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RESULTS ON SINGULAR DISTRIBUTION PRODUCTS OF MIKUSIŃSKI TYPE

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Abstract. Results on products of Schwartz distributions are obtained when they have coinciding point singularities and only sums of the products exist in the distribution space. These results follow the pattern of a well-know distributional product published by Jan Mikusiński in 1966, and are named Mikusiński type products. The formulas are derived as the distributions are embedded in Colombeau algebra of generalized functions. This algebra possesses optimal properties regarding the distributional multiplication, and its notion of "association" allows one to obtain the results in terms of distributions.

1. Introduction

The problem of multiplication of Schwartz distributions has been for a long time objective of many research studies. This is due to the large employment of distributions in the natural sciences and other mathematical fields, where products of distributions with coinciding singularities often appear. Starting with the historically first work of König [14], various constructions of differential algebras that include distributions have been

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proposed by Berg [2], Antonevich and Radyno [1], Egorov [9], and other authors.

In the last years, the associative differential algebra $\mathcal{G}$ of generalized functions of J.-F. Colombeau [4] became very popular in tackling distributional problems. The distributions are linearly embedded in $\mathcal{G}$ and the multiplication is compatible with the operations of differentiation and products with C^∞-differentiable functions. Moreover, the "association" in $\mathcal{G}$, being a faithful generalization of the equality of distributions in the space $\mathcal{D}'(\mathbb{R}^m)$, yields results in terms of distributions (and numerical factors). With applications in mind, this approach is followed here: we evaluate particular products of distributions with coinciding singularities — as embedded in Colombeau algebra — by their associated distributions.

Recall now the well-known result of Mikusiński published in [15]:

$$x^{-1} \,.\, x^{-1} \;-\; \pi^2\, \delta(x) \,.\, \delta(x) \;=\; x^{-2}, \quad x \in \mathbb{R}. \tag{1}$$

Though, neither of the products on the left-hand side here exists, their difference still has a correct meaning in $\mathcal{D}'(\mathbb{R}^m)$. Another formula of this type in dimension one — in a nonstandard approach to Distribution theory — was given in [17]:

$$H \,.\, \delta'(x) \;+\; \delta(x) \,.\, \delta(x) \;\overset{*}{=}\; \delta'(x)\,/2. \tag{2}$$

(H is the Heaviside function and "$\overset{*}{=}$" stands for the equality up to an infinitesimal quantity.)

Formulas of that type can be found in the mathematical and physical literature. We proposed the name "products of Mikusiński type" for such equations in previous papers [6], [8], where generalization of (2) and the basic Mikusiński formula (1) were derived in Colombeau algebra (see equations (9) and (11) below). In this paper, we continue the study in [6] and [7] obtaining further results on Mikusiński type products, or M-type products for short, in the algebra $\mathcal{G}(\mathbb{R})$. Some of the results are extended to the case of several variables as well. Singular products of piecewise differentiable functions with derivatives of the δ-function are also evaluated in $\mathcal{G}(\mathbb{R})$.

2. Fundamentals of Colombeau theory

We recall the basic definitions of Colombeau algebra of generalized functions.

Notation 1. If $\mathbb{N}_0$ stands for the nonnegative integers and $p = (p_1, p_2, \ldots, p_m)$ is a multiindex in $\mathbb{N}_0^m$, we let $|p| = \sum_{i=1}^m p_i$ and $p! = p_1!...p_m!$. Then, if $x = (x_1, \ldots, x_m)$ is in $\mathbb{R}^m$, denote $x^p = (x_1^{p_1}, x_2^{p_2}, \ldots, x_m^{p_m})$ and $\partial^p = \partial^{|p|}/\partial x_1^{p_1} \ldots \partial x_m^{p_m}$. Also, by $x < 0$ is meant: $x_1 \le 0, \ldots, x_m \le 0$ and $x \ne 0$.

Further, if q is in $\mathbb{N}_0$, we put $A_q(\mathbb{R}) = \{\varphi(x) \in \mathcal{D}(\mathbb{R})\colon \int_{\mathbb{R}} x^j \varphi(x)\,dx = \delta_{0j}$ for $0 \le j \le q$, where $\delta_{00} = 1$, $\delta_{0j} = 0$ for $j > 0\}$. This also extends to $\mathbb{R}^m$ as an m-fold product: $A_q(\mathbb{R}^m) = \{\varphi(x) \in \mathcal{D}(\mathbb{R}^m)\colon \varphi(x_1, \dots, x_m) = \prod_{i=1}^m \chi(x_i)$ for some χ in $A_q(\mathbb{R})\}$. Finally, we denote $\varphi_\varepsilon = \varepsilon^{-m}\varphi(\varepsilon^{-1}x)$ for φ in $A_q(\mathbb{R}^m)$ and $\varepsilon > 0$.

