\bar r)|t_j^n - t_k^m| + C_1 \sup_{t\in[-r,t_j^n]}\|u_n(t) - u_m(t)\|.
\end{aligned}
$$

Similarly, for $0 \le t_j^n \le t_k^m$, we obtain

$$
\|L_{t_j^n}(\bar z_j^n) - L_{t_k^m}(\bar z_k^m)\| \le r_1(\bar r)|t_j^n - t_k^m| + C_1 \sup_{t\in[-r,t_j^n]}\|u_n(t) - u_m(t)\|.
$$

Therefore

$$
\begin{aligned}
&\|G(t_j^n,(\bar z_j^n)_{t_j^n}, L_{t_j^n}(\bar z_j^n)) - G(t_k^m,(\bar z_k^m)_{t_k^m}, L_{t_k^m}(\bar z_k^m))\| \\
&\le C_2[\sup_{t\in[-r,t_j^n]}\|u_n(t) - u_m(t)\| + C_{10}(|t_j^n - t_k^m| + h_m) + r_1(\bar r)(|t_j^n - t_k^m|) \\
&\quad + C_1 \sup_{t\in[-r,t_j^n]}\|u_n(t) - u_m(t)\|] + r_2(\bar r, C_4)|t_j^n - t_k^m|.
\end{aligned}
$$

Also, we arrive at

$$\begin{aligned}&\|(\bar{z}^n_{j-1})_{t^n_{j-1}} - (\bar{z}^m_{k-1})_{t^m_{k-1}}\|_{PC}\\ &\qquad\le \sup_{t\in[-r,t^n_j]}\|u_n(t)-u_m(t)\| + C_{10}(|t^n_j - t^m_k| + h_n + 2h_m).\end{aligned}$$

Put $(1-wh_n)^j(1-wh_n)^k = \gamma_{j,k}$ and $a_{j,k} = \|z^n_j - z^m_k\|$. Then, we have

$$\begin{aligned}&(1-wh_n)^j(1-wh_m)^k\|z^n_j - z^m_k\| = \gamma_{j,k}a_{j,k} \qquad (2)\\ &\le \frac{h_n}{h_n+h_m}\gamma_{j,k-1}a_{j,k-1} + \frac{h_m}{h_n+h_m}\gamma_{j-1,k}a_{j-1,k} + \frac{h_nh_m}{h_n+h_m}\Big\{C_{11}|t^n_j - t^m_k|\\ &\quad + r_0(\bar{r},\bar{r},\bar{r})[\sup_{t\in[-r,t^n_j]}\|u_n(t)-u_m(t)\| + C_{10}(|t^n_j - t^m_k| + h_n + 2h_m)]\\ &\quad + \{C_2[\sup_{t\in[-r,t^n_j]}\|u_n(t)-u_m(t)\| + C_{10}(|t^n_j - t^m_k| + h_m) + r_1(\bar{r})|t^n_j - t^m_k|\\ &\quad + C_1\sup_{t\in[-r,t^n_j]}\|u_n(t)-u_m(t)\|] + r_2(\bar{r},C_4)|t^n_j - t^m_k|\Big\}\\ &\le \frac{h_n}{h_n+h_m}\gamma_{j,k-1}a_{j,k-1} + \frac{h_m}{h_n+h_m}\gamma_{j-1,k}a_{j-1,k}\\ &\quad + \frac{h_nh_m}{h_n+h_m}\{C_{13}D_{j-1,k} + C_{12}\sup_{t\in[-r,t^n_j]}\|u_n(t)-u_m(t)\| + \varepsilon_{n,m}\},\end{aligned}$$

where

$$\begin{aligned}C_{13} &= C_{11} + r_0(\bar{r},\bar{r},\bar{r})C_{10} + C_2(C_{10} + r_1(\bar{r})) + r_2(\bar{r},C_4)\\ C_{12} &= r_0(\bar{r},\bar{r},\bar{r}) + C_2(1+C_1),\\ \varepsilon_{n,m} &= C_{13}h_n + C_{10}[r(\bar{r_0},\bar{r},\bar{r})(h_n+2h_m) + C_2h_m].\end{aligned}$$

Here we use the fact that

$$\||t^n_j - t^m_k| \le |(t^n_j - t^m_k) - h_n| + h_n = |t^n_{j-1} - t^m_k| + h_n \le D_{j-1,k} + h_n.$$

On the other hands, from (A4)

$$\begin{aligned}&(1-wh_n)\|z^n_i - \phi(0)\| \le \|z^n_i - h_n(\frac{z^n_i - z^n_{i-1}}{h_n} - G(t^n_i,(\bar{z}^n_i)_{t^n_i}, L_{t^n_i}(\bar{z}^n_i)) - \phi(0)\|\\ &\quad + h_n|A(0,\phi)\phi(0)| + h_nr_0(\|(\bar{z}^n_{i-1})t^n_{i-1}\|_{PC}, \|\phi\|_{PC}, \|\phi(0)\|)\\ &\quad \cdot[|t^n_i - 0|(1 + |A(0,\phi)\phi(0)|) + \|(\bar{z}^n_{i-1})_{t^n_{i-1}} - \phi\|_{PC}]\\ &\le\|z^n_{i-1} - \phi(0)\| + h_n\|G(t^n_i,(\bar{z}^n_i)_{t^n_i}, L_{t^n_i}(\bar{z}^n_i))\| + h_n|A(0,\phi)\phi(0)|\\ &\quad + h_nr_0(\bar{r},\bar{r},\bar{r})[T(1+|A(0,\phi)\phi(0)|) + 2\bar{r}]\\ &\le\|z^n_{i-1} - \phi(0)\|\\ &\quad + h_n[C_5 + |A(0,\phi)\phi(0)| + r_0(\bar{r},\bar{r},\bar{r})(T(1+|A(0,\phi)\phi(0)|) + 2\bar{r})]\\ &\equiv\|z^n_{i-1} - \phi(0)\| + C_{14}h_n,\end{aligned}$$

