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SOME MODIFICATIONS OF DENSITY TOPOLOGIES

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Abstract. In this paper we introduce two topologies on the plane connected with the notions of density and I-density. Their definitions are based on the notion of a regular density point. We investigate connections between them and the density and I-density topologies on the plane and on the real line. We consider axioms of separation and functions continuous with respect to these topologies.

1. Introduction

The aim of this paper is to introduce some topologies connected with the notions of density and I-density. The motivation to consider such topologies comes from the notion of a regular approximate differential ([2, 408–409]), the definition of which is based on the notion of a regular density point.

For terminology and definitions concerning density points and the density topologies on the real line and on the plane, see [4]. Information on I-density points and the I-density topologies on the real line and on the plane are contained in [7], [8], [9], [1], [3].

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2. The measure density case

Let $\mathbb{R}$ denote the real line, $\mathbb{R}^2$ - the plane, $\mathbb{R}_+$ - the set of positive real numbers, $\mathbb{Q}$ - the set of rational numbers, $\mathbb{N}$ - the set of positive integers.

Denote by m_k the Lebesgue measure and by $\mathcal{L}_k$ the family of all Lebesgue measurable sets on $\mathbb{R}^k$, $k = 1, 2$. Let $d_k(A, p)(\underline{d_k}(A, p))$ denote the ordinary density (lower ordinary density) of a measurable set $A \subset \mathbb{R}^k$ at a point $p \in \mathbb{R}^k$, $k = 1, 2$. If a plane set is contained in a line, then we shall consider its linear measure and we use the linear density d_1.

Let A' denote the complement of A. If $A \subset \mathbb{R}$ then $A' = \mathbb{R} \setminus A$. If $A \subset \mathbb{R}$ and $x_0 \in \mathbb{R}$ then $A + x_0 = \{a + x_0 : \ a \in A\}$ and $-A = \{-a : \ a \in A\}$.

Let $(x_0, y_0) \in \mathbb{R}^2,\ t \in \mathbb{R}_+$. Then we denote

$$R((x_0, y_0), t) = ([x_0-t, x_0+t] \times [y_0-t, y_0+t]) \setminus ((x_0-t, x_0+t) \times (y_0-t, y_0+t))$$

and

$$\mathcal{R}(x_0, y_0) = \{B \subset \mathbb{R}^2 : \ B = \bigcup_{t \in T} R((x_0, y_0), t) \quad \text{for some} \quad T \subset \mathbb{R}_+\}.$$

If $x_0 = y_0 = 0$, we write $R(t)$ instead of $R((x_0, y_0), t)$, and $\mathcal{R}$ instead of $\mathcal{R}(x_0, y_0)$.

If $A \subset \mathbb{R}^2$ then let

$$A^+_{[R]}(x_0, y_0) = \{t \in \mathbb{R}_+ : \ R((x_0, y_0), t) \subset A\}$$

and

$$A_{[R]}(x_0, y_0) = A^+_{[R]}(x_0, y_0) \cup (-A^+_{[R]}(x_0, y_0)).$$

Obviously, $A_{[R]}(x_0, y_0) = -A_{[R]}(x_0, y_0)$. If $x_0 = y_0 = 0$, we write $A_{[R]}$ instead of $A_{[R]}(x_0, y_0)$. If $A \subset \mathbb{R}^2$ and $x \in \mathbb{R}$ then $A_x = \{y \in \mathbb{R} : (x, y) \in A\}$.

Let $\mathcal{O}$ denote the Euclidean topology on the plane. If $\mathcal{T}$ is an arbitrary topology on the plane then by $C_{\mathcal{T}}$ we shall denote the family of all functions $f : \mathbb{R}^2 \to \mathbb{R}$ continuous with respect to the topology $\mathcal{T}$ on the domain and the Euclidean topology on the range.

We shall say that two sets $A, B \subset \mathbb{R}^2$ are equivalent ($A \sim B$) if and only if $m_2(A \triangle B) = 0$ where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Let $\mathcal{D}_1$ ($\mathcal{D}_2$) denote the density topology on the real line (the ordinary density topology on the plane).

A function f is called Baire*1 ([5]) if for every perfect set P there exists a portion Q of P (that is a nonempty set of the form $U \cap P$ where U is a nondegenerate interval) such that $f|Q$ is continuous.

Definition 2.1. We say that a point $(x_0, y_0) \in \mathbb{R}^2$ is a regular density point of a set $A \subset \mathbb{R}^2$ if and only if there exists a measurable set $B \subset A$ such that $d_2(B, (x_0, y_0)) = 1$ and $B \in \mathcal{R}(x_0, y_0)$. In this case we write $d_r(A, (x_0, y_0)) = 1$.

For $A \in \mathcal{L}_2$ let

$$\Phi(A) = \{(x, y) \in \mathbb{R}^2 : \; d_2(A, (x, y)) = 1\}$$

and

$$\Phi_r(A) = \{(x, y) \in \mathbb{R}^2 : \; d_r(A, (x, y)) = 1\}.$$

Obviously, $\Phi_r(A) \subset \Phi(A)$ for $A \in \mathcal{L}_2$. Observe that the operator Φ_r has the following properties:

1^0 $\Phi_r(\emptyset) = \emptyset$, $\Phi_r(\mathbb{R}^2) = \mathbb{R}^2$;
2^0 if $A \subset B$, then $\Phi_r(A) \subset \Phi_r(B)$;
3^0 $\Phi_r(A \cap B) = \Phi_r(A) \cap \Phi_r(B)$

for $A, B \in \mathcal{L}_2$. It suffices to show that

$$\Phi_r(A) \cap \Phi_r(B) \subset \Phi_r(A \cap B)$$

(the remaining properties follow immediately from the definition of Φ_r). Let $(x_0, y_0) \in \Phi_r(A) \cap \Phi_r(B)$. Then there exist two sets $C, D \in \mathcal{L}_2 \cap \mathcal{R}(x_0, y_0)$ such that $C \subset A$, $D \subset B$, $d_2(C, (x_0, y_0)) = d_2(D, (x_0, y_0)) = 1$. Obviously, $C \cap D \subset A \cap B$, $C \cap D \in \mathcal{R}(x_0, y_0)$ and $d_2(C \cap D, (x_0, y_0)) = 1$ since $\Phi(C) \cap \Phi(D) = \Phi(C \cap D)$ (see [6, Theorem 22.4]). Consequently, $(x_0, y_0) \in \Phi_r(A \cap B)$.

Observe that Φ_r is not a so called "lower density operator" (see [6, Theorem 22.4]) since the Lebesgue Density Theorem does not hold for Φ_r. Indeed, let

$$A = [-1, 1] \times ((\mathbb{R} \setminus \mathbb{Q}) \cap [-1, 1]).$$

Then $m_2(A) = 4$ and $\Phi_r(A) = \emptyset$, so $m_2(A \triangle \Phi_r(A)) = 4$.

We require the "lower density" has the same value for equivalent sets, but our operator has not this property. There exist two measurable sets $C, D \subset \mathbb{R}^2$ (namely, $C = [-1, 1]^2$, $D = A$) such that $C \sim D$ (that is $m_2(C \triangle D) = 0$) and $\Phi_r(C) = (-1, 1)^2 \neq \Phi_r(D)$.

Put

$$\mathcal{T}_r = \{A \in \mathcal{L}_2 : \; A \subset \Phi_r(A)\}.$$

Theorem 2.2. *The family $\mathcal{T}_r$ is a topology on the plane, essentially stronger than the Euclidean topology and essentially weaker than the density topology $\mathcal{D}_2$ on the plane.*

Proof. Obviously, $\emptyset, \mathbb{R}^2 \in \mathcal{T}_r$. By property 3^0 the family $\mathcal{T}_r$ is closed under finite intersections. It suffices to show that $\mathcal{T}_r$ is closed also under arbitrary unions. Let $\{A_\alpha, \alpha \in \Lambda\}$ be a subfamily of $\mathcal{T}_r$. Then $A_\alpha \subset \Phi_r(A_\alpha)$ for $\alpha \in \Lambda$.

We first prove that $\bigcup_{\alpha\in\Lambda} A_\alpha \in \mathcal{L}_2$. We have $A_\alpha \subset \Phi_r(A_\alpha) \subset \Phi(A_\alpha)$ and $A_\alpha \in \mathcal{L}_2$, so A_α is open in the density topology for $\alpha \in \Lambda$. Hence $\bigcup_{\alpha\in\Lambda} A_\alpha$ is also open in the density topology and, consequently, it is a measurable set.

Now, we observe that $\bigcup_{\alpha\in\Lambda} A_\alpha \subset \bigcup_{\alpha\in\Lambda} \Phi_r(A_\alpha) \subset \Phi_r(\bigcup_{\alpha\in\Lambda} A_\alpha)$ by monotonicity of Φ_r and by the inclusion $A_\alpha \subset \Phi_r(A_\alpha)$ for $\alpha \in \Lambda$.

The inclusions $\mathcal{O} \subset \mathcal{T}_r \subset \mathcal{D}_2$ are obvious. Put

$$A = [(\mathbb{R}\setminus\mathbb{Q})\times\mathbb{R}] \cup [\mathbb{R}\times(\mathbb{R}\setminus\mathbb{Q})] \quad \text{and} \quad B = (\mathbb{R}\setminus\mathbb{Q})\times\mathbb{R}.$$

Then $A \in \mathcal{T}_r \setminus \mathcal{O}$, since $\mathbb{R}^2 \setminus A \subset \mathbb{Q}\times\mathbb{Q}$, and $B \in \mathcal{D}_2 \setminus \mathcal{T}_r$, since $\mathbb{R}^2 \setminus B = \mathbb{Q}\times\mathbb{R}$. □

Now we indicate connections between the regular density and the linear density.

Let p_{x_0} denote the line $x = x_0$ and p^{y_0} - the line $y = y_0$.

Lemma 2.3. *Let $B \in \mathcal{R}$. The set B is measurable ($B \in \mathcal{L}_2$) if and only if $B_{[R]} \in \mathcal{L}_1$.*

Proof. Let $B \in \mathcal{L}_2$ and $B \in \mathcal{R}$. Put

$$Z = \{t \in \mathbb{R}_+ : B_t \notin \mathcal{L}_1\}.$$

By the Fubini theorem $m_1(Z) = 0$. Let $\{t_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of numbers from $\mathbb{R}_+ \setminus Z$ tending to zero. Then $B_{t_n} \in \mathcal{L}_1$ for $n \in \mathbb{N}$, so $B_{t_n} \cap [t_n, \infty) \in \mathcal{L}_1$ and

$$B^+_{[R]} = \bigcup_{n\in\mathbb{N}} (B_{t_n} \cap [t_n, \infty)).$$

Consequently, $B^+_{[R]}$ and also $B_{[R]}$ are measurable sets on the real line.