Definition 1. Let $\mathcal{E}[\mathbb{R}^m]$ be the algebra of functions $f(\varphi, x) : A_0(\mathbb{R}^m) \times \mathbb{R}^m \to \mathbb{C}$ that are infinitely differentiable, by a fixed "parameter" φ. The generalized functions of Colombeau are elements of the quotient algebra

$$\mathcal{G} \equiv \mathcal{G}(\mathbb{R}^m) = \mathcal{E}_{\mathrm{M}}[\mathbb{R}^m] \,/\, \mathcal{I}[\mathbb{R}^m].$$

Here $\mathcal{E}_{\mathrm{M}}[\mathbb{R}^m]$ is the subalgebra of "moderate" functions such that for each compact subset K of $\mathbb{R}^m$ and $p \in \mathbb{N}_0^m$ there is a $q \in \mathbb{N}$ such that, for each $\varphi \in A_q(\mathbb{R}^m)$,

$$\sup_{x \in K} |\partial^p f(\varphi_\varepsilon, x)| = O(\varepsilon^{-q}), \text{ as } \varepsilon \to 0_+.$$

In turn, the ideal $\mathcal{I}[\mathbb{R}^m]$ of $\mathcal{E}_{\mathrm{M}}[\mathbb{R}^m]$ is the set of functions such that for each compact subset K of $\mathbb{R}^m$ and any $p \in \mathbb{N}_0^m$ there is a $q \in \mathbb{N}$ such that, for every $r \ge q$ and $\varphi \in A_r(\mathbb{R}^m)$,

$$\sup_{x \in K} |\partial^p f(\varphi_\varepsilon, x)| = O(\varepsilon^{r-q}), \text{ as } \varepsilon \to 0_+.$$

(The Landau symbol $O(\varepsilon)$ stands for an arbitrary function of asymptotic order less or equal to that of ε, as $\varepsilon \to 0_+$.)

The algebra $\mathcal{G}$ contains the distributions on $\mathbb{R}^m$, canonically embedded as a $\mathbb{C}$-vector subspace by the map $i : \mathcal{D}'(\mathbb{R}^m) \to \mathcal{G} : u \mapsto \widetilde{u} = \{\widetilde{u}(\varphi, x) = (u * \check{\varphi})(x)\}$, where $\check{\varphi}(x) = \varphi(-x)$ and φ is running the set $A_q(\mathbb{R}^m)$. The equality in Colombeau algebra $\mathcal{G}$ is very strict, so the next weaker concept for "association" is introduced.

Definition 2. **(a)** Two generalized functions $f, g \in \mathcal{G}$ are said to be associated, denoted $f \approx g$, if for some representatives $f(\varphi_\varepsilon, x), g(\varphi_\varepsilon, x)$ of theirs and each $\psi(x) \in \mathcal{D}(\mathbb{R}^m)$ there is a $q \in \mathbb{N}_0$ such that $\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}^m} [f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)]\psi(x)\,dx = 0$, for all $\varphi(x) \in A_q(\mathbb{R}^m)$.

(b) A generalized function $f \in \mathcal{G}$ is said to admit some $u \in \mathcal{D}'(\mathbb{R}^m)$ as an "ssociated distribution", denoted $f \approx u$, if for some representative of f and for each $\psi(x) \in \mathcal{D}(\mathbb{R}^m)$ there is a $q \in \mathbb{N}_0$ such that, for all $\varphi(x) \in A_q(\mathbb{R}^m)$, $\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}^m} f(\varphi_\varepsilon, x)\psi(x)\,dx = \langle u, \psi \rangle$.

These definitions are independent of the representative chosen. The distribution associated, if it exists, is unique. The image in $\mathcal{G}$ of every distribution is associated with the latter [4], the association thus being a generalization of the equality of distributions in $\mathcal{D}'(\mathbb{R}^m)$. Now, we give this.

Definition 3. By product of some distributions in Colombeau algebra $\mathcal{G}$, sometimes called "Colombeau product", is meant the product of their embeddings in $\mathcal{G}$, whenever the result admits an associated distribution.

The following coherence result holds [16, Proposition 10.3]: If the regularized model product of two distributions exists, then their Colombeau product also exists and coincides with the former. On the other hand, in the general setting of Colombeau algebra $\mathcal{G}(\mathbb{R}^m)$ [4], as well as in the algebra $\mathcal{G}(\mathbb{R})$ on the real line, this assertion turns into an equivalence, according to a result by Jelínek [12]; cf. also a recent study by Boie [3]. However, in the setting of algebra $\mathcal{G}(\mathbb{R}^m)$ with parameter functions φ defined as m-fold tensor products, the Colombeau product extends the model product: simple examples of Colombeau products do not exist as model products [16].

3. Preliminary results

We recall now several results that will be needed later. The $\approx$-association is consistent with the linear operations in Colombeau algebra, but it holds only the following "weak" version of the formula for partial derivatives ∂_i, $i = 1, \dots, m$ of the Colombeau product of distributions on $\mathbb{R}^m$.

Lemma 1 ([6]). *Let the embeddings of the distributions u, v and the distribution w satisfy $\widetilde{u}\,.\,\widetilde{v} \approx w$. Then it holds*

$$\widetilde{\partial_i u}\ .\ \widetilde{v}\ +\ \widetilde{u}\ .\ \widetilde{\partial_i v} \approx \partial_i w, \qquad i = 1, 2, \dots, m. \tag{3}$$

Note that, in general, only the sum on the left-hand side of (3) has an associated distribution, but not the individual summands in it; hence the name "weak". Clearly, this assertion extends to M-type products in $\mathcal{G}(\mathbb{R}^m)$ as well.