where $C_{14} = C_5 + |A(0,\phi)\phi(0)| r_0(\bar{r},\bar{r},\bar{r})[T(1+|A(0,\phi)\phi(0)|)+2\bar{r}]$. Applying this inequality for $i = 1, 2, \dots, j$, we have

$$\begin{aligned}\|z_j^n - \phi(0)\| &\le C_{14} h_n \sum_{i=1}^{j} (1 - wh_n)^{-i} \\ &\le C_{14}(jh_n)(1 - wh_n)^{-j} \le C_{14} D_{j,0}(1 - wh_n)^{-j}.\end{aligned}$$

Thus $a_{j,0} \equiv (1 - wh_n)^j \|z_j^n - \phi(0)\| \le C_{14} D_{j,0} \le C_{15} D_{j,0}$, for $j = 0, 1, \dots, n$, where $C_{15} = \max\{C_{14}, C_{13}\}$. In the same way, we see that $a_{0,k} \le C_{15} D_{0,k}$, for $k = 0, 1, \dots, m$. This means that (1) holds for the pairs $(j, 0)$ and $(0, k)$. Assume that (1) holds for the pairs $(j-1, k)$ and $(j, k-1)$. We want to show that (1) holds for the pairs (j, k) as well. By (2),

$$\begin{aligned}\gamma_{j,k} a_{j,k} \le & \frac{h_n}{h_n + h_m}[C_{15} D_{j,k-1} + E_j + jh_n(C_{15} D_{j,k-1} + \varepsilon_{n,m})] \\ & + \frac{h_m}{h_n + h_m}[C_{15} D_{j-1,k} + E_{j-1} + (j-1)h_n(C_{15} D_{j-1,k} + \varepsilon_{n,m})] \\ & + \frac{h_n h_m}{h_n + h_m}[C_{12} \sup_{t \in [-r, t_j^n]} \|u_n(t) - u_m(t)\| + C_{15} D_{j-1,k} + \varepsilon_{n,m}] \\ \le & C_{15} D_{j,k} + E_j + jh_n(C_{15} D_{j,k} + \varepsilon_{n,m}).\end{aligned}$$

Here, we used

$$\frac{h_n}{h_n + h_m} D_{j,k-1} + \frac{h_m}{h_n + h_m} D_{j-1,k} \le D_{j,k}.$$

Consequently, we show that (1) is true for all (j, k) with $0 \le j \le n$ and $0 \le k \le m$. □

We are now ready for the proof of the existence of a limit solution of (FDE,ϕ).

By the simlar method for the proof of Theorem 2 in Kartsatos–Liu [4], we have the following theorem.

Theorem 2. *The limit $u(t) = \lim_{n\to\infty} u_n(t)$ exists uniformly on $[-r, T]$ and $u(t)$ is a Lipschitz continuous function on $[-r, T]$ with Lipschitz constant C_{10}. This function $u(t)$ is called a "limit solution" of (FDE,ϕ).*

Proof. Let $\{t_j^n\}$, $\{t_k^m\}$ be two partitions of $[0, T]$, where $t_j^n = jh_n = j(T/n)$, $j = 0, 1, \dots, n$ and $t_k^m = kh_m = k(T/m)$, $k = 0, 1, \dots, m$. Let $t \in (t_{j-1}^n, t_j^n] \cap (t_{k-1}^m, t_k^m]$. Then

$$|t_j^n - t_k^m| \le |t_j^n - t| + |t - t_k^m| \le h_n + h_m.$$

By Lemma 4, we have

$$(1-wh_n)^j(1-wh_m^k\|u_n(t)-u_m(t)\| = (1-wh_n)^j(1-wh_m)^k\|z_j^n - z_m^k\|$$
$$\le C_{15}\{(h_n+h_m)^2+(h_n+h_m)T\}^{1/2} + C_{12}\sum_{i=1}^{j}\sup_{t\in[-r,t_i^n]}\|u_n(t)-u_m(t)\|h_n$$
$$+ T\{C_{15}[(h_n+h_m)^2+(h_n+h_m)T]^{1/2}+\varepsilon_{n,m}\}.$$

We define the function $F_{n,m}$ as follows:

$$F_{n,m}(t) = \begin{cases} 0, & \text{for } t=0,\\ \sup_{s\in[-r,t_l^n]}\|u_n(s)-u_m(s)\|, & \text{for } t\in(t_{l-1}^n, t_l^n],\\ & \text{for some } l=1,2,\dots,n. \end{cases}$$

Fix $t\in(0,T]$. Then $t\in(t_{l-1}^n, t_l^n]$, for some $l=1,2,\dots,n$. Thus

$$F_{n,m}(t) = \max\{\sup_{s\in[-r,t]}\|u_n(s)-u_m(s)\|,\ \sup_{s\in[t,t_l^n]}\|u_n(s)-u_m(s)\|\}.$$

