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EXISTENCE AND CONTROLLABILITY RESULTS FOR NONLINEAR DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

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Abstract. In this paper we investigate the existence and controllability of mild solutions to the first order semilinear evolution inclusions in Banach spaces with nonlocal conditions. We shall rely of a fixed point theorem for condensing maps due to Martelli.

1. Introduction

The purpose of this paper is to study the existence and controllability of solutions for nonlinear initial value problems for semilinear evolution inclusions together with nonlocal conditions.

In Section 3, we study the existence of a solution for a nonlinear initial value problem (IVP for short) for a semilinear evolution inclusion, together with a nonlocal condition, of the form:

$$y' - A(t,y)y \in F(t,y), \quad t \in J := [0,b], \tag{1}$$

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$$y(0) + f(y) = y_0, \tag{2}$$

where $F : J \times E \longrightarrow 2^E$ is a bounded, closed, convex valued multivalued map, $f : C(C(J, E), E)$, $y_0 \in E$, $A(t, y)$ is a continuous operator on E for each $(t, y) \in J \times E$, and E is a real Banach space with the norm $\|\cdot\|$.

Such problems together with classical initial conditions have been studied repeatedly in the literature. In [14] Marino, using a fixed point theorem due to Martelli [15], studied the same type of problem together with classical local initial conditions and together with nonlinear boundary conditions. Also, Anguraj and Balachandran [1] studied problem (1)–(2) together with classical initial conditions in $\mathbb{R}^n$, using the Bohnenblust-Karlin extension of Kakutani's theorem on fixed points for set valued mappings.

The work on evolution nonlocal initial value problems was initiated by Byszewski. In [7] and [6], applying a method of C_0-semigroups and the Banach fixed point theorem, he proved the existence and uniqueness of mild, strong and classical solutions of the first order evolution nonlocal initial value problem. For the importance of nonlocal conditions in different fields, the interested reader is refereed to [7] and the references cited therein.

Initial value problems together with nonlocal conditions have been studied by some authors. For example, we refer to the papers of Balachandran and Chandrasekaran [4], Balachandran and Ilamaran [3], Byszewski [7], [6] and Ntouyas and Tsamatos [16].

This paper is a generalization of the previous papers to IVP for nonlinear differential inclusions with nonlocal conditions. The existence of solutions for (1)–(2) will be achieved using a fixed point theorem for condensing maps proved by Martelli [15].

In Section 4, we shall establish sufficient conditions for the controllability of a semilinear evolution system in Banach spaces together with a nonlocal initial condition. More precisely, we consider nonlocal semilinear problem of the form:

$$y' - A(t, y)y \in F(t, y) + (Bu)(t), \quad t \in J = [0, b], \tag{3}$$

$$y(0) + f(y) = y_0, \tag{4}$$

where A, F and y_0 are as in problem (1)–(2), the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space, and B is a bounded linear operator from U to E.

Controllability results of nonlinear integrodifferential systems in Banach spaces, by using the Schauder fixed point theorem, were studied by Balachandran, Balasubramaniam and Dauer in [2]. Han and Park [11], applying a Banach fixed point theorem, proved boundary controllability of differential equations together with nonlocal conditions.

In this paper, we study the controllability of systems (3)–(4), relied, as in the first part, on a fixed point theorem for condensing maps due to Martelli [15].

2. Preliminaries and basic assumptions

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout the paper.

$C(J, E)$ is the Banach space of continuous functions from J into E normed by

$$\|y\|_\infty = \max\{|y(t)| : t \in J\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E with norm

$$\|N\| := \sup\{|Ny| : |y| = 1\}.$$

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [17]).

$L^1(J, E)$ denotes the linear space of equivalence classes of measurable functions $y : J \longrightarrow E$ such that $\int_0^b |y(s)|\, ds < \infty$.

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G : X \longrightarrow 2^X$ has convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(V) = \bigcup_{x\in V} G(x)$ is bounded in X for any bounded set V of X (i.e. $\sup_{x\in V}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if, for each $x_* \in X$, the set $G(x_*)$ is a nonempty, closed subset of X, and if, for each open set V of X containing $G(x_*)$, there exists an open neighbourhood A of x_* such that $G(A) \subseteq V$.