Notation 2. We denote the "normed" powers of the variable $x \in \mathbb{R}^m$ for an arbitrary $p \in \mathbb{N}_0^m$ that are supported only in one quadrant of the Euclidean space $\mathbb{R}^m$ by:

$$\nu_+^p \equiv \nu_+^p(x) = \{x^p/p!,\ x > 0,\ \ = 0 \ \text{elsewhere}\ \},$$
$$\nu_-^p \equiv \nu_-^p(x) = \{(-x)^p/p!,\ x < 0,\ \ = 0 \ \text{elsewhere}\ \}.$$

Denote further their "even" and "odd" compositions as: $\nu_0^p = \nu_+^p + \nu_-^p$, $\nu_0^p = \nu_+^p - \nu_-^p$, which will be jointly denoted as ν_σ^p, $\sigma = (0, 1)$.

In dimension one, these notations correspond to the normed even and odd distributions $|x|^p$ and $|x|^p \operatorname{sgn} x$ ($x \in \mathbb{R}$, $p \in \mathbb{N}$), as introduced in [10].

Observe that ν_σ^p are indeed even and odd functions of x for $\sigma = 0$ or 1: $\nu_\sigma^p(-x) = (-1)^\sigma \nu_\sigma^p(x)$. Finally, one has

$$\partial_x \nu_\pm^{p+1} = \pm\, \nu_\pm^p,\ \partial_x \nu_0^{p+1} = \nu_1^p,\ \partial_x \nu_1^{p+1} = \nu_0^p \tag{4}$$

(with no number coefficients). With the above notation, it now holds the following.

Proposition 1 ([5])**.** *For an arbitrary $p \in \mathbb{N}_0^m$, the embeddings $\widetilde{\delta^{(p)}}(x), \widetilde{\nu_\pm^p}$ in $\mathcal{G}(\mathbb{R}^m)$ of the distributions $\delta^{(p)}(x), \nu_\pm^p$ satisfy:*

$$\widetilde{\nu_+^p} \,.\, \widetilde{\delta^{(p)}}(x) \approx (-1)^{|p|}\, 2^{-m}\, \delta(x), \qquad \widetilde{\nu_-^p} \,.\, \widetilde{\delta^{(p)}}(x) \approx 2^{-m}\, \delta(x). \tag{5}$$

We note that these equations are known in distribution theory but they have been only derived as regularized products in dimension one, using symmetric mollifiers.

Combining now equations (5) and taking into account Notation 2, we obtain for each $p \in \mathbb{N}_0^m$:

$$\widetilde{\nu_0^{2p+1}} \,.\, \widetilde{\delta^{(2p+1)}(x)} \approx 0, \qquad \widetilde{\nu_1^{2p}} \,.\, \widetilde{\delta^{(2p)}(x)} \approx 0. \tag{6}$$

Remark 1. Note that ν_0^{2p} and ν_1^{2p+1} coincide with the C^∞ function $\nu^q(x) = x^q/q!$ for $q = 2p, 2p+1$, correspondingly, except for $x = 0$, and $\nu_\sigma^q(\pm 0) = \nu^q(0)$ for $\sigma = 0, 1$ and any $q \in \mathbb{N}_0^m$. Thus, by classical theorems in Distribution theory [13, Chapter 6], their products with $\delta^{(q)}(x)$ coincide with $\nu^q(x) \,.\, \delta^{(q)}(x)$, which exist in the space $\mathcal{D}'(\mathbb{R}^m)$.

We next give some results on M-type products of the above distributions in $\mathcal{G}(\mathbb{R})$. Denoting $\check{H} := H(-x)(= \nu_-^0)$, one easily checks that $(\nu_-)' = -\check{H}$ and $(\check{H})' = -\delta$. Then, combining the results of Lemma 1 and Proposition 1, the following equations can be proved in $\mathcal{G}(\mathbb{R})$ for an arbitrary $p \in \mathbb{N}$:

$$\begin{aligned} &\widetilde{\nu_+^p} . \widetilde{\delta^{(p+1)}}(x) + \widetilde{\nu_+^{p-1}} . \widetilde{\delta^{(p)}}(x) \approx \frac{(-1)^p}{2}\, \delta', \\ &\widetilde{\nu_-^p} . \widetilde{\delta^{(p+1)}}(x) - \widetilde{\nu_-^{p-1}} . \widetilde{\delta^{(p)}}(x) \approx \frac{1}{2}\, \delta'. \end{aligned} \tag{7}$$

When $p = 0$ in equation (5), we obtain by Lemma 1:

$$\widetilde{H} . \widetilde{\delta}'(x) + \widetilde{\delta}^{\,2}(x) \approx \frac{1}{2}\, \delta', \qquad \widetilde{\check{H}} . \widetilde{\delta}'(x) - \widetilde{\delta}^{\,2}(x) \approx \frac{1}{2}\, \delta'. \tag{8}$$

Note that the first equation in (8) coincides with (2), but it was derived in [6] with no auxiliary requirements on the mollifiers, such as to be even functions as required in [17]. Furthermore, the next proposition directly generalizes the M-type products (8) for the distributions $\nu_\pm^p$ and $\delta^{(p+1)}(x)$.