If $s\in[t,t_l^n]$, then by Lemma 2, we get

$$\|u_n(s)-u_m(s)\| \le \|u_n(t)-u_m(t)\| + \|u_n(s)-u_n(t)\| + \|u_m(s)-u_m(t)\|$$
$$\le \sup_{s\in[-r,t]}\|u_n(s)-u_m(s)\| + 2C_{10}h_n + C_{10}(h_n+h_m),$$

which yields

$$\sup_{s\in[t,t_j^n]}\|u_n(s)-u_m(s)\| \le \sup_{s\in[-r,t]}\|u_n(s)-u_m(s)\| + 3C_{10}h_n + C_{10}h_m.$$

Therefore, we see that

$$F_{n,m}(t) \le \sup_{s\in[-r,t]}\|u_n(s)-u_m(s)\| + 3C_{10}h_n + C_{10}h_m,$$

for every $t\in[0,T]$. Hence, for $t\in(t_{j-1}^n, t_j^n]$,

$$C_{12}\sum_{l=1}^{j}\sup_{s\in[-r,t_j^n]}\|u_n(s)-u_m(s)\|h_n = C_{12}\sum_{l=1}^{j}\int_{t_{l-1}}^{t_l^n}F_{n,m}(\tau)d\tau$$
$$= C_{12}\int_0^t F_{n,m}(\tau)d\tau + C_{12}\int_t^{t_j^n}F_{n,m}(\tau)d\tau$$
$$\le C_{12}\int_0^t \sup_{s\in[-r,r]}\|u_n(s)-u_m(s)\|d\tau$$
$$+ 3C_{10}C_{12}Th_n + C_{10}C_{12}Th_m + 2\bar{r}C_{12}h_n.$$

Here, we have used the fact that

$$\int_t^{t_j^n} F_{n,m}(\tau)d\tau = \int_t^{t_j^n} \sup_{s\in[-r,t_j^n]} \|u_n(s)-u_m(s)\|d\tau$$
$$\leq \int_t^{t_j^n} 2\bar{r}dz \leq 2\bar{r}h_n.$$

It follows that

$$(1-wh_n)^j(1-wh_m)^k\|u_n(t)-u_m(t)\|$$
$$\leq \delta_{n,m} + C_{12}\int_0^t \sup_{s\in[-r,\tau]} \|u_n(s)-u_m(s)\|d\tau,$$

for every $t \in (t_{j-1}^n, t_j^n] \cap (t_{k-1}^m.t_k^m]$, where

$$\begin{aligned}\delta_{n,m} =& C_{15}\{(h_n+h_m)^2 + (h_n+h_m)T\}^{1/2} \\ &+ T\{C_{15}[(h_n+h_m)^2 + (h_n+h_m)T]^{1/2} + \varepsilon_{n,m}\} \\ &+ 3C_{10}C_{12}Th_n + C_{10}C_{12}Th_n + 2\bar{r}C_{12}h_n.\end{aligned}$$

Thus, for every $t \in [0,T]$,

$$\gamma_{j,k}\|u_n(t)-u_m(t)\| \leq \delta_{n,m} + C_{12}\int_0^t \sup_{s\in[-r,\tau]} \|u_n(s)-u_m(s)\|d\tau.$$

Taking $n,m\to\infty$ in the above inequality,

$$e^{-2wT}\overline{\lim_{n,m\to\infty}}\|u_n(t)-u_m(t)\| \leq C_{12}\int_0^t \overline{\lim_{n,m\to\infty}} \sup_{s\in[-r,\tau]} \|u_n(s)-u_m(s)\|d\tau.$$

By Grownwall's inequality, we have

$$\lim_{n,m\to\infty} \sup_{s\in[-r,t]} \|u_n(s)-u_m(s)\| = 0.$$

This implies that $u_n(t)$ converges to a function $u(t)$, $t \in [-r,T]$, uniformly on $[-r,T]$. Also,

$$\|u(t)-u(s)\| \leq C_{10}|t-s|, \qquad \text{for } \ t,s \in [-r,T],$$

which proves the Lipschitz continuity of the function $u(t)$ on $[-r,T]$ with Lipschitz constant C_{10}. □

3. The uniqueness of limit solutions and the existence of strong solutions

In this section, from estimating the difference $u(t) - \hat{u}(t)$, where u and $\hat{u}$ are limite soluions of (FDE,ϕ) and (FDE,$\hat{\phi}$), respectively, we study the uniqueness of limit solution and the existence of strong solution in a reflexive Banach space X.

Theorem 3. Let ϕ, $\hat{\phi}$ satisfy *(ϕ1)*. If $u, \hat{u}$ are limit solutions of *(FDE,ϕ)* and *(FDE,$\hat{\phi}$)*, respectively, then, for $0 \le s \le t \le T$, we have

$$\begin{aligned} \|u(t) - \hat{u}(t)\| \le e^{2wT}\{\|u(s) - \hat{u}(s)\| + r_0(\bar{r}, \bar{r}, \bar{r}) \int_s^t \|u_z - \hat{u}_z\|_{PC} dz \\ + \int_s^t \langle u(z) - \hat{u}(z), G(z, u_z, L_z(u)) - G(z, \hat{u}_z, L_z(\hat{u}))\rangle_+ dz\}. \end{aligned} \tag{3}$$