Suppose now that $B_{[R]} \in \mathcal{L}_1$. Obviously, $B_{[R]} \times \mathbb{R},\ \mathbb{R} \times B_{[R]} \in \mathcal{L}_2$. Put

$$C_I = \{(x,y) \in \mathbb{R}^2 : \ |y| \le |x|\}$$

and

$$C_{II} = \{(x,y) \in \mathbb{R}^2 : \ |x| \le |y|\}.$$

Clearly, $C_I, C_{II} \in \mathcal{L}_2$ and

$$B = [(B_{[R]} \times \mathbb{R}) \cap C_I] \cup [(\mathbb{R} \times B_{[R]}) \cap C_{II}].$$

Consequently, $B \in \mathcal{L}_2$. □

Definition 2.4. We say that $x \in \mathbb{R}$ is an inner density point of $A \subset \mathbb{R}$ if and only if there exists a set $B \in \mathcal{L}_1$ such that $B \subset A$ and $d_1(B, x) = 1$.

Theorem 2.5. *If* $(0,0)$ *is a regular density point of a set* $A \subset \mathbb{R}^2$ *then* $(0,0)$ *is an inner (linear) density point of* $A \cap p_0$ *and* $A \cap p^0$*.*

Proof. We shall show that if $d_r(A,(0,0)) = 1$ then $(0,0)$ is an inner density point of $A \cap p_0$.

By assumption there exists a measurable set $B \subset A$, $B \in \mathcal{R}$, such that $d_2(B,(0,0)) = 1$. It is sufficient to prove that $d_1(B_{[R]},0) = 1$ since $B_{[R]} \times \{0\} \subset A \cap p_0$.

Suppose, contrary to our claim, that $\underline{d_1}(B_{[R]},0) < 1$. So there exist a positive number ε and a decreasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers tending to zero such that

$$\lim_{n\to\infty} \frac{m_1(B_{[R]} \cap [-t_n,t_n])}{2t_n} < 1-\varepsilon.$$

Changing the numeration, if necessary, we may assume that for every $n \in \mathbb{N}$

$$m_1(B_{[R]} \cap [-t_n,t_n]) < (1-\varepsilon)2t_n.$$

By the symmetry of $B_{[R]}$ ($B_{[R]} = -B_{[R]}$) we have

$$m_1(B_{[R]} \cap [0,t_n]) < (1-\varepsilon)t_n,$$

so

$$m_1((B_{[R]})' \cap [0,t_n]) \geq \varepsilon t_n.$$

Put

$$C_n = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x \quad \text{and} \quad x \in [0,t_n]\}.$$

Thus

$$m_2(B' \cap C_n) \geq \frac{\varepsilon^2 t_n^2}{2}.$$

Consequently,

$$m_2(B \cap C_n) \leq m_2(C_n) - \frac{\varepsilon^2 t_n^2}{2} = \frac{t_n^2(1-\varepsilon^2)}{2},$$

so

$$\frac{m_2(B \cap [-t_n,t_n]^2)}{4t_n^2} = \frac{8m_2(B \cap C_n)}{4t_n^2} \leq 1-\varepsilon^2 < 1.$$

Therefore

$$\liminf_{t\to 0^+} \frac{m_2(B \cap [-t,t]^2)}{4t^2} \leq \liminf_{n\to\infty} \frac{m_2(B \cap [-t_n,t_n]^2)}{4t_n^2} < 1,$$

a contradiction.

The analogous considerations can be carried out for $A \cap p^0$. □

Theorem 2.6. *Let* $B \in \mathcal{R}$*. If* $d_1(B_{[R]},0) = 1$ *then* $d_2(B,(0,0)) = 1$*.*

Proof. From our assumption we have

$$\lim_{h\to 0^+}\frac{m_1(B_{[R]}\cap[0,h])}{h}=1,$$

so for each $\varepsilon>0$ there exists a positive number δ such that

$$\frac{m_1(B_{[R]}\cap[0,h])}{h}>1-\varepsilon$$

for $h\in(0,\delta)$. Therefore

$$m_1((B_{[R]})'\cap[0,h])<\varepsilon h$$

for $h\in(0,\delta)$. Put

$$C_h=\{(x,y)\in\mathbb{R}^2:\ 0\le y\le x\quad\text{and}\quad x\in[0,h]\}.$$

Then

$$m_2(B'\cap C_h)<\frac{1}{2}(h+h-\varepsilon h)\varepsilon h=\frac{1}{2}(2-\varepsilon)\varepsilon h^2.$$

Hence

$$\frac{m_2(B'\cap[-h,h]^2)}{4h^2}<(2-\varepsilon)\varepsilon<2\varepsilon$$

for $h\in(0,\delta)$. Consequently,

$$\lim_{h\to 0^+}\frac{m_2(B'\cap[-h,h]^2)}{4h^2}=0,$$

so $d_2(B,(0,0))=1$. □

Corollary 2.7. *If $B\in\mathcal{R}$ then the following conditions are equivalent:*

a) $d_1(B_{[R]},0)=1$,
b) $d_2(B,(0,0))=1$,
c) $d_r(B,(0,0))=1$.

Of course, the same results can be obtained for any point (x_0,y_0) taken instead of $(0,0)$.

Let $\mathcal{D}\times\mathcal{D}$ denote the product density topology on the plane.

Theorem 2.8. $\mathcal{T}_r\subsetneq\mathcal{D}\times\mathcal{D}$.