Proposition 2 ([6]). *For an arbitrary p in $\mathbb{N}_0$, the embeddings in $\mathcal{G}(\mathbb{R})$ of the distributions $\nu_\pm^p$ and $\delta^{(p+1)}(x)$ satisfy:*

$$
\begin{aligned}
&(-1)^p\, \widetilde{\nu_+^p} \,.\, \widetilde{\delta^{(p+1)}}(x) + \widetilde{\delta}^{\,2}(x) \approx \frac{p+1}{2}\,\delta', \\
&\widetilde{\nu_-^p} \,.\, \widetilde{\delta^{(p+1)}}(x) - \widetilde{\delta}^{\,2}(x) \approx \frac{p+1}{2}\,\delta'.
\end{aligned}
\tag{9}
$$

Accordingly, combining equations (9), we get for any p in $\mathbb{N}_0$:

$$
\begin{aligned}
&\widetilde{\nu_0^{2p+1}} \,.\, \widetilde{\delta^{(2p+2)}}(x) - 2\widetilde{\delta}^{\,2}(x) \approx 0, \\
&\widetilde{\nu_1^{2p}} \,.\, \widetilde{\delta^{(2p+1)}}(x) + 2\widetilde{\delta}^{\,2}(x) \approx 0.
\end{aligned}
\tag{10}
$$

We finally recall the generalization of basic Mikusiński equation (1) for arbitrary $p, q \in \mathbb{N}$ derived in [8] in Colombeau algebra of tempered generalized functions on $\mathbb{R}$:

$$
\widetilde{x^{-p}} \,.\, \widetilde{x^{-q}} \;-\; \pi^2 \frac{(-1)^{p+q}}{(p-1)!\,(q-1)!}\, \widetilde{\delta^{(p-1)}}(x) \,.\, \widetilde{\delta^{(q-1)}}(x) \;\approx\; x^{-p-q}. \tag{11}
$$

4. Further results on Mikusiński type distributional products

We now proceed to particular M-type products of the distributions $\nu_\pm^p, \nu_\sigma^p$, and $\delta^{(p)}(x)$ in Colombeau algebra. Applying first the weak rule (3) for differentiation of Colombeau products to equations (8), we get:

$$
\begin{aligned}
&\widetilde{H} \,.\, \widetilde{\delta''}(x) + 3\,\widetilde{\delta}(x) \,.\, \widetilde{\delta'}(x) \;\approx \frac{1}{2}\,\delta'', \\
&\widetilde{\check{H}} \,.\, \widetilde{\delta''}(x) - 3\,\widetilde{\delta}(x) \,.\, \widetilde{\delta'} \approx \frac{1}{2}\,\delta''.
\end{aligned}
\tag{12}
$$

Differentiating then equations (7) and (9) for $p = 1$, according to that rule, and combining the results, we obtain:

$$
\begin{aligned}
&\widetilde{\nu_+} \,.\, \widetilde{\delta'''}(x) - 5\,\widetilde{\delta}(x) \,.\, \widetilde{\delta'}(x) \approx -\frac{3}{2}\,\delta'', \\
&\widetilde{\nu_-} \,.\, \widetilde{\delta'''}(x) - 5\,\widetilde{\delta}(x) \,.\, \widetilde{\delta'} \approx \frac{3}{2}\,\delta''.
\end{aligned}
\tag{13}
$$

Moreover, as shown by the next two propositions, the M-type products given by equations (12) and (13) can be directly generalized for the distributions $\nu_\pm^p$ and $\delta^{(p+2)}(x)$ for each $p \in \mathbb{N}_0$.

Proposition 3. *For an arbitrary* $p \in \mathbb{N}_0$*, the embeddings in* $\mathcal{G}(\mathbb{R})$ *of the distributions* ν_+^p *and* $\delta^{(p+2)}(x)$ *satisfy*

$$(-1)^p \,\widetilde{\nu_+^p} \,.\, \widetilde{\delta^{(p+2)}}(x) + (2p+3)\,\widetilde{\delta}(x)\,.\,\widetilde{\delta}'(x) \approx \frac{(p+1)(p+2)}{4}\,\delta''(x). \tag{14}$$