Proof. By the definition of the limit solution, there exists an h_n–approximate solution $u_n(t)$ such that, for $j = 1, 2, \dots, n$

$$\frac{x_j^n - x_{j-1}^n}{h_n} \in A(t_j^n, (\bar{x}_{j-1}^n)_{s_{j-1}^n}) x_j^n + G(s_j^n, (\bar{x}_j^n)_{s_j^n}, L_{s_j^n}(\bar{x}_j^n)),$$

$x_0^n = \phi(0)$ and $\lim_{n\to\infty} u_n(t) = u(t)$, where $h_n = s_j^n - s_{j-1}^n$ and $u_n(t) \equiv \bar{x}_n^n(t)$. Similarly, there exists an h_m–approximate solution $\hat{u}_m(t)$ such that

$$\frac{y_k^m - y_{k-1}^m}{h_m} \in A(t_k^m, (\bar{y}_{k-1}^m)_{t_{k-1}^m}) y_k^m + G(t_k^m, (\bar{y}_k^m)_{t_k^m}, L_{t_k^m}(\bar{y}_k^m)),$$

$y_0^m = \hat{\phi}(0)$ and $\lim_{m\to\infty} \hat{u}_m(t) = \hat{u}(t)$, where $h_m = t_k^m - t_{k-1}^m$ and $\hat{u}_m(t) \equiv \bar{y}_m^m(t)$. Notice that

$$\begin{aligned} \langle y, z\rangle_+ \le \langle y, z\rangle_\lambda \equiv \frac{\|y + \lambda z\| - \|y\|}{\lambda} \\ \le \langle u, v\rangle_\lambda + \|z - v\| + \frac{2\|y - u\|}{\lambda}, \end{aligned}$$

and

$$\begin{aligned} \|(\bar{x}_j^n)_{s_{j-1}^n} - (\bar{y}_k^m)_{t_{k-1}^m}\|_{PC} \\ = \|(u_n)_{s_{j-1}^n} - (\hat{u}_m)_{t_{k-1}^m}\|_{PC} \\ \le \|(u_n)_{s_{j-1}^n} - (\hat{u}_m)_{s_{j-1}^n}\|_{PC} + \|(\hat{u}_m)_{s_{j-1}^n} - (\hat{u}_m)_{t_{k-1}^m}\|_{PC} \\ \le \|(u_n)_{s_{j-1}^n} - (\hat{u}_m)_{s_{j-1}^n}\|_{PC} + C_{10}(|s_j^n - t_k^m| + h_n + 2h_m), \end{aligned}$$

From Lemma 3, we have

$$\begin{aligned}\hat{\gamma}_{j,k}a_{j,k} \leq & \frac{h_n}{h_n+h_m}a_{j,k-1}\hat{\gamma}_{j,k-1} + \frac{h_m}{h_n+h_m}a_{j-1,k}\hat{\gamma}_{j-1,k} \\ & + \frac{h_nh_m}{h_n+h_m}\{r_0(\bar{r},\bar{r},\bar{r})\|(u_n)_{s^n_{j-1}} - (\hat{u}_m)_{s^n_{j-1}}\|_{PC} \\ & + C_{16}|s^n_j - t^m_k| + \delta^n_j + \hat{\delta}^m_k + \rho(|s^n_j - t^m_k|) \\ & + C_{10}r_0(\bar{r},\bar{r},\bar{r})(h_n+2h_m)\},\end{aligned}$$

where $C_{16} = C_{11} + C_{10}r_0(\bar{r},\bar{r},\bar{r})$, $\hat{\gamma}_{j,k} = (1-wh_n)^{j-p}(1-wh_m)^{k-q}$,

$$\begin{aligned}\delta^n_j &\equiv \langle x^n_j - \hat{u}(s^n_j), G(s^n_j, (\bar{x}^n_j)_{s^n_j}, L_{s^n_j}(\bar{x}^n_j)) - G(s^n_j, \hat{u}_{s^n_j}, L_{s^n_j}(\hat{u}))\rangle_\lambda \\ &= \langle u_n(s^n_j) - \hat{u}(s^n_j), G(s^n_j, (u_n)_{s^n_j}, L_{s^n_j}(u_n)) - G(s^n_j, \hat{u}_{s^n_j}, L_{s^n_j}(\hat{u}))\rangle_\lambda,\end{aligned}$$

$$\begin{aligned}\hat{\delta}^m_k &\equiv \|G(t^m_k, (\bar{y}^m_k)_{t^m_k}, L_{t^m_k}(\bar{y}^m_k)) - G(t^m_k, \hat{u}_{t^m_k}, L_{t^m_k}(\hat{u}))\| + \frac{2}{\lambda}\|y^m_k - \hat{u}(t^m_k)\| \\ &= \|G(t^m_k, (\hat{u}_m)_{t^m_k}, L_{t^m_k}(\hat{u}_m)) - G(t^m_k, \hat{u}_{t^m_k}, L_{t^m_k}(\hat{u}))\| + \frac{2}{\lambda}\|y^m_k - \hat{u}(t^m_k)\|\end{aligned}$$

and

$$\rho(t) \equiv \sup_{|s-r|\leq t}[\frac{2}{\lambda}\|\hat{u}(s) - \hat{u}(r)\| + \|G(s, \hat{u}_s, L_s(\hat{u})) - G(r, \hat{u}_r, L_r(\hat{u}))\|], \qquad t \in [0,T].$$