G is said to be completely semicontinuous if $G(V)$ is relatively compact for every bounded subset $V \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \longrightarrow BCC(E)$ is said to be measurable if, for each $x \in E$, the function $t \longmapsto Y(t) = d(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\}$ is measurable. Other equivalent definitions of measurability for multivalued maps can be found in [12]. For the proofs of the above results and for more details on multivalued maps we refer the interested reader to the books of Deimling [9], and Hu and Papageorgiou [12].

An upper semi-continuous map $G : X \longrightarrow 2^X$ is said to be condensing if for any bounded $V \subseteq X$, with $\alpha(V) \neq 0$, we have $\alpha(G(V)) < \alpha(V)$, where α denotes the Kuratowski measure of noncompacteness. For properties of the Kuratowski measure, we refer to Darbo [8] and Banas and Goebel [5].

We remark that a completely continuous multivalued map is the easiest example of a condensing map.

Let us list the basic hypotheses:

(H1) $A : J \times E \to B(E)$ is a continuous function such that

$$\forall r > 0 \ \exists r_1 = r_1(r) > 0 \quad \text{such that}$$
$$|v| \leq r_1 \Rightarrow \|A(t, v)\| \leq r, \ \forall t \in J, \ \forall v \in E.$$

Remark 2.1. From (H1) we are able to claim the existence, for any fixed $u \in C(J, E)$, of a unique function $U_u : J \times J \to B(E)$, defined and continuous on $J \times J$, such that

$$U_u(t, s) = I + \int_s^t A_u(w)U_u(w, s)dw \tag{5}$$

(evolution operator of A), where I stands for the identity operator on E and $A_u(t) := A(t, u(t))$.

From (5), one has

$$U_u(t, t) = I, \quad U_u(t, s)U_u(s, r) = U_u(t, r), \quad (t, s, r) \in J \times J \times J.$$

Moreover,

$$(\partial U_u(t, s)/\partial t) = A_u(t)U_u(t, s) \ \text{ for almost all } \ t \in J, \forall s \in J.$$

(H2) $F : J \times E \longrightarrow BCC(E); (t, y) \longmapsto F(t, y)$ is measurable with respect to t for each $y \in E$, u.s.c. with respect to y for each $t \in J$, and for each fixed $y \in C(J, E)$ the set

$$S_{F,y} = \Big\{g \in L^1(J, E) : \ g(t) \in F(t, y(t)) \text{ for a.e. } \ t \in J\Big\}$$

is nonempty.

(H3) There exists a constant $L > 0$ such that $|f(y)| \leq L$ for each $y \in C(J, E)$;

(H4) For each bounded $Q \subset C(J, E)$, and for each $y \in Q$ and $t \in J$ the set

$$\Big\{U_y(t, 0)y_0 - U_y(t, 0)f(y) + \int_0^t U_y(t, s)g(s)ds : g \in S_{F,y}\Big\}$$

is relatively compact in E.

Remark 2.2.

1) If $\dim E < \infty$ then, for each $y \in C(J,E)$, $S_{F,y} \neq \emptyset$ (see Lasota and Opial [13]).
2) If $\dim E = \infty$ then $S_{F,y}$ is nonempty if and only if the function $Y : J \longrightarrow \mathbb{R}$, defined by
$$Y(t) := \inf\{|v| : v \in F(t,y)\},$$
belongs to $L^1(J,\mathbb{R})$ (see Hu and Papageorgiou [12]).
3) If we assume that $U_y(t,s), (t,s) \in J \times J$ is completely continuous then (H4) is satisfied.
4) From (H1), it follows that $u \in C(J,E)$ implies $A_u \in C(J,B(E))$ and
$$\|u_n - u^*\|_\infty \to 0 \text{ implies that}$$
$$\|A_{u_n} - A_{u^*}\|_\infty := \max\{\|A_{u_n}(t) - A_{u^*}(t)\| : t \in J\} \to 0,$$
as $n \to \infty$.