Proof. For an arbitrary $\psi(x)$ in $\mathcal{D}(\mathbb{R})$, denote first $V := \langle\, \widetilde{\delta}(\varphi_\varepsilon, x)\widetilde{\delta}'(\varphi_\varepsilon, x), \psi(x)\,\rangle$. Then, we get on the change $-x/\varepsilon = t$ and applying Taylor theorem:

$$\begin{aligned}
V &= -\frac{1}{\varepsilon^3}\int_{-\varepsilon b}^{-\varepsilon a} \varphi\left(-\frac{x}{\varepsilon}\right)\,\varphi'\left(-\frac{x}{\varepsilon}\right)\,\psi(x)\,dx\\
&= -\frac{1}{\varepsilon^2}\int_{-\varepsilon b}^{-\varepsilon a} \varphi(t)\,\varphi'(t)\,\psi(-\varepsilon t)\,dx\\
&= \int_a^b \left[-\frac{\psi(0)}{\varepsilon^2} + \frac{\psi'(0)}{\varepsilon}\,t - \frac{\psi''(0)}{2}\,t^2\right]\varphi(t)\,\varphi'(t)\,dt + O(\varepsilon)\\
&= -\frac{\psi'(0)}{2\,\varepsilon}\int_a^b \varphi^2(t)\,dt + \frac{\psi''(0)}{2}\int_a^b t\,\varphi^2(t)\,dt \;+\; O(\varepsilon).
\end{aligned} \tag{15}$$

It is taken into account here that, if $\operatorname{supp}\varphi(x) \subseteq [a,b]$ for some a, b in $\mathbb{R}$, then $\operatorname{supp}\varphi(-x/\varepsilon) \subseteq [-\varepsilon b, -\varepsilon a]$.

Denoting further $V_p := \langle\, \widetilde{\nu_+^p}(\varphi_\varepsilon, x)\,.\,\widetilde{\delta^{(p+2)}}(\varphi_\varepsilon, x),\, \psi(x)\,\rangle$, we obtain

$$\begin{aligned}
p!\; V_p =& \frac{(-1)^{p+2}}{\varepsilon^{p+3}}\int_{-b\varepsilon}^{-a\varepsilon}\left(\int_{-x/\varepsilon}^{b}(x+\varepsilon t)^p\varphi(t)\,dt\right)\varphi^{(p+2)}\left(-\frac{x}{\varepsilon}\right)\psi(x)\,dx\\
=& \frac{1}{\varepsilon^2}\int_a^b \psi(-\varepsilon y)\varphi^{(p+2)}(y)\int_y^b (y-t)^p\varphi(t)\,dt\,dy\\
=& \frac{\psi(0)}{\varepsilon^2}\int_a^b \varphi(t)\int_a^t (y-t)^p\varphi^{(p+2)}(y)\,dy\,dt\\
&- \frac{\psi'(0)}{\varepsilon}\int_a^b \varphi(t)\int_a^t y\,(y-t)^p\varphi^{(p+2)}(y)\,dy\,dt\\
&+ \frac{\psi''(0)}{2}\int_a^b \varphi(t)\int_a^t y^2\,(y-t)^p\varphi^{(p+2)}(y)\,dy\,dt + O(\varepsilon)\\
=:& \;\frac{\psi(0)}{\varepsilon^2}I_1 \;-\; \frac{\psi'(0)}{\varepsilon}\,I_2 \;+\; \frac{\psi''(0)}{2}\,I_3 \;+\; O(\varepsilon).
\end{aligned} \tag{16}$$

On a multiple integration by parts, the integrated term being zero each time, we calculate successively:

$$I_1 = (-1)^p\,p!\,\int_a^b \varphi(y)\int_a^t \varphi''(y)\,dy\,dt = (-1)^p\,p!\,\frac{\varphi^2(t)}{2}\bigg|_a^b \;=\; 0,$$

$$\begin{aligned}
I_2 = & \int_a^b \varphi(t) \int_a^t (y-t)^{p+1} \varphi^{(p+2)}(y)\, dy\, dt \\
& + \int_a^b t\varphi(t) \int_a^t (y-t)^p \varphi^{(p+2)}(y)\, dy\, dt \\
= & (-1)^{p+1}\,(p+1)! \int_a^b \varphi^2(t)\, dt + (-1)^p\, p! \int_a^b t\,\varphi(t)\,\varphi'(t)\, dt \\
= & \frac{1}{2}\,(-1)^{p+1}\, p!\,(2p+3) \int_a^b \varphi^2(t)\, dt, \\
I_3 = & \int_a^b \varphi(t) \int_a^t (y-t)^{p+2} \varphi^{(p+2)}(y) dy\, dt \\
& + \int_a^b \varphi(t) \int_a^t (2yt-t^2)(y-t)^p \varphi^{(p+2)}(y) dy\, dt \\
= & 2 \int_a^b t\,\varphi(t) \int_a^t (y-t)^{p+1} \varphi^{(p+2)}(y)\, dy\, dt \\
& + \int_a^b t^2 \varphi(t) \int_a^t (y-t)^p \varphi^{(p+2)}(y)\, dy\, dt \\
= & (-1)^{p+1}\, 2\,(p+1)! \int_a^b t\,\varphi^2(t)\, dt + (-1)^p\, p! \int_a^b t^2\, \varphi(t)\,\varphi'(t)\, dt \\
= & (-1)^{p+1}\,(p)!\; (2p+3) \int_a^b t\,\varphi^2(t)\, dt.
\end{aligned}$$