Notice that $\rho(t)$ is a nondecreasing function on $[0,T]$. For $p \in \{0,1,\ldots,n\}$, $q \in \{0,1,\ldots,m\}$, let $j = p,\ldots,n$, $k = q,\ldots,m$. Then

$$\begin{aligned}|s^n_j - t^m_k| &\leq |(s^n_j - s^n_p) - (t^m_k - t^m_q) - h_n| + |s^n_p - t^m_q| + h_n \\ &\leq C_{j-1,k} + |s^n_p - t^m_q| + h_n,\end{aligned}$$

where $\hat{D}_{j,k} \equiv \{[(s^n_j - s^n_p) - (t^m_k - t^m_q)]^2 + (s^n_j - s^n_p)h_n + (t^m_k - t^m_q)h_m\}^{1/2}$ and $C_{j,k} \equiv \hat{D}_{j,k} + \hat{D}^2_{j,k}$.

Let $\delta \in (0, T/2)$ and assume that n and m as sufficiently large so that we have $\max\{h_n, h_m\} < \delta$. Then, by the proof of Lemma 2.4 of Kobayashi [7],

$$\begin{aligned}\rho(|s^n_j - t^m_k|) &\leq \frac{\rho(T)}{\delta}|\,|s^n_j - t^m_k| - h_n| + \rho(2\delta) \\ &\leq \frac{\rho(T)}{\delta}(C_{j-1,k} + |s^n_p - t^m_q|) + \rho(2\delta).\end{aligned}$$

Thus

$$\begin{aligned}\hat{\gamma}_{j,k}a_{j,k} \leq & \frac{h_n}{h_n+h_m}a_{j,k-1}\hat{\gamma}_{j,k-1} + \frac{h_m}{h_n+h_m}a_{j-1,k}\hat{\gamma}_{j-1,k} \\ & + \frac{h_nh_m}{h_n+h_m}\{r_0(\bar{r},\bar{r},\bar{r})\|(u_n)_{s^n_{j-1}} - (u_m)_{s^n_{j-1}}\|_{PC}\end{aligned} \tag{4}$$

$$+ (C_{16} + \frac{\rho(T)}{\delta})(C_{j-1,k} + |s_p^n - t_q^m|)$$
$$+ \rho(2\delta) + \delta_j^n + \hat{\delta}_k^m + 2C_{16}(h_n + h_m)\}.$$

From (A4), we obtain,

$$\begin{aligned}(1 - wh_n)\|x_j^n - x_p^n\| \le& \|x_{j-1}^n - x_p^n\| + h_n\|G(t_j^n, (\bar{x}_j^n)_{t_j^n}, L_{t_j^n}(\bar{x}_j^n))\| \\ &+ h_n|A(t_p^n, (\bar{x}_{p-1}^n)_{s_{p-1}^n})x_p^n| \\ &+ h_n r_0(\bar{r}, \bar{r}, \bar{r})[|s_j^n - s_p^n|(1 + C_8 + C_5) + C_{10} + C_{10}hn] \\ \le& \|x_{j-1}^n - x_p^n\| + h_n|A(s_p^n, (\bar{x}_{p-1}^n)_{s_{p-1}^n})x_p^n| \\ &+ C_{16}|s_j^n - s_p^n|h_n + h_nC_{17},\end{aligned}$$

where $C_{17} = C_5 + C_{10}Tr_0(\bar{r}, \bar{r}, \bar{r})$. Hence, we have

$$\begin{aligned}&(1 - wh_n)^{j-p}\|x_j^n - x_p^n\| \\ &\le h_n\{(|A(s_p^n, (\bar{x}_{p-1}^n)_{s_{p-1}^n})x_p^n| + C_{17})(j - p) + C_{17}\sum_{i=0}^{j-p-1}|s_{j-i}^n - s_p^n|\} \\ &\le |s_j^n - s_p^n|(|A(s_p^n, (\bar{x}_p^n)_{s_p^n})x_p^n| + C_{17}) + C_{18}|s_j^n - t_p^n|^2 \\ &\le C_{j,q}C_{18},\end{aligned}$$

where $C_{18} = |A(s_p^n, (\bar{x}_p^n)_{s_p^n})x_p^n| + C_{16} + C_{17} + |A(s_q^m, (\bar{x}_q^m)_{s_q^m})x_q^m|$, which yields

$$\begin{aligned}(1 - wh_n)^{j-p}\|x_j^n - y_q^m\| \le& (1 - wh_n)^{j-p}\|x_j^n - x_p^n\| \\ &+ (1 - wh_n)^{j-p}\|x_p^n - y_q^m\| \le C_{18} \cdot C_{j,q} + \|x_p^n - y_q^m\|,\end{aligned}$$

and similarly,

$$(1 - wh_m)^{k-q}\|x_p^n - y_k^m\| \le C_{18}C_{p,k} + \|x_p^n - y_q^m\|.$$

We claim that for $p \in \{0, 1, \ldots, n\}$ and $q \in \{0, 1, \ldots, m\}$, we have the estimate

$$\begin{aligned}&(1 - wh_n)^{j-p}(1 - wh_m)^{k-q}\|x_j^n - y_k^m\| \le \|x_p^n - y_q^m\| + C_{19}C_{j,k} \\ &+ \sum_{i=p}^{j}\delta_i^n h_n + \sum_{i=q}^{k}\hat{\delta}_i^m h_m + r_0(\bar{r}, \bar{r}, \bar{r})\sum_{i=p}^{j}\|(u_n)_{s_{i-1}^n} - (u_m)_{s_{i-1}^n}\|_{PC}h_n \\ &+ jh_n[(\delta^{-1}\rho(T) + C_{19})(C_{j,k} + |s_p^n - t_q^m|) + \rho(2\delta) + C_{19}(h_n + h_m)],\end{aligned} \tag{5}$$