Proof. Let $A\in\mathcal{T}_r$ and $(x_0,y_0)\in A$. There exists a measurable set $B\subset A$ such that $B\in\mathcal{R}(x_0,y_0)$ and $d_2(B,(x_0,y_0))=1$. From Theorem 2.5 it follows that

$$d_1(B_{[R]}(x_0,y_0)+x_0,x_0)=1$$

and

$$d_1(B_{[R]}(x_0,y_0)+y_0,y_0)=1.$$

Simultaneously, $(B_{[R]}(x_0,y_0)+x_0)\times(B_{[R]}(x_0,y_0)+y_0)\subset A$, so (x_0,y_0) is the interior point of A in the topology $\mathcal{D}\times\mathcal{D}$. From the arbitrariness of (x_0,y_0)

it follows that $A \in \mathcal{D} \times \mathcal{D}$. Put $B = (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$. Obviously, $B \in \mathcal{D} \times \mathcal{D}$. Simultaneously, $\Phi_r(B) = \emptyset$, so $B \notin \mathcal{T}_r$. □

Theorem 2.9. *Let $A \in \mathcal{L}_2$. If $m_1(\operatorname{proj}_y A) = 0$ and $m_1(\operatorname{proj}_x A) = 0$ then A is closed in the topology $\mathcal{T}_r$.*

Proof. Let $(x_0, y_0) \in A'$. Assume first that $(x_0, y_0) = (0, 0)$. Put

$$Z_1 = \operatorname{proj}_x A \ \cup \ \operatorname{proj}_y A$$

and

$$Z = Z_1 \ \cup \ (-Z_1).$$

Obviously, Z is symmetric with respect to zero and $m_1(Z) = 0$. Note that if $t \notin Z$ then

$$t \notin \operatorname{proj}_x A \cup \operatorname{proj}_y A \cup (-\operatorname{proj}_x A) \cup (-\operatorname{proj}_y A),$$

so

$$p_t \cap A = \emptyset, \quad p_{-t} \cap A = \emptyset, \quad p^t \cap A = \emptyset \ \text{ and } \ p^{-t} \cap A = \emptyset.$$

Therefore $R(t) \subset A'$ for $t \in Z' \cap \mathbb{R}_+$, hence $\bigcup_{t \in Z' \cap \mathbb{R}_+} R(t) \subset A'$. Put

$$B = \bigcup_{t \in Z' \cap \mathbb{R}_+} R(t).$$

Clearly, $B_{[R]} = Z'$, so $d_1(B_{[R]}, 0) = 1$. By Corollary 2.7 we have $d_2(B, (0, 0)) = 1$ and, consequently, $d_r(A', (0, 0)) = 1$.

Now let $(x_0, y_0) \neq (0, 0)$. Put

$$Z = \operatorname{proj}_x A \cup (2x_0 - \operatorname{proj}_x A) \cup (x_0 - y_0 + \operatorname{proj}_y A) \cup (y_0 + x_0 - \operatorname{proj}_y A).$$

Obviously, $m_1(Z) = 0$. If $t \notin Z$ then

$$t \notin \operatorname{proj}_x A, \ t \notin 2x_0 - \operatorname{proj}_x A, \ t \notin x_0 - y_0 + \operatorname{proj}_y A, \ t \notin y_0 + x_0 - \operatorname{proj}_y A,$$

hence

$$p_t \cap A = \emptyset, \ p_{2x_0 - t} \cap A = \emptyset, \ p^{y_0 - x_0 + t} \cap A = \emptyset, \ p^{y_0 + x_0 - t} \cap A = \emptyset.$$

Therefore $R((x_0, y_0), t) \subset A'$ for $t \in Z' \cap \mathbb{R}_+$. Put

$$B = \bigcup_{t \in Z' \cap \mathbb{R}_+} R((x_0, y_0), t).$$

Clearly, $B \subset A'$ and $B_{[R]}(x_0, y_0) = Z'$, so $d_1(B_{[R]}(x_0, y_0), 0) = 1$. By Corollary 2.7 (applied to the set B moved by the vector $[-x_0, -y_0]$) we have $d_2(B, (x_0, y_0)) = 1$ and, consequently, $d_r(A', (x_0, y_0)) = 1$. From the arbitrariness of $(x_0, y_0) \in A'$ it follows that $A' \in \mathcal{T}_r$. □

Corollary 2.10. *The space $(\mathbb{R}^2, \mathcal{T}_r)$ is not separable.*

Corollary 2.11. *The space* $(\mathbb{R}^2, \mathcal{T}_r)$ *is not compact.*

Proof. The $\mathcal{T}_r$ -compact sets are the finite sets. If E is infinite, then it contains a countable infinite subset E_1, which is not compact in the Euclidean topology. If E is $\mathcal{T}_r$-compact then E_1 would be also $\mathcal{T}_r$ - compact which gives a contradiction. □

It is natural to investigate the separation axioms for the topology $\mathcal{T}_r$. Obviously, $\mathcal{T}_r$ is stronger than the Euclidean topology on the plane, so it is a Hausdorff (T_2) topology.

Observe, that $\mathcal{T}_r$ is not normal. Indeed, let L and R denote the sets of left and right end points of component intervals of the complement of Cantor set $C \subset \mathbb{R}$, respectively. The sets L and R are countable, so they are closed in $\mathcal{T}_r$ and $L \cap R = \emptyset$. Suppose that $(\mathbb{R}^2, \mathcal{T}_r)$ is a normal (T_4) space. Then there exists a continuous function $f : (\mathbb{R}^2, \mathcal{T}_r) \to ([0,1], \mathcal{O})$ such that $f(L \times \{0\}) = \{1\}$ and $f(R \times \{0\}) = \{0\}$. Clearly, each point of C is an accumulation point of R and L, so f is discontinuous everywhere on $C \times \{0\}$ as a function from the plane with the Euclidean topology. Simultaneously, $\mathcal{T}_r \subset \mathcal{D}_2$ and, consequently, each function continuous with respect to $\mathcal{T}_r$ on the domain is in the first class of Baire — a contradiction.