Replacing these latter terms in (16), we get

$$\begin{aligned}
(-1)^p\, V_p = & (2p+3)\; \left[\frac{\psi'(0)}{2\,\varepsilon} \int_a^b \varphi^2(t)\, dt + \frac{\psi''(0)}{2} \int_a^b t\,\varphi^2(t)\, dt \right] \\
& + \frac{1}{4}(p+1)(p+2)\psi''(0) + O(\varepsilon).
\end{aligned}$$

In view of equality (15), we finally write

$$(-1)^p\, V_p + \langle (2p+3)\, \widetilde{\delta}(x)\, \widetilde{\delta}'(x), \psi(x) \rangle = \frac{(p+1)(p+2)}{4} \langle\, \delta''(x), \psi(x)\, \rangle + O(\varepsilon).$$

Passing therefore to the limit, as $\varepsilon \to 0_+$, and applying Definition 2, we obtain the M-type product (14) for any $p \in \mathbb{N}$. □

Proposition 4. *For an arbitrary $p \in \mathbb{N}_0$, the embeddings in $\mathcal{G}(\mathbb{R})$ of the distributions ν_-^p and $\delta^{(p+2)}(x)$ satisfy*

$$\widetilde{\nu_-^p} \,.\, \widetilde{\delta^{(p+2)}}(x) \;-\; (2p \;+\; 3)\, \widetilde{\delta}(x) \,.\, \widetilde{\delta}'(x) \;\approx\; \frac{(p+1)(p+2)}{4}\, \delta''(x). \tag{17}$$

Proof. Since $x_-^p = (-x)_+^p$ for any $p \in \mathbb{N}_0$, equation (17) is obtained on replacing $x \to -x$ in (14) and taking into account that $\delta^{(p)}(-x) = (-1)^p \delta^{(p)}(x)$. □

Remark 2. Further extension of the results of Propositions 3–4 to the products $\nu_\pm^p \, . \, \delta^{(p+q)}(x)$ for $q = 3, 4, \ldots$ or for arbitrary $q \in \mathbb{N}$, is also possible (though more difficult) to prove, but the results are not M-type products any more: one gets for the balancing term not single product but a sum of such products.

Combining now equations (14), (17), we get this.

Corollary 1. *For an arbitrary $p \in \mathbb{N}$, the embeddings in $\mathcal{G}(\mathbb{R})$ of the distributions ν_σ^p and $\delta^{(p+2)}(x)$ satisfy:*

$$\widetilde{\nu_0^{2p-1}} \, . \, \widetilde{\delta^{(2p+1)}(x)} - 2\,(4p+1)\, \widetilde{\delta}(x) \, . \, \widetilde{\delta'}(x) \approx 0, \tag{18}$$

$$\widetilde{\nu_1^{2p-2}} \, . \, \widetilde{\delta^{(2p)}(x)} + 2\,(4p-1)\, \widetilde{\delta}(x) \, . \, \widetilde{\delta'}(x) \approx 0. \tag{19}$$

We will now demonstrate that a class of M-type distributional products can be extended to the many-variable case. Namely, the following general assertion holds.

Theorem 1. *Let u_k, v_k $(k = 1, 2)$ be distributions in $\mathcal{D}'(\mathbb{R}^m)$, such that $u_k(x) = \prod_{i=1}^m u_k^i(x_i)$ and $v_k(x) = \prod_{i=1}^m v_k^i(x_i)$. If all u_k^i, v_k^i are distributions in $\mathcal{D}'(\mathbb{R})$ and their embeddings in $\mathcal{G}(\mathbb{R})$ satisfy: $\widetilde{u_1^i} \, . \, \widetilde{v_1^i} - \widetilde{u_2^i} \, . \, \widetilde{v_2^i} \approx 0$, $i = 1, \ldots, m$, then the embeddings in $\mathcal{G}(\mathbb{R}^m)$ of the tensor-product distributions u_k, v_k satisfy: $\widetilde{u_1} \, . \, \widetilde{v_1} - \widetilde{u_2} \, . \, \widetilde{v_2} \approx 0$.*

Proof. By the linearity of Definition 2, we have $\widetilde{u_1^i} \, . \, \widetilde{v_1^i} \approx \widetilde{u_2^i} \, . \, \widetilde{v_2^i}$, which holds in $\mathcal{G}(\mathbb{R})$ for each $i = 1, \ldots, m$. Suppose further we have restricted ourselves to the subspace of test-functions $\psi(x) = \prod_{i=1}^m \psi_i(x_i)$, with each ψ_i in $\mathcal{D}(\mathbb{R})$. Then, in view of the tensor-product structure of both the distributions $u_k, v_k \in \mathcal{D}'(\mathbb{R}^m)$ and the parameter functions $\varphi \in A_0(\mathbb{R}^m)$, on applying a Fubini-type theorem for tensor-product distributions [11, § 4.3], we get for $I := \langle \widetilde{u_1}(\varphi_\varepsilon, x)\,.\,\widetilde{v_1}(\varphi_\varepsilon, x), \psi(x)\rangle$:

$$\begin{aligned} I =& \langle \prod_{i=1}^{m} \widetilde{u_1}^i(\chi_\varepsilon, x_i)\,.\,\prod_{i=1}^{m} \widetilde{v_1}^i(\chi_\varepsilon, x_i), \prod_{i=1}^{m} \psi_i(x_i)\rangle \\ =& \prod_{i=1}^{m} \langle \widetilde{u_1}^i(\chi_\varepsilon, x_i)\,.\,\widetilde{v_1}^i(\chi_\varepsilon, x_i), \psi_i(x_i)\rangle \\ =& \prod_{i=1}^{m} [\langle \widetilde{u_2}^i(\chi_\varepsilon, x_i)\,.\,\widetilde{v_2}^i(\chi_\varepsilon, x_i), \psi_i(x_i)\rangle + o^i(1)] \\ =& \langle \widetilde{u_2}(\varphi_\varepsilon, x)\,.\,\widetilde{v_2}(\varphi_\varepsilon, x), \psi(x)\rangle + o(1). \end{aligned}$$

Here, each Landau symbol $o(1)$ stands for an arbitrary function of asymptotic order less than any constant, or equivalently, that tends to 0, as $\varepsilon \to 0$. Thus, we have

$$\lim_{\varepsilon\to 0} \langle [\widetilde{u_1}(\varphi_\varepsilon, x)\widetilde{v_1}(\varphi_\varepsilon, x) \; - \; \widetilde{u_2}(\varphi_\varepsilon, x)\widetilde{v_2}(\varphi_\varepsilon, x)],\; \psi(x)\rangle \;=\; 0.$$

Now since the set of test-functions $\psi(x) = \prod_{i=1}^{m} \psi_i(x_i)$ is a dense subset of $\mathcal{D}(\mathbb{R}^m)$ [11, § 4.3], it follows, by Definition 2, that the product in consideration holds for the embeddings of the tensor-product distributions u_k, v_k. The proof is complete. □

Remark 3. Observe that the distributions ν_σ^p and $\delta^{(q)}(x)$ have a tensor-product structure (with coinciding components u^i). Thus, by Theorem 1, the M-type products (10), (18), and (19) hold when the distributions involved belong to the space $\mathcal{D}'(\mathbb{R}^m)$.

5. Singular products of piecewise differentiable functions

We first recall a result proved in [7], starting with the following.

Notation 3. Let $C_d^k(\mathbb{R}\setminus\{0\})$ be the class of k-times differentiable functions on $\mathbb{R}\setminus\{0\}$ for some $k \in \mathbb{N}_0$, such that each function $f(x)$ and its derivatives have discontinuities of first order at the point $x = 0$, i.e. for each $i = 0, \ldots, k$, the values $f^{(i)}(0_+)$ and $f^{(i)}(0_-)$ exist but generally differ from each other. Denote then:

$$\begin{aligned} m_i =& \frac{1}{2}\left[f^{(i)}(0_+) + f^{(i)}(0_-)\right], \\ h_i =& f^{(i)}(0_+) - f^{(i)}(0_-) \text{ (the jump at 0)}, \quad i = 0, \ldots, k. \end{aligned}$$

In Distribution theory, functions in $C_d^k(\mathbb{R}\setminus\{0\})$ cannot be multiplied with a distribution having singular support that includes the point $x=0$. Nonetheless, their Colombeau products with the δ-function and its derivative exist, as proved in [7]:

Proposition 5. *For each function $f(x)\in C_d^1(\mathbb{R}\setminus\{0\})$, the embedding in $\mathcal{G}(\mathbb{R})$ satisfy:*

$$\begin{aligned}&\widetilde{f}(x)\,.\,\widetilde{\delta}(x)\approx m_0\,\delta(x),\\&\widetilde{f}(x)\,.\,\widetilde{\delta}'(x)+h_0\,\widetilde{\delta}^2(x)\approx m_0\,\delta'(x)-m_1\,\delta(x).\end{aligned}\tag{20}$$

Remark 4. The choice of the point $x=0$ is no loss of generality, and moreover these equations can be modified for the case of finite number of discontinuities. Observe that whenever the jump h_0 of the function is zero, then the M-type product in (20) becomes an "ordinary" Colombeau product.

We now extend these results, connecting them with such given above. Recall that any function on $\mathbb{R}$ can be canonically represented as a sum of its even and odd parts:

$$f(x)=\sum_{\sigma=0,1}f_\sigma(x),\qquad\text{where}$$

$$f_0(x):=\frac{1}{2}\,[f(x)+\check{f}(x)]\quad\text{and}\quad f_1(x):=\frac{1}{2}\,[f(x)-\check{f}(x)]$$

are indeed even and odd functions: $f_\sigma(-x)=(-1)^\sigma\,f_\sigma(x)$, $\sigma=(0,1)$.

One then checks that, for any function $f(x)$ in $C_d^k(\mathbb{R}\setminus\{0\})$ and $i\le(k-1)/2$,

$$\begin{aligned}&m\Big(f_0^{(2i)}\Big)=m_{2i},\quad m\Big(f_1^{(2i+1)}\Big)=m_{2i+1},\\&h\Big(f_1^{(2i)}\Big)=h_{2i},\qquad h\Big(f_0^{(2i+1)}\Big)=h_{2i+1},\end{aligned}\tag{21}$$

the rest four combinations being all zero. Now from (20) and (21), it follows this.