provided that $j = p, \ldots, n,\ k = q, \ldots, m$, where $C_{19} = \max\{2C_{16}, C_{18}\}$. We know that (5) is true for $p \le j \le n$ and $k = q$. By the same proof, (5) holds also for $j = p$ and $q \le k \le m$. To apply induction, let us assume that (5) holds for the pairs $(j - 1, k)$, $(j, k - 1)$, where $p + 1 \le j \le n$ and $q + 1 \le k \le m$. Applying (4) and using

$$\frac{h_n}{h_n + h_m}C_{j,k-1} + \frac{h_m}{h_n + h_m}C_{j-1,k} \le C_{j,k},$$

we have (5). By (4),

$$
\begin{aligned}
&(1-wh_n)^{j-p}(1-wh_m)^{k-q}\|x_j^n - y_k^m\| = a_{j,k}\hat{\gamma}_{j,k}\\
\leq&\frac{h_n}{h_n+h_m}a_{j,k-1}\hat{\gamma}_{j,k-1} + \frac{h_m}{h_n+h_m}a_{j-1,k}\hat{\gamma}_{j-1,k}\\
&+\frac{h_nh_m}{h_n+h_m}\{r_0(\bar{r},\bar{r},\bar{r})\|(u_n)_{s^n_{j-1}} - (u_m)_{s^n_{j-1}}\|_{PC}\\
&+(C_{16}+\frac{\rho(T)}{\delta})(C_{j-1,k}+|s^n_p - t^m_q|)+\rho(2\delta)+\delta^n_j+\hat{\delta}^m_k+2C_{16}(h_n+h_m)\}\\
\leq&\frac{h_n}{h_n+h_m}\{\|x^n_p - y^m_q\| + C_{19}C_{j,k-1} + \sum_{i=p}^{j}\delta^n_i h_n + \sum_{i=q}^{k-1}\hat{\delta}^m_i h_m\\
&+r_0(\bar{r},\bar{r},\bar{r})\sum_{i=p}^{j}\|(u_n)_{s^n_{i-1}} - (u_m)_{s^n_{i-1}}\|_{PC}h_n\\
&+jh_n[(\frac{\rho(T)}{\delta}+C_{19})(C_{j,k-1}+|s^n_p - t^m_q|)+\rho(2\delta)+C_{19}(h_n+h_m)]\}\\
&+\frac{h_m}{h_n+h_m}\{\|x^n_p - y^m_q\| + C_{19}C_{j-1,k} + \sum_{i=p}^{j-1}\delta^n_i h_m + \sum_{i=q}^{k}\hat{\delta}^m_i h_m\\
&+r_0(\bar{r},\bar{r},\bar{r})\sum_{i=p}^{j-1}\|(u_n)_{s^n_{i-1}} - (u_m)_{s^n_{i-1}}\|_{PC}h_n\\
&+(j-1)h_n[(\frac{\rho(T)}{\delta}+C_{19})(C_{j-1,k}+|s^n_p - t^m_q|)+\rho(2\delta)+C_{19}(h_n+h_m)]\}\\
&+\frac{h_nh_m}{h_n+h_m}\{r_0(\bar{r},\bar{r},\bar{r})\|(u_n)_{s^n_{j-1}} - (u_m)_{s^n_{j-1}}\|_{PC}\\
&+(C_{16}+\frac{\rho(T)}{\delta})(C_{j-1,k}+|s^n_p - t^m_q|)+\rho(2\delta)+\delta^n_j+\hat{\delta}^m_k+2C_{16}(h_n+h_m)\}\\
\leq&\|x^n_p - y^m_q\| + C_{19}\{\frac{h_n}{h_n+h_m}C_{j,k-1} + \frac{h_m}{h_n+h_m}C_{j-1,k}\}\\
&+\sum_{i=p}^{j}\delta^n_i h_n + \sum_{i=q}^{k}\hat{\delta}^m_i h_m + r_0(\bar{r},\bar{r},\bar{r})\|(u_n)_{s^n_{j-1}} - (u_m)_{s^n_{j-1}}\|_{PC}
\end{aligned}
$$

$$
+jh_n[(\frac{\rho(T)}{\delta}+C_{19})(C_{j,k}+|s^n_p - t^m_q|)+\rho(2\delta)+C_{19}(h_n+h_m)].
$$

We have shown that (5) holds for the pair (j,k). Therefore, (5) holds for $p \leq j \leq n$ and $q \leq k \leq m$ with $p \in \{0,1,\ldots,n\}$ and $q = \{0,1,\ldots,m\}$.