Now we shall prove that the family of all functions continuous with respect to $\mathcal{T}_r$ on the domain is not contained in the class of Baire*1 functions introduced by O'Malley ([5]).

We say that any of the sets $\bigcup_{n\in\mathbb{N}}(a_n, b_n)$ or $\bigcup_{n\in\mathbb{N}}[a_n, b_n]$ is a right (left) interval set at a point $x \in \mathbb{R}$ if $a_{n+1} < b_{n+1} < a_n < b_n$ $(a_{n+1} > b_{n+1} > a_n > b_n)$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = x$. We say that a set $E \subset \mathbb{R}$ is an interval set at a point $x \in \mathbb{R}$ if it is a union of a right interval set and a left interval set at x.

For our purpose we shall use Lemma 2.1.8 from [3] in the following form:

Lemma 2.12. *There exists a perfect set* $C \subset \mathbb{R}$ *such that for every* $x \in C$ *there is an interval set* E *at* x *such that* $C \subset E \cup \{x\}$ *and* $d_1(E, x) = 0$.

Theorem 2.13. *There exists a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *continuous with respect to the topology* $\mathcal{T}_r$ *on the domain, which is not in Baire**1 *class.*

Proof. Let us consider the set C from Lemma 2.12. We may assume that $C \subset [0,1]$. Let

$$[0,1] \setminus C = \bigcup_{k=1}^{\infty} (a_k, b_k).$$

Clearly, $a_1 \in C$. By Lemma 2.12 there exists an interval set $E_1 \subset \Big[a_1 - (b_1 - a_1)/2, a_1 + (b_1 - a_1)/2\Big]$ symmetric with respect to a_1, which is a union of open intervals such that

$$C \cap \left[a_1 - \frac{b_1 - a_1}{2}, a_1 + \frac{b_1 - a_1}{2}\right] \subset E_1 \cup \{a_1\}$$

and

$$d_1(E_1, a_1) = 0. \tag{1}$$

Put

$$H_1 = \left[a_1 - \frac{b_1 - a_1}{2}, a_1 + \frac{b_1 - a_1}{2}\right] \setminus (E_1 \cup \{a_1\}).$$

Obviously, H_1 is an interval set at the point a_1, which is a union of closed intervals

$$H_1 = \bigcup_{n=1}^{\infty} [c_n^{(1)},\ d_n^{(1)}]$$

and

$$H_1 \subset \left[a_1 - \frac{b_1 - a_1}{2}, a_1 + \frac{b_1 - a_1}{2}\right] \setminus C \subset \bigcup_{k=1}^{\infty} (a_k, b_k).$$

Simultaneously, $d_1(H_1, a_1) = 1$ by (1).

Since $a_2 \notin \bigcup_{k=1}^{\infty} (a_k, b_k)$, so $a_2 \notin H_1 \cup \{a_1\}$. Put $\alpha_2 = \operatorname{dist}(a_2, H_1 \cup \{a_1\})$. The set $H_1 \cup \{a_1\}$ is closed and bounded, so $\alpha_2 > 0$.

Let

$$r_2 = \frac{1}{2} \min(b_2 - a_2, \alpha_2).$$

Obviously, $r_2 > 0$ and $(a_2 - r_2,\ a_2 + r_2) \cap H_1 = \emptyset$. By Lemma 2.12 there exists a left interval set E_2 at the point a_2, which is a union of open intervals and fulfils the conditions

$$E_2 \subset [a_2 - r_2, a_2], \qquad C \cap [a_2 - r_2, a_2] \subset E_2 \cup \{a_2\}$$

and

$$d_1(E_2, a_2) = 0. \tag{2}$$

Put

$$\widetilde{H}_2 = [a_2 - r_2, a_2] \setminus (E_2 \cup \{a_2\}).$$

Obviously, $\widetilde{H}_2$ is a left interval set at the point a_2, which is a union of closed intervals

$$\widetilde{H}_2 = \bigcup_{n=1}^{\infty} [c_n^{(2)},\ d_n^{(2)}] \subset [a_2 - r_2, a_2] \setminus C \subset \bigcup_{k=1}^{\infty} (a_k, b_k).$$

The set $2a_2 - \widetilde{H}_2$ is a right interval set at the point a_2. Put $H_2 = \widetilde{H}_2 \cup (2a_2 - \widetilde{H}_2)$. Then H_2 is an interval set at the point a_2, symmetric with respect to a_2, $H_2 \subset \bigcup_{k=1}^{\infty} (a_k, b_k)$ and $d_1(H_2, a_2) = 1$ by (2).

Suppose now that we have defined the interval sets $H_1, H_2, ..., H_m$ at the points $a_1, a_2, ..., a_m$ respectively, each of them is a countable union of closed intervals, $H_i \subset \bigcup_{k=1}^{\infty}(a_k, b_k)$, $d_1(H_i, a_i) = 1$ and H_i is symmetric with respect to a_i, $i = 1, ..., m$. Since $a_{m+1} \notin \bigcup_{k=1}^{\infty}(a_k, b_k)$, therefore $a_{m+1} \notin \bigcup_{i=1}^{m}(H_i \cup \{a_i\})$. Put $\alpha_{m+1} = \mathrm{dist}(a_{m+1}, \bigcup_{i=1}^{m} H_i)$ and