A fuction $y \in C(J,E)$ is called a mild solution of (1)–(2) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y(t))$ a.e. on J, and
$$y(t) = U_y(t,0)y_0 - U_y(t,0)f(y) + \int_0^t U_y(t,s)v(s)ds.$$

The following lemmas are crucial in the proof of our main theorem:

Lemma 2.1 ([13])**.** *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying* (H2) *and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$. Then the operator*
$$\Gamma \circ S_F : C(I,X) \longrightarrow BCC(C(I,X)),\ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$
is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2.2 ([10])**.** *Suppose that $\varphi_1,\ \varphi_2 \in C(J,\mathbb{R}),\ \varphi_3 \in L^1(J,\mathbb{R}),\ \varphi_3(t) \geq 0$ a.e. on J and $\varphi_1(t) \leq \varphi_2(t) + \int_0^t \varphi_3(s)\varphi_1(s)ds$. Then*
$$\varphi_1(t) \leq \varphi_2(t) + \int_0^t \varphi_3(s)\varphi_2(s)exp\Big(\int_s^t \varphi_3(\tau)d\tau\Big)ds.$$

Lemma 2.3 ([15])**.** *Let X be a Banach space and let $N : X \longrightarrow BCC(X)$ be an u.s.c. and condensing map. If the set*
$$\Omega := \{y \in X : \lambda y \in N(y) \ \ for\ some \ \ \lambda > 1\}$$
is bounded then N has a fixed point.

3. Existence results

Now, we are able to state and prove our main theorem, concerning the existence results for the nonlocal IVP (1)–(2).

Theorem 3.1. *Let $f : C(J, E) \longrightarrow E$ be a continuous function. Assume that hypotheses* (H1)–(H4) *are satisfied. Moreover, assume that:*

(H5) $\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq p(t)\psi(|y|)$ *for almost all* $t \in J$ *and for all* $y \in E$, *where* $p \in L^1(J, \mathbb{R}_+)$ *and* $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ *is continuous, increasing and such that*

$$M \int_0^b p(s)ds < \int_c^\infty \frac{du}{\psi(u)};$$

where $c := M|y_0| + ML$ *and* $M := \sup\{\|U_y(t, s)\|; (t, s) \in J \times J\}$.

Then problem (1)–(2) *has at least one mild solution on* J.

Proof. We transform problem (1)–(2) into a fixed point problem. Consider the multivalued map $N : C(J, E) \longrightarrow 2^{C(J,E)}$, defined by

$$N(y) := \Big\{ h \in C(J, E) : h(t) = U_y(t, 0)y_0 - U_y(t, 0)f(y) + \int_0^t U_y(t, s)g(s)ds \ : g \in S_{F,y} \Big\},$$

where

$$S_{F,y} = \Big\{ g \in L^1(J, E) : g(t) \in F(t, y(t)) \ \text{ for a.e. } \ t \in J \Big\}.$$

Remark 3.1. It is clear that the fixed points of N are mild solutions to (1)–(2).

We shall show that N is completely continuous with bounded, closed, convex values and it is upper semicontinuous. The proof will be given in several steps.

Step 1. *$N(y)$ is convex for each $y \in C(J, E)$.*

Indeed, if h_1, h_2 belong to $N(y)$ then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$h_i(t) = U_y(t, 0)y_0 - U_y(t, 0)f(y) + \int_0^t U_y(t, s)g_i(s)ds, \ \ (i = 1, 2).$$

Let $0 \leq k \leq 1$. Then, for each $t \in J$, we have

$$(kh_1 + (1-k)h_2)(t) = U_y(t,0)y_0 - U_y(t,0)f(y) + \int_0^t U_y(t,s)[kg_1(s) + (1-k)g_2(s)]ds.$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$kh_1 + (1-k)h_2 \in N(y).$$

Step 2. *N maps bounded sets into bounded sets.*

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $h \in N(y), y \in B_r = \{y \in C(J,E) : \|y\|_\infty \leq r\}$, one has $\|h\|_\infty \leq \ell$.