Now, we prove (3). We consider points $s \in (s_{p-1}^n, s_p^n] \cap (t_{q-1}^m, t_q^m]$ and $t \in (s_{j-1}^n, s_j^n] \cap (t_{k-1}^m, t_k^m]$. Letting $m \to \infty$ and $n \to \infty$ in (5), we get

$$\|u(t) - \hat{u}(t)\| \le \lim_{m,n\to\infty} [(1 - wh_n)^{j-p}(1 - wh_m)^{k-q}]^{-1} \tag{6}$$

$$\cdot \{\|u(s) - \hat{u}(s)\| + \overline{\lim_{n\to\infty}} \sum_{i=p}^{j} \delta_i^n h_n + \overline{\lim_{m\to\infty}} \sum_{i=q}^{k} \hat{\delta}_i^m h_m$$

$$+ r_0(\bar{r}, \bar{r}, \bar{r}) \overline{\lim_{m,n\to\infty}} \sum_{i=p}^{j} \|(u_n)_{s_{i-1}^n} - (u_m)_{s_{i-1}^n}\|_{PC} h_n + T\rho(2\delta)\},$$

for every $\delta \in (0, T/2)$. Since

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=p}^{j} \delta_i^n h_n &= \lim_{n\to\infty} \sum_{i=p}^{j} h_n \langle u_n(s_i^n) - \hat{u}(s_i^n), G(s_i^n, (u_n)_{s_i^n}, L_{s_i^n}(u_n)) \\ &\quad - G(s_i n, \hat{u} s_i^n, L_{s_i^n}(\hat{u})\rangle_\lambda \\ &= \int_s^t [u(\tau) - \hat{u}(\tau), G(\tau, u_\tau, L_\tau(u)) - G(\tau, \hat{u}_\tau, L_\tau(\hat{u}))]_\lambda d\tau,\end{aligned}$$

$$\begin{aligned}\lim_{m\to\infty} \sum_{i=q}^{k} \hat{\delta}_i^m h_m &= \lim_{m\to\infty} \sum_{i=q}^{k} h_m [\|G(t_i^m, (\hat{u}_m)_{t_i^m}, L_{t_i^m}(\hat{u}_m)) \\ &\quad - G(t_i^m, \hat{u}_{t_i^m}, L_{t_i^m}(\hat{u}))\| + \frac{2}{\lambda}\|\hat{u}_m(t_i^m) - \hat{u}(t_i^m)\|] = 0,\end{aligned}$$

and

$$\lim_{m\to\infty} \sum_{i=p}^{j} \|(u_n)_{s_{i-1}^n} - (u_m)_{s_{i-1}^n}\|_{PC} h_n = \int_s^t \|u_\tau - \hat{u}_\tau\|_{PC} d\tau,$$

letting $\delta \downarrow 0$ and then, as $\lambda \downarrow 0$ in (6), we obtain the desired result. □

Theorem 4. There exists a unique limit solution of *(FDE,ϕ)*.

Proof. Let u, $\hat{u}$ are two limit solutions of (FDE,ϕ). From Theorem 3,

$$\begin{aligned}\|u(t) - \hat{u}(t)\| \le & e^{2wT}\{r_0(\bar{r}, \bar{r}, \bar{r}) \int_0^t \|u_\tau - \hat{u}_\tau\|_{PC} d\tau \\ & + \int_0^t \|G(\tau, u_\tau, L_\tau(u)) - G(\tau, \hat{u}_\tau, L_\tau(\hat{u}))\| d\tau\},\end{aligned}$$

which yields

$$\|u(t) - \hat{u}(t)\| \le e^{2wT}(r_0(\bar{r}, \bar{r}, \bar{r}) + C_2 + C_1C_2) \int_0^t (1 + C_2 + C_1C_2)\|u - \hat{u}\|_\tau d\tau.$$

Moreover, for each $t \in [0, T]$, the above inequality holds and so

$$\|u - \hat{u}\|_t \le e^{2wT}(r_0(\bar{r}, \bar{r}, \bar{r}) + C_2 + C_1C_2) \int_0^t \|u - \hat{u}\|_\tau d\tau.$$

Applying the Graonwall's inequality, $u \equiv \hat{u}$ on $[-r, T]$. This proves the uniqueness of the limit solution for (FDE,ϕ). □

Lemma 5. Let $u(t)$ be a limit solution. Then, we have the following inequality

$$\|u(t) - x\| - \|u(s) - x\| \le \int_s^t \{\langle u(\tau) - x, G(\tau, u_\tau, L_\tau(u)) + y\rangle_+ + w\|u(s) - x\| + \theta(\tau, \beta)\} d\tau \tag{7}$$

for $0 \le s \le t \le T$, $y \in A(\beta, u_\beta)x$, $\beta \in [0, T]$, where

$$\theta(\tau, \beta) \equiv r_0(\bar{r}, \bar{r}, \|x\|) \cdot (1 + \|y\| + C_{10})|\tau - \beta|.$$

Proof. Let $u_n(t)$ be an h_n-approximate solution with $\lim_{n\to\infty} u_n(t) = u(t)$. Then there exists $v_j^n \in A(t_j^n, (\bar{z}_j^n)_{t_{j-1}^n})z_j^n$ such that

$$(z_j^n - x) - (z_{j-1}^n - x) = h_n(G(t_j^n, (\bar{z}_j^n)_{t_j^n}, L_{t_j^n}(\bar{z}_j^n)) + v_j^n).$$