Corollary 2. *For each function $f(x)\in C_d^1(\mathbb{R}\backslash\{0\})$, the embeddings in $\mathcal{G}(\mathbb{R})$ of its even and odd parts satisfy:*

$$\widetilde{f_0}(x)\,.\,\widetilde{\delta}(x)\approx m_0\,\delta(x),\qquad\widetilde{f_1}(x)\,.\,\widetilde{\delta}(x)\approx 0,\tag{22}$$

$$\widetilde{f_0}(x)\,.\,\widetilde{\delta}'(x)\approx m_0\,\delta'(x),\quad\widetilde{f_1}(x)\,.\,\widetilde{\delta}'(x)+h_0\,\widetilde{\delta}^2(x)\approx -m_1\,\delta'(x).\tag{23}$$

We next give more M-type products in $\mathcal{G}(\mathbb{R})$ for the even and odd (parts of) functions in $C^2_d(\mathbb{R}\setminus\{0\})$.

Proposition 6. *The embedding in $\mathcal{G}(\mathbb{R})$ of the even and odd parts of any function $f(x)\in C^2_d(\mathbb{R}\setminus\{0\})$, satisfy:*

$$\widetilde{f_0}(x)\,.\,\widetilde{\delta''}(x) - h_1\,\widetilde{\delta}^{\,2}(x) \;\approx m_0\,\delta''(x) + m_2\,\delta(x), \tag{24}$$

$$\widetilde{f_1}(x)\,.\,\widetilde{\delta''}(x) + 3\,h_0\,\widetilde{\delta}\,.\,\widetilde{\delta'} \approx -2\;m_1\,\delta'(x). \tag{25}$$

Proof. Differentiating equation (20) in $\mathcal{G}$, where it holds $\partial_x\widetilde{u} = \widetilde{\partial_x u}$ for the imbedding in $\mathcal{G}$ of anu distribution $u\in\mathcal{D}'(\mathbb{R})$, we get

$$\widetilde{f}(x)\,.\,\widetilde{\delta''}(x) + \widetilde{f'}(x)\,.\,\widetilde{\delta'}(x) + 2\,h_0\,\widetilde{\delta}(x)\,.\,\widetilde{\delta'}(x) \approx m_0\,\delta''(x) - m_1\,\delta'(x).$$

The left-hand side of this equation is to be considered as a single entity. Recall now [10, § 1.2] that the distributional derivative of any function $f\in C^1_d(\mathbb{R}\setminus\{0\})$ is given by $f' = f'_{cl} + h_0\,\delta$, where f'_{cl} is the classical derivative of f for $x\neq 0$. We thus obtain

$$\widetilde{f}(x)\,.\,\widetilde{\delta''}(x) + \widetilde{f'_{cl}}(x)\,.\,\widetilde{\delta'}(x) + 3\,h_0\,\widetilde{\delta}(x)\,.\,\widetilde{\delta'}(x) \approx m_0\,\delta''(x) - m_1\,\delta'(x).$$

For the second term here, equation (20) yields

$$\widetilde{f'_{cl}}(x)\,.\,\widetilde{\delta'}(x) + \;h_1\,\widetilde{\delta}^{\,2}(x) \approx m_1\,\delta'(x) - m_2\,\delta(x).$$

Replacing this in the above equation, we get

$$\begin{aligned}&\widetilde{f}(x)\,.\,\widetilde{\delta''}(x) - \;h_1\,\widetilde{\delta}^{\,2}(x) + 3\,h_0\,\widetilde{\delta}(x)\,.\,\widetilde{\delta'}(x)\\ &\approx m_0\,\delta''(x) - 2\,m_1\,\delta'(x) + m_2\,\delta(x).\end{aligned} \tag{26}$$

Applying then the last equation successively to the even and odd parts f_σ of $f(x)$ and taking into account that (21) gives

$$h(f_0) = m(f'_0) = 0, \qquad h(f'_1) = m(f_1) = m(f''_1) = 0,$$

equation (26) splits into the M-type products (24), (25). The proof is complete. □

Examples . The next equations are obtained, replacing the function $f(x)$ successively with:

(a) $\nu^0_1 \equiv \operatorname{sgn} x$ in (25): $\quad \widetilde{\operatorname{sgn}}\,x\,.\,\widetilde{\delta''}(x) \;+\; 6\,\widetilde{\delta}(x)\,.\,\widetilde{\delta'}(x) \;\approx\; 0,$

(b) $\nu_0 \equiv |x|$ in (24): $\quad |x|\,.\,\widetilde{\delta'}(x) \;-\; 2\,\widetilde{\delta}^{\,2}(x) \approx\; 0,$

(c) $\nu^2_1 \equiv \frac{1}{2}|x|^2 \operatorname{sgn} x$ in (25): $\quad |x|^2\;\operatorname{sgn}\,x\,.\,\widetilde{\delta''}(x) \;\approx\; 0.$

Note that these equations coincide correspondingly with: (19) for $p=1$, the first equation in (10) for $p=1$, and with the second equation in (6) for $p=0$.

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