If $h \in N(y)$ then there exists $g \in S_{F,y}$ such that, for each $t \in J$, we have

$$h(t) = U_y(t,0)y_0 - U_y(t,0)f(y) + \int_0^t U_y(t,s)g(s)ds.$$

By (H3) and (H5), we have, for each $t \in J$, that

$$\begin{aligned}|h(t)| \leq &\|U_y(t,0)\||y_0| + \|U_y(t,0)\||f(y)| + \int_0^t \|U_y(t,s)g(s)\|\, ds \\ \leq &M|y_0| + ML + M \sup_{y\in[0,r]} \psi(y) \left(\int_0^t p(s)ds\right).\end{aligned}$$

Then, for each $h \in N(B_r)$, we have

$$\|h\|_\infty \leq M|y_0| + ML + M \sup_{t\in J}\Big(\int_0^t p(s)ds\Big) \max_{y\in B_r} \sup_{y\in[0,r]} \psi(y) := \ell.$$

Step 3. *N sends bounded sets into equicontinuous sets of $C(J,E)$.*

Let $t_1, t_2 \in J, t_1 < t_2$ and B_r be a bounded set in $C(J,E)$.

For each $y \in B_r$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = U_y(t,0)y_0 - U_y(t,0)f(y) + \int_0^t U_y(t,s)g(s)ds.$$

Thus

$$\begin{aligned}|h(t_2) - h(t_1)| \leq &\|(U_y(t_2,0) - U_y(t_1,0))y_0\| + \|(U_y(t_2,0) - U_y(t_1,0))\|L \\ &+ \left\|\int_0^{t_1} [U_y(t_2,s) - U_y(t_1,s)]g(s)ds\right\| \\ &+ \left\|\int_{t_1}^{t_2} U_y(t_2,s)g(s)ds\right\|\end{aligned}$$

$$
\begin{aligned}
&\le \|(U_y(t_2,0)-U_y(t_1,0))y_0\| + \|(U_y(t_2,0)-U_y(t_1,0))\|L \\
&\quad + \left\| \int_0^{t_1} [U_y(t_2,s)-U_y(t_1,s)]g(s)ds \right\| + M\int_{t_1}^{t_2} \|g(s)\|ds \\
&\le \|(U_y(t_2,0)-U_y(t_1,0))y_0\| + \|(U_y(t_2,0)-U_y(t_1,0))\|L \\
&\quad + \left(\int_0^b p(s)ds\right) \sup_{y\in[0,r]} \psi(y)\|U_y(t_2,s)-U_y(t_1,s)\| \\
&\quad + M \sup_{y\in[0,r]} \psi(y) \left(\int_{t_1}^{t_2} p(s)ds\right).
\end{aligned}
$$

As $t_2 \longrightarrow t_1$ then the right-hand side of the above inequality tends to zero.

Step 4. *$U_u(t,s)$ is continuous with respect to u, i.e.*

$$\|u_n - u^*\|_\infty \longrightarrow 0 \Rightarrow \|U_{u_n} - U_{u^*}\|_\infty \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Indeed, let $\|u_n - u^*\|_\infty \longrightarrow 0$. Then there exits $r > 0$ such that $\|u_n\|_\infty$, $\|u_*\|_\infty \le r$. Moreover, if $s \le t$ (analogously if $t < s$) we have

$$
\begin{aligned}
\|U_{u_n} - U_{u^*}\|_\infty \le& \int_s^t \|U_{u_n}(w,s)\| \cdot \|A_{u_n}(w) - A_{u^*}(w)\|dw \\
&+ \int_s^t \|A_{u^*}\| \cdot \|U_{u_n}(w,s) - U_{u^*}(w,s)\|dw \\
\le& M\int_s^t \|A_{u_n}(w) - A_{u^*}(w)\|dw \\
&+ \int_s^t \|A_{u^*}\| \cdot \|U_{u_n}(w,s) - U_{u^*}(w,s)\|dw.
\end{aligned}
$$