$$r_{m+1} = \frac{1}{2}\min(b_{m+1} - a_{m+1}, \alpha_{m+1}).$$

Obviously, $r_{m+1} > 0$ and $(a_{m+1} - r_{m+1}, a_{m+1} + r_{m+1}) \cap \bigcup_{i=1}^{m} H_i = \emptyset$. By Lemma 2.12 there exists a left interval set E_{m+1} at the point a_{m+1}, which is a union of open intervals and fulfils the conditions

$$E_{m+1} \subset [a_{m+1} - r_{m+1}, a_{m+1}], \qquad C \cap [a_{m+1} - r_{m+1}, a_{m+1}] \subset E_{m+1} \cup \{a_{m+1}\}$$

and

$$d_1(E_{m+1},\ a_{m+1}) = 0. \tag{3}$$

Put

$$\widetilde{H}_{m+1} = [a_{m+1} - r_{m+1}, a_{m+1}] \setminus (E_{m+1} \cup \{a_{m+1}\}).$$

Obviously, $\widetilde{H}_{m+1}$ is a left interval set at the point a_{m+1}, which is a union of closed intervals

$$\widetilde{H}_{m+1} = \bigcup_{n=1}^{\infty}[c_n^{(m+1)}, d_n^{(m+1)}] \subset [a_{m+1} - r_{m+1}, a_{m+1}] \setminus C \subset \bigcup_{k=1}^{\infty}(a_k, b_k).$$

The set $2a_{m+1} - \widetilde{H}_{m+1}$ is a right interval set at the point a_{m+1}. Put $H_{m+1} = \widetilde{H}_{m+1} \cup (2a_{m+1} - \widetilde{H}_{m+1})$. Then H_{m+1} is an interval set at the point a_{m+1}, symmetric with respect to $a_{m+1}, H_{m+1} \subset \bigcup_{k=1}^{\infty}(a_k, b_k)$ and $d_1(H_{m+1}, a_{m+1}) = 1$ by (3).

So we have defined the sequence $\{H_i\}_{i \in N}$ of interval sets at the points a_i, respectively, such that H_i is symmetric with respect to $a_i, H_i \subset \bigcup_{k=1}^{\infty}(a_k, b_k)$ and $d_1(H_i, a_i) = 1$ for $i \in \mathbb{N}$.

Define the function $\widetilde{g}_i$ on $H_i \cup C$ by

$$\widetilde{g}_i(t) = \begin{cases} \dfrac{1}{2^i} & \text{for} \quad t \in H_i \\ 0 & \text{for} \quad t \in C, \end{cases}$$

and extend $\widetilde{g}_i$ on $\mathbb{R} \setminus (H_i \cup C)$ in such a way that it is piecewice linear on $\mathbb{R}$ and bounded by $1/2^i$. Let

$$g_i(t) = \begin{cases} \dfrac{1}{2^i} & \text{for} \quad t = a_i, \\ \widetilde{g}_i(t) & \text{for} \quad t \in \mathbb{R} \setminus \{a_i\}, \end{cases}$$

for $i \in \mathbb{N}$. Obviously $0 \leq g_i(t) \leq 1/2^i$ for $t \in \mathbb{R}$, $i \in \mathbb{N}$. Put

$$\overline{g}_i(t) = g_i(t + a_i)$$

for $i \in \mathbb{N},\ t \in \mathbb{R}$, and let

$$\overline{f}_i(x,y) = \overline{g}_i(\max(|x|,|y|)), \qquad f_i(x,y) = \overline{f}_i(x - a_i,\ y)$$

for $(x,y) \in \mathbb{R}^2$, $i \in \mathbb{N}$. Finally, define

$$f(x,y) = \sum_{i=1}^{\infty} f_i(x,y)$$

for $(x,y) \in \mathbb{R}^2$.

First we observe that f is not in Baire*1 class. Let $D = C \times \{0\}$. Clearly, D is a perfect set. Let $U = (p,q) \times (r,s)$ be an arbitrary interval such that $D \cap U \neq \emptyset$. Then there exists $i \in \mathbb{N}$ such that $(a_i, 0) \in D \cap U$ (this is possible because C' is dense in C). Since C is perfect, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \underset{n\to\infty}{\longrightarrow} a_i,\ x_n \neq a_k$ for $k \in \mathbb{N}$ and $x_n \in C \cap (p,q)$ for $n \in \mathbb{N}$. Then $f(x_n, 0) = 0$ and $f(a_i, 0) \geq 1/2^i > 0$, therefore $f\big|D$ is not continuous.

It remains to prove that $f : (\mathbb{R}^2, \mathcal{T}_r) \to (\mathbb{R}, \mathcal{O})$ is continuous. First, we shall show that $\overline{f}_i : (\mathbb{R}^2, \mathcal{T}_r) \to (\mathbb{R}, \mathcal{O})$ is continuous for $i \in \mathbb{N}$. Fix $i \in \mathbb{N}$. The functions $\widetilde{g}_i$ and g_i are continuous on the real line except for a point a_i, hence $\overline{g}_i$ is continuous at each point of the real line except for 0. Consequently, $\overline{f}_i$ is continuous with respect to the Euclidean topology at each point of the plane except for $(0,0)$, as the superposition of continuous functions. We have $d_1(H_i, a_i) = 1, H_i$ is symmetric with respect to a_i and $g_i(t) = 1/2^i$ for $t \in H_i \cup \{a_i\}$, so $\overline{f}_i$ is constant on the set $E = \{(x,y) \in \mathbb{R}^2 : \max(|x|,|y|) \in \ \operatorname{int}\ H_i - a_i\}$. From Corollary 2.7 it follows that $d_r(E,(0,0)) = 1$, so $\overline{f}_i$ is continuous at the point $(0,0)$ with respect to the topology $\mathcal{T}_r$. Thus for each $i \in \mathbb{N}$ the function f_i is continuous with respect to the topology $\mathcal{T}_r$ on the domain. Consequently, f is continuous with respect to $\mathcal{T}_r$ as a sum of uniformly convergent series of functions continuous with respect to $\mathcal{T}_r$. □

3. The category density case

Denote by $\mathcal{B}_k$ the family of all sets having the Baire property and by I_k — the family of all sets of the first category on $\mathbb{R}^k,\ k = 1,2$.