Then

$$\begin{aligned} \|z_j^n - x\| - \|z_{j-1}^n - x\| \le& \langle z_j^n - x, z_j^n - x\rangle_- - \langle z_j^n - x, z_{j-1}^n - x\rangle_+ \\ \le& \langle z_j^n - x, z_j^n - z_{j-1}^n\rangle_- \\ =& h_n\langle z_j^n - x, G(t_j^n, (\bar{z}_j^n)_{t_j^n}, L_{t_j^n}(\bar{z}_j^n)) + v_j^n\rangle_- \\ \le& h_n\langle z_j^n - x, v_j^n - y\rangle_- \\ &+ h_n\langle z_j^n - x, y + G(t_j^n, (\bar{z}_j^n)_{t_j^n}, L_{t_j^n}(\bar{z}_j^n))\rangle_+, \end{aligned} \tag{8}$$

for any $y \in A(\beta, u_\beta)x$, $\beta \in [0, T]$. By (A2), we arrive at

$$\begin{aligned} \langle z_j^n - x, v_j^n - y\rangle_- \le& w\|z_j^n - x\| + r_0(\|(\bar{z}_{j-1}^n)_{t_{j-1}^n}\|_{PC}, \|u_\beta\|_{PC}, \|x\|) \\ &\cdot [|t_j^n - \beta|(1 + \|y\|) + \|(\bar{z}_{j-1}^n)_{t_{j-1}^n} - u_\beta\|_{PC}] \end{aligned}$$

and

$$\begin{aligned} \|(\bar{z}_j^n)_{t_{j-1}^n} - u_\beta\|_{PC} =& \|(u_n)_{t_{j-1}^n} - u_\beta\|_{PC} \\ \le& \sup_{\theta\in[-r,0]} \|u_n(t_{j-1}^n + \theta) - u_n(\beta + \theta)\| \\ &+ \sup_{\theta\in[-r,0)} \|u_n(\beta + \theta) - u(\beta + \theta)\| \\ \le& C_{10}|t_j^n - \beta| + 2C_{10}h_n + \|u_n - u\|_T. \end{aligned}$$

Combining this with (8), we have

$$\begin{aligned}\|z_j^n - x\| - \|z_{j-1}^n - x\| \leq & h_n w \|z_j^n - x\| + h_n r_0(\bar{r}, \bar{r}, \|x\|) \\ & \cdot [|t_j^n - \beta|(1 + \|y\| + C_{10}) + 2C_{10}h_n + \|u_n - u\|_T] \\ & + h_n \langle z_j^n - x, G(t_j^n, (\bar{z}_j^n)_{t_j^n}, L_{t_j^n}(\bar{z}_j^n)) + y\rangle_+.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\|z_j^n - x\| - \|z_k^n - x\| \leq \sum_{i=k+1}^{j} & h_n\{\langle z_i^n - x, G(t_i^n, (\bar{z}_i^n)_{t_i^n}, L_{t_i^n}(\bar{z}_i^n)) + y\rangle_+ \\ & + w\|z_i^n - x\| + \theta(t_i^n, \beta) + 2r_0(\bar{r}, \bar{r}, \|x\|)C_{10}h_n \\ & + r_0(\bar{r}, \bar{r}, \|x\|)\|u_n - u\|_T\},\end{aligned}$$

where $\theta(t_i^n, \beta) = r_0(\bar{r}, \bar{r}, \|x\|)(1 + \|y\| + C_{10})|t_i^n - \beta|$. Now, let $s \in (t_{k-1}^n, t_k^n]$ and $t \in (t_{j-1}^n, t_j^n]$ and let $n \to \infty$ to obtain the desired result. □

Theorem 5. Let X be reflexive. Then *(FDE,ϕ)* has a strong solution.

Proof. By Theorem 2, there exists a limit solution $u(t)$ of (FDE,ϕ) which is Lipschitz continuous with Lipschitz constant C_{10} on $[-r, T]$. Thus, $u(t)$ is absolutely continuous on $[-r, T]$. Since X is reflexive, $u(t)$ is differentiable almost everywhere on $[0, T]$. Now, let $u(t)$ be differentiable at $t = t_0$ and $h > 0$. Letting $s = \beta = t_0$ and $t = t_0 + h$ in (7), we have

$$\begin{aligned}& \|u(t_0 + h) - x\| - \|u(t_0) - x\| \\ & \leq \int_{t_0}^{t_0+h} \{\langle u(\tau) - x, G(\tau, u_\tau, L_\tau(u)) + y_+ + w\|u(\tau) - x\| + \theta(\tau, t_0)\} d\tau,\end{aligned}$$

for $y \in A(t_0, u_{t_0})x$, where $\theta(\tau, t_0) = r_0(\bar{r}, \bar{r}, \|x\|)(1 + \|y\| + C_{10})|\tau - t_0|$. Dividing by h and then letting $h \downarrow 0$, we know that

$$\langle u(t_0) - x, u'(t_0)\rangle_+ \leq w\|u(t_0) - x\| + \langle u(t_0) - x, G(t_0, u_{t_0}, L_{t_0}(u)) + y\rangle_+.$$

Thus

$$\langle u(t_0) - x, (u'(t_0) - G(t_0, u_{t_0}, L_{t_0}(u)) - wu(t_0)) - (y - wx)\rangle_- \leq 0.$$

By the maximality of $A(t_0, u_{t_0})$, we have

$$u'(t_0) \in A(t_0, u_{t_0})u(t_0) + G(t_0, u_{t_0}, L_{t_0}(u)).$$

This means that our limit solution is actually a strong solution. □

Remark 1. Under the condition (A2) with $|t - \tau| = |h(t) - h(\tau)|$, where $h : [0, T] \to X$ is continuous, we have the unique uniform continuous limit solution for (FDE,ϕ) eventhough the estimate is complexive.

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