Applying Lemma 2.2, we obtain

$$
\begin{aligned}
\|U_{u_n} - U_{u^*}\|_\infty \le& M\int_s^t \|A_{u_n}(w) - A_{u^*}(w)\|dw \\
&+ M\int_s^t \|A_{u^*}(w)\| \left[\int_s^t \|A_{u_n}(\tau) - A_*(\tau)\|d\tau\right] \\
&\times \exp\Big(\int_w^t \|A_{u^*}(z)\|dz\Big)dw \\
\le& bM\|A_{u_n} - A_{u^*}\|_\infty \\
&+ b^2M\|A_{u^*}\|_\infty \|A_{u_n} - A_{u^*}\|_\infty \exp(b\|A_{u^*}\|_\infty) \\
\le& \|A_{u_n} - A_{u^*}\|_\infty Mb(1 + br_1\exp(br_1)).
\end{aligned}
$$

The conclusion follows from Remark 2.2.

As a consequence of Steps 2–4 and hypothesis (H4), together with the Ascoli-Arzelá theorem, we can conclude that N is completely continuous and, therefore, a condensing map.

Step 5. *N has a closed graph.*

Let $y_n \longrightarrow y^*$, $h_n \in N(y_n)$ and $h_n \longrightarrow h^*$. We shall prove that $h^* \in N(y^*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = U_{y_n}(t,0)y_0 - U_{y_n}(t,0)f(y_n) + \int_0^t U_{y_n}(t,s)g_n(s)ds.$$

We have to prove that there exists $g^* \in S_{F,y^*}$ such that

$$h^*(t) = U_{y^*}(t,0)y_0 - U_{y^*}(t,0)f(y^*) + \int_0^t U_{y^*}(t,s)g^*(s)ds.$$

Consider the linear continuous operator

$$\Gamma : L^1(J,E) \longrightarrow C(J,E)$$

$$g \longmapsto \Gamma(g)(t) = \int_0^t U_y(t,s)g(s)ds.$$

Clearly, we have that

$$\|(h_n - U_{y_n}(t,0)y_0 + U_{y_n}(t,0)f(y_n)) - (h^* - U_{y^*}(t,0)y_0 + U_{y^*}(t,0)f(y^*))\|_\infty \to 0,$$

as $n \to \infty$.

From Lemma 2.1 , it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - U_{y_n}(t,0)y_0 + U_{y_n}(t,0)f(y_n) \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y^*$, it follows, from Lemma 2.1, that

$$h^*(t) - U_{y^*}(t,0)y_0 + U_{y^*}(t,0)f(y^*) = \int_0^t U_{y^*}(t,s)g^*(s)ds$$

for some $g^* \in S_{F,y^*}$.

Step 6. *The set*

$$\Omega := \{y \in C(J,E) : \lambda y \in N(y), \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus, there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}U_y(t,0)y_0 - \lambda^{-1}U_y(t,0)f(y) + \lambda^{-1}\int_0^t U_y(t,s)g(s)ds, \quad t \in J.$$

Consequently, by (H3) and (H5), we have, for each $t \in J$, that

$$|y(t)| \leq M|y_0| + ML + M\int_0^t p(s)\psi(|y(s)|)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we obtain

$$v(0) = M|y_0| + ML \quad \text{and} \quad |y(t)| \leq v(t), \quad t \in J,$$

and

$$v'(t) = Mp(t)\psi(|y(t)|), \ \ t \in J.$$

Applying the nondecreasing character of ψ we get

$$v'(t) \leq Mp(t)\psi(v(t)), \quad t \in J.$$

The above inequality implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq M\int_0^b p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

Therefore, there exists a constant d such that $v(t) \leq d, \ t \in J$, and hence $\|y\|_\infty \leq d$, where d depends only on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J,E)$. As a consequence of Lemma 2.3, we deduce that N has a fixed point, which is a mild solution of (1)–(2). □

4. Controllability results

Definition 4.1. A fuction $y \in C(J,E)$ is called a mild solution of (3)–(4) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y(t))$ a.e. on J, and

$$y(t) = U_y(t,0)y_0 - U_y(t,0)f(y) + \int_0^t U_y(t,s)[(Bu)(s) + v(s)]\,ds. \tag{6}$$

Definition 4.2. The nonlocal problem (3)–(4) is said to be nonlocally controllable on the interval J, if for every $y_0, y_1 \in E$, there exists a control $u \in L^2(J,U)$, such that the mild solution $t \to y(t)$ of (3)(4) satisfies $y(b) + f(y) = y_1$.