If a point $p \in \mathbb{R}^k$ is the I_k-ordinary density point (I_k-ordinary dispersion point) of a set $A \in \mathcal{B}_k$, then we shall write $d_{I_k}(A,p) = 1$ ($d_{I_k}(A,p) = 0$), $k = 1,2$. If a plane set is contained in a real line, then we shall consider its linear I-density d_{I_1}.

We shall say that two sets $A, B \subset \mathbb{R}^2$ are equivalent ($A \sim B$) if and only if $A \triangle B \in I_2$. Let $\mathcal{D}_{I_1}$ ($\mathcal{D}_{I_2}$) denote the I-density topology on the real line (the ordinary I-density topology on the plane). See [1].

If $A \subset \mathbb{R}$ then $n \cdot A = \{n \cdot a : a \in A\}$. If $A \subset \mathbb{R}^2$ then $(n,n) \cdot A = \{(n \cdot x, n \cdot y) : (x,y) \in A\}$.

Definition 3.1. We say that a point $(x_0, y_0) \in \mathbb{R}^2$ is a regular I-density point of a set $A \subset \mathbb{R}^2$ if and only if there exists a set having the Baire property $B \subset A$ such that $d_{I_2}(B, (x_0, y_0)) = 1$ and $B \in \mathcal{R}(x_0, y_0)$. In this case we write $d_{rI}(A, (x_0, y_0)) = 1$.

For $A \in \mathcal{B}_2$ let

$$\Phi_I(A) = \{(x,y) \in \mathbb{R}^2 : \ d_{I_2}(A, (x,y)) = 1\}$$

and

$$\Phi_{rI}(A) = \{(x,y) \in \mathbb{R}^2 : \ d_{rI}(A, (x,y)) = 1\}.$$

Obviously, $\Phi_{rI}(A) \subset \Phi_I(A)$ for $A \in \mathcal{B}_2$ and the operator Φ_{rI} has properties analogous to those for Φ_r considered in measure case, i.e.

1^0 $\Phi_{rI}(\emptyset) = \emptyset, \Phi_{rI}(\mathbb{R}^2) = \mathbb{R}$;
2^0 if $A \subset B$, then $\Phi_{rI}(A) \subset \Phi_{rI}(B)$;
3^0 $\Phi_{rI}(A \cap B) = \Phi_{rI}(A) \cap \Phi_{rI}(B)$

for $A, B \in \mathcal{B}_2$. The set $A = [-1,1] \times ((\mathbb{R} \setminus \mathbb{Q}) \cap [-1,1])$ is of the second category on the plane and $\Phi_{rI}(A) = \emptyset$, so $A \triangle \Phi_{rI}(A) \notin \mathcal{I}_2$ and the theorem analogous to the Lebesgue Density Theorem for Φ_{rI} does not hold and neither does the equality of values of Φ_{rI} for equivalent sets.

Put

$$\mathcal{T}_{rI} = \{A \in \mathcal{B}_2 : \ A \subset \Phi_{rI}(A)\}.$$

Theorem 3.2. *The family $\mathcal{T}_{rI}$ is a topology on the plane, essentially stronger than the Euclidean topology and essentially weaker than the I-density topology $\mathcal{D}_{I_2}$ on the plane.*

The proof is analogous to the proof of Theorem 2.2.

Lemma 3.3. *Let $B \in \mathcal{R}$. The set B has the Baire property ($B \in \mathcal{B}_2$) if and only if $B_{[R]} \in \mathcal{B}_1$.*

The proof based upon the Kuratowski-Ulam Theorem (see [6, Theorems 15.1 and 15.2]) is analogous to the proof of Lemma 2.3.

Definition 3.4. We say that $x \in \mathbb{R}$ is an inner I-density point of $A \subset \mathbb{R}$ if and only if there exists a set $B \in \mathcal{B}_1$ such that $B \subset A$ and $d_{I_1}(B, x) = 1$.

Theorem 3.5. *If $(0,0)$ is a regular I-density point of a set $A \subset \mathbb{R}^2$ then $(0,0)$ is an inner (linear) I-density point of $A \cap p_0$ and $A \cap p^0$.*

Proof. We shall show that if $d_{rI}(A,(0,0)) = 1$ then $(0,0)$ is an inner I-density point of $A \cap p_0$.