Theorem 4.1. *Let $f : C(J,E) \longrightarrow E$ be a continuous function. Assume that hypotheses* (H1)–(H4) *are satisfied. Moreover, we assume that:*

(H6) *The linear operator* $W : L^2(J,U) \to E$, *defined by*

$$Wu = \int_0^b U_y(b,0)Bu(s)\,ds,$$

has an invertible operator W^{-1}, *which takes values in* $L^2(J,U)/kerW$, *and there exist positive constants* M_1 *and* M_2 *such that* $\|B\| \leq M_1$ *and* $\|W^{-1}\| \leq M_2$.

(H7) $\|F(t,y)\| \leq p(t)\psi(|y|)$ *for almost all* $t \in J$ *and all* $y \in E$, *where* $p \in L^1(J,\mathbb{R}_+)$ *and* $\psi : \mathbb{R}_+ \longrightarrow (0,\infty)$ *is continuous and increasing with*

$$M\int_0^b p(s)ds < \int_{\overline{c}}^{\infty} \frac{du}{\psi(u)};$$

where $\overline{c} = M(|y_0| + L + M_0 b)$, $M = \sup\{\|U_y(t,s)\|; (t,s) \in J \times J\}$ *and*

$$M_0 = M_1 M_2 \Big[|y_1| + L + M|y_0| + ML + M\int_0^b p(s)\psi(|y(s)|)\,ds\Big].$$

Then problem (3)–(4) *is nonlocally controllable on* J.

Proof. Applying hypothesis (H6), for an arbitrary function $y(\cdot)$, define the control

$$u_y(t) = W^{-1}\left[y_1 - f(y) - U_y(b,0)y_0 + U_y(b,0)f(y) - \int_0^b U_y(b,s)g(s)\,ds\right](t),$$

where

$$g \in S_{F,y} = \Big\{g \in L^1(J,E) :\ g(t) \in F(t,y(t)) \text{ for a.e. } \ t \in J\Big\}.$$

Now, we shall show that, when using this control, the operator $N : C(J,E) \longrightarrow 2^{C(J,E)}$, defined by

$$N(y)(t) := \Big\{h \in C(J,E) : h(t) = U_y(t,0)y_0 - U_y(t,0)f(y) + \int_0^t U_y(t,s)[(Bu_y)(s) + g(s)]\,ds, \quad t \in J\Big\},$$

has a fixed point. This fixed point is then a solution of problem (3)-(4).

Clearly $y_1 - f(y) \in N(y)(b)$.

We shall show that N is completely continuous with bounded, closed, convex values and it is upper semicontinuous. The proof will be given in several steps, in a way parallel to that of Theorem 3.1.

Step 1. *$N(y)$ is convex for each $y \in C(J,E)$.*

This is trivial, since $S_{F,y}$ is convex (because F has convex values) and, therefore, it is omitted.

Step 2. *N is bounded on bounded sets of $C(J,E)$.*

Let $B_r := \{y \in C(J,E) : \|y\|_\infty \le r\}$. Then, if $h \in N(y)$ then there exists $g \in S_{F,y}$ such that

$$h(t) = U_y(t,0)y_0 - U_y(t,0)f(y) + \int_0^t U_y(t,s)[Bu_y(s) + g(s)]\,ds.$$

By (H3), (H6) and (H7) we have, for each $t \in J$, that

$$\begin{aligned}|h(t)| \le & \|U_y(t,0)\||y_0| + \|U_y(t,0)\||f(y)| \\ & + \int_0^t \|U_y(t,s)[Bu_y(s) + g(s)]\|\,ds \\ \le & M|y_0| + ML + MM_0b + M \sup_{y\in[0,r]} \psi(y) \left(\int_0^t p(s)ds\right)\end{aligned}$$

or

$$\begin{aligned}\|N(y)\|_\infty \le & M|y_0| + ML + MM_0b \\ & + M \sup_{t\in J}\Big(\int_0^t p(s)ds\Big) \max_{y\in B_r} \sup_{y\in[0,r]} \psi(y).\end{aligned}$$

Step 3. *N sends bounded sets into equicontinuous sets of $C(J,E)$.*

Let $t_1, t_2 \in J, t_1 < t_2$ and B_r be a bounded set in $C(J,E)$. Then

$$\begin{aligned}|h(t_2) - h(t_1)| \le & \|(U_y(t_2,0) - U_y(t_1,0))y_0\| + \|(U_y(t_2,0) - U_y(t_1,0))\|L \\ & + \Big\|\int_0^{t_1} [U_y(t_2,\eta) - U_y(t_1,\eta)]BW^{-1}\Big[y_1 - f(y) \\ & - U_y(b,0)y_0 + U_y(b,0)f(y) - \int_0^b U_y(b,s)g(s)ds\Big](\eta)d\eta\Big\| \\ & + \Big\|\int_0^{t_1} [U_y(t_2,s) - U_y(t_1,s)]g(s)ds\Big\| \\ & + \Big\|\int_{t_1}^{t_2} U_y(t_2,\eta)BW^{-1}\Big[y_1 - f(y) - U_y(b,0)y_0 \\ & + U_y(b,0)f(y) - \int_0^b U_y(b,s)g(s)ds\Big](\eta)d\eta\Big\| \\ & + \Big\|\int_{t_1}^{t_2} U_y(t_2,s)g(s)ds\Big\|\end{aligned}$$

$$\begin{aligned}
\leq & \|(U_y(t_2,0)-U_y(t_1,0))y_0\| + \|(U_y(t_2,0)-U_y(t_1,0))\|L \\
& + \int_0^{t_1} \|U_y(t_2,\eta)-U_y(t_1,\eta)\| M_1 M_2 \Big[|y_1| + L + M|y_0| + ML \\
& + M\int_0^b p(s)\psi(|y(s)|)\,ds\Big] d\eta \\
& + \int_0^{t_1} \|U_y(t_2,\eta)-U_y(t_1,\eta)\| p(s)\psi(|y(s)|)\,ds \\
& + \int_{t_1}^{t_2} \|U_y(t_2,\eta)\| M_1 M_2 \Big[|y_1| + L + M|y_0| + ML \\
& + M\int_0^b p(s)\psi(|y(s)|)\,ds\Big] d\eta \\
& + \int_{t_1}^{t_2} \|U_y(t_2,s)\| p(s)\psi(|y(s)|)\,ds.
\end{aligned}$$

Therefore, $N(B_r)$ is relatively compact.

Step 4. *$U_u(t,s)$ is continuous with respect to u, i.e.*

$$\|u_n - u^*\|_\infty \longrightarrow 0 \Rightarrow \|U_{u_n} - U_{u^*}\|_\infty \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

This was proved is Step 4 of Section 3.

Step 5. *N has a closed graph.*

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = U_{y_n}(t,0)y_0 - U_{y_n}(t,0)f(y_n) + \int_0^t U_{y_n}(t,s)[(Bu_{y_n})(s) + g_n(s)]ds,$$

where

$$\begin{aligned}
u_{y_n}(t) = W^{-1}\Big[& y_1 - f(y) - U_{y_n}(b,0)y_0 + U_{y_n}(b,0)f(y_n) \\
& - \int_0^b U_{y_n}(b,s)g_n(s)\,ds\Big](t).
\end{aligned}$$

We have to prove that there exists $g_* \in S_{F,y_*}$ such that

$$h_*(t) = U_{y_*}(t,0)y_0 - U_{y_*}(t,0)f(y_*) + \int_0^t U_{y_*}(t,s)[(Bu_{y_*})(s) + g_*(s)]ds.$$

where

$$u_{y_*}(t) = W^{-1}\Big[y_1 - f(y_*) - U_{y_*}(b,0)y_0 + U_{y_*}(b,0)f(y_*) - \int_0^b U_{y_*}(b,s)g_*(s)\,ds\Big](t).$$