By our assumption there exists a set $B \in \mathcal{B}_2 \cap \mathcal{R}$, $B \subset A$ such that $d_{I_2}(B,(0,0)) = 1$. By the definition of an I_2-density point, for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ of $\{n_m\}_{m\in\mathbb{N}}$ such that

$$\limsup_p \ (n_{m_p}, n_{m_p}) \cdot B' \cap [-1,1]^2 \in I_2. \tag{4}$$

Suppose that $d_{I_1}(B_{[R]}, 0) \neq 1$. Then there exists an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers such that for every subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ of $\{n_m\}_{m\in\mathbb{N}}$ we have

$$\limsup_p \ n_{m_p} \cdot (B_{[R]})' \cap [-1,1] = \bigcap_{k=1}^{\infty} \bigcup_{p=k}^{\infty} n_{m_p} \cdot (B_{[R]})' \cap [-1,1] \notin I_1.$$

Note that if $x \in \limsup_p \ n_{m_p} \cdot (B_{[R]})' \cap [-1,1]$ then

$$(\{x\} \times (-x,x)) \subset \bigcap_{k=1}^{\infty} \bigcup_{p=k}^{\infty} (n_{m_p}, n_{m_p}) \cdot B' \cap [-1,1]^2,$$

so the set

$$\Big[\bigcap_{k=1}^{\infty} \bigcup_{p=k}^{\infty} (n_{m_p}, n_{m_p}) \cdot B' \cap [-1,1]^2 \Big]_x$$

is of the second category on the line for all $x \in \limsup_p \ n_{m_p} \cdot (B_{[R]})' \cap [-1,1]$, which together with (4) gives a contradiction with the Kuratowski–Ulam theorem. Analogously for $A \cap p^0$. $\square$

Theorem 3.6. *Let $B \in \mathcal{R}$. If $d_{I_1}(B_{[R]}, 0) = 1$ then $d_{I_2}(B,(0,0)) = 1$.*

Proof. By our assumption, for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ of $\{n_m\}_{m\in\mathbb{N}}$ such that

$$\limsup_p \ n_{m_p} \cdot (B_{[R]})' \cap [-1,1] \in I_1.$$

Using the notation from the proof of Lemma 2.3 we observe that

$$\Big[(\limsup_p \ n_{m_p} \cdot (B_{[R]})' \cap [-1,1]) \times [-1,1] \Big] \cap C_I$$
$$= (\limsup_p \ (n_{m_p}, n_{m_p}) \cdot B') \cap [-1,1]^2 \cap C_I.$$

By Theorem 15.3 in [6] we have

$$(\limsup_p \ n_{m_p} \cdot (B_{[R]})' \cap [-1,1]) \times [-1,1] \in I_2$$

hence

$$\left[(\limsup_p \; n_{m_p} \cdot (B_{[R]})' \cap [-1,1]) \times [-1,1]\right] \cap C_I \in I_2$$

and

$$(\limsup_p \; (n_{m_p}, n_{m_p}) \cdot B') \cap [-1,1]^2 \cap C_I \in I_2.$$

As similar arguments apply to the set C_{II}, we have

$$\begin{aligned}
&(\limsup_p \; (n_{m_p}, n_{m_p}) \cdot B') \cap [-1,1]^2 \\
&= \left[(\limsup_p \; (n_{m_p}, n_{m_p}) \cdot B') \cap [-1,1]^2 \cap C_I\right] \\
&\cup \left[(\limsup_p \; (n_{m_p}, n_{m_p}) \cdot B') \cap [-1,1]^2 \cap C_{II}\right] \in I_2.
\end{aligned}$$

□

Corollary 3.7. *If $B \in \mathcal{R}$ then the following conditions are equivalent:*

a) $d_{I_1} \; (B_{[R]}, 0) = 1$,
b) $d_{I_2} \; (B, (0,0)) = 1$,
c) $d_{rI} \; (B, (0,0)) = 1$.

The same results can be obtained for any point (x_0, y_0).

Let $\mathcal{D}_{I_1} \times \mathcal{D}_{I_1}$ denote the product I-density topology on the plane.

Theorem 3.8. $\mathcal{T}_{rI} \subsetneqq \mathcal{D}_{I_1} \times \mathcal{D}_{I_1}$.

The proof is analogous to the proof of Theorem 2.8.

Theorem 3.9. *Let $A \in \mathcal{B}_2$. If $\operatorname{proj}_x A \in I_1$ and $\operatorname{proj}_y A \in I_1$ then A is closed in the topology $\mathcal{T}_{rI}$.*

The proof is analogous to the proof of Theorem 2.9 and is based on the fact that if

$$Z = \operatorname{proj}_x A \cup \operatorname{proj}_y A \cup (-\operatorname{proj}_x A) \cup (-\operatorname{proj}_y A) \in I_1$$

then $\mathbb{R} \setminus Z$ is a residual set, hence $d_{I_1}(\mathbb{R} \setminus Z, 0) = 1$.

Corollary 3.10. *The space $(\mathbb{R}^2, \mathcal{T}_{rI})$ is not separable.*

Corollary 3.11. *The space $(\mathbb{R}^2, \mathcal{T}_{rI})$ is not compact.*

Considering the separation axioms we see $\mathcal{T}_{rI}$ is a Hausdorff (T_2) topology but it is not regular (T_3). Indeed, let $F = (\mathbb{Q} \times \mathbb{Q}) \setminus \{(0,0)\}$. Since F is countable, it is closed in the topology $\mathcal{T}_{rI}$. We cannot separate this set from the point $(0,0)$ in $\mathcal{T}_{rI}$ because we cannot separate the set $\mathbb{Q} \setminus \{0\}$ from the point 0 in the I-density topology on the line (Theorem 3.5).

Since $\mathcal{T}_{rI}$ topology is weaker than I-density topology on the plane, so each function continuous with respect to $\mathcal{T}_{rI}$ on the domain is I-approximately continuous and, consequently, it is in the first class of Baire.

Lemma 2.12 is also true for I-density, and the construction of the function from Theorem 2.13 works in the category case. Thus we have the following theorem.

Theorem 3.12. *There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ continuous with respect to the topology $\mathcal{T}_{rI}$ in the domain, which is not in Baire^{*1} class.*

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