Set

$$\overline{u}_y(t) = W^{-1}\Big[y_1 - f(y) - U_y(b,0)y_0 + U_y(b,0)f(y)\Big].$$

Since $f,\ W^{-1}$ are continuous and $U_{y_n}(t,s) \to U_{y_*}(t,s),\ (t,s) \in J\times J$, then $\overline{u}_{y_n}(t) \longrightarrow \overline{u}_{y_*}(t)$ for $t \in J$.
Clearly we have that

$$\|\Big(h_n - U_{y_n}(t,0)y_0 + U_{y_n}(t,0)f(y_n)) - \int_0^t U_{y_n}(b,s)(B\overline{u}_{y_n})(s)ds\Big) - \Big(h_* - U_{y_*}(t,0)y_0 + U_{y_*}(t,0)f(y_*)) - \int_0^t U_{y_*}(b,s)(B\overline{u}_{y_*})(s)ds\Big)\|_\infty \to 0,$$

as $n \to \infty$.

Consider the operator

$$\Gamma : L^1(J,E) \longrightarrow C(J,E),$$

$$g \longmapsto \Gamma(g)(t) = \int_0^t U_y(t,s)\Big[BW^{-1}\Big(\int_0^b U_y(b,\tau)g(\tau)d\tau\Big)(s) + g(s)\Big]ds.$$

Clearly, L is linear and continuous. Indeed, one has

$$\|\Gamma g\|_\infty \le M(bMM_1M_2 + 1)\|g\|_{L^1}.$$

From Lemma 2.2, it follows that $\Gamma\circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - U_{y_n}(t,0)y_0 + U_{y_n}(t,0)f(y_n) - \int_0^t U_{y_n}(b,0)(B\overline{u}_{y_n})(s)ds \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows, from Lemma 2.2, that

$$h_*(t) - U_{y_*}(t,0)y_0 + U_{y_*}(t,0)f(y_*) - \int_0^t U_{y_*}(t,s)(B\overline{u}_{y_*})(s)ds = \int_0^t U_{y_*}(t,s)\Big[W^{-1}\Big(\int_0^b U_{y_*}(b,\tau)g_*(\tau)d\tau\Big)(s) + g_*(s)\Big]ds$$

for some $g_* \in S_{F,y_*}$.

Step 6. *The set*

$$\Omega := \{y \in C(J,E) : \lambda y = N(y),\ \text{for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus, there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) =& \lambda^{-1} U_y(t,0) y_0 - \lambda^{-1} U_y(t,0) f(y) \\ &+ \lambda^{-1} \int_0^t U_y(t,\eta) B W^{-1} \Big[y_1 - f(y) - U_y(b,0) y_0 + U_y(b,0) f(y) \\ &- \int_0^b U_y(b,s) f(s, y(s))\, ds \Big](\eta) d\eta \\ &+ \lambda^{-1} \int_0^t U_y(t,s) g(s)\, ds, \quad t \in J. \end{aligned}$$

Consequently, by (H3), (H6) and (H7), for each $t \in J$, we have that

$$|y(t)| \leq M|y_0| + ML + MM_0 b + M \int_0^t p(s)\psi(|y(s)|) ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we obtain

$$v(0) = M|y_0| + ML + MM_0 b, \quad |y(t)| \leq v(t), \quad t \in J,$$

and

$$v'(t) = Mp(t)\psi(|y(t)|), \ \ t \in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq Mp(t)\psi(v(t)), \ \ t \in J.$$

This implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq M \int_0^b p(s) ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

The above inequality implies that there exists a constant d' such that $v(t) \leq d',\ t \in J$, and hence $|y|_\infty \leq d'$, where d' depends only on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J, E)$. As a consequence of Lemma 2.2, we deduce that N has a fixed point, which is a mild solution of (3)–(4). Thus, system (3)–(4) is nonlocally controllable on J. □

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