

THE EXISTENCE AND UNIQUENESS OF SOLUTION OF ONE COUPLED PLATE THERMOMECHANICS PROBLEM

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Abstract. The thermo-elastic plate system of equations is analysed. The sufficient conditions of existence, uniqueness and continuity dependence on initial data of the Cauchy problem solutions for differential-operational equation of mixed type (a part of the equation of hyperbolic type, and a part of parabolic type) are given in this paper. If the operational coefficients are suitably chosen, the investigated equation can be used to obtain a differential equation describing vibrations of a plate — the modified Germain-Lagrange equation of hyperbolic type. Moreover, in order to define the temperature field, one can use a three-dimensional equation of thermal conductivity (a parabolic equation).

1. Introduction

Let H be a Hilbert space, let L, M be a self-adjoint operators negatively defined in H with the domains $D(M) \subset D(L)$, and let C be an operator bounded in H . It is assumed that the operator L commutes with C and

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with the resolvent M (the definition of commuting is given in reference [5]). Let us consider the following system on the interval $[0, T]$

$$\begin{aligned} W''(t) + L^2W(t) + \alpha LC\theta(t) &= q_1(t), \\ \theta'(t) - M\theta(t) - \beta C^*LW'(t) &= q_2(t), \end{aligned} \quad (1)$$

($\alpha, \beta > 0$) with unknown functions W and θ .

We take $H = L_2(\Omega)$ as the space of functions defined in the bounded domain with the piecewise smooth boundary $\partial\Omega$ in the space $\Omega = \Omega_1 \times [-h/2, h/2] \subset \mathbb{R}^3$ ($\Omega_1 \subset \mathbb{R}^2$, $h > 0$) and having the summing up square norm. As M we take the operator which is realization of the formal differential expression

$$\frac{\lambda_0^2 \partial^2}{\partial x^2} + \frac{\lambda_0^2 \partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (\lambda_0 \neq 0),$$

and with the attached boundary condition $\theta|_{\partial\Omega} = 0$. As L operator we take the operator being a closure of a tensor product of the operators L_1 and E , where L_1 is generated in $L_2(\Omega_1)$ by the expression

$$\gamma \nabla^2 = \gamma \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad (\gamma > 0)$$

with the boundary condition $W|_{\partial\Omega_1} = 0$. E is the identity operator in $L_2(-h/2, h/2)$, i.e. L is the closure of the operator generated in $L_2(\Omega)$ by the expression

$$\gamma \nabla^2 = \gamma \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and defined on smooth functions from $L_2(\Omega)$ which are equal to zero on $\partial\Omega_1 \times [-h/2, h/2]$. The operator C is defined by the formula

$$C\theta = \int_{-h/2}^{h/2} z\theta(x, y, z) dz.$$

Then

$$C^*g = z \int_{-h/2}^{h/2} g(x, y, z) dz.$$

If L, M, C are chosen in this way, and the numbers α, β, γ are properly chosen, one can get from (1) the known plate equation governing its vibrations (see, for instance, [1]), with the following parameters: $F = 0$, $k_x = k_y = 0$, $P_x = P_y = 0$; it is called the thermal conductivity equation.

Furthermore, in order to simplify the considerations we assume in (1) that $\alpha = \beta = 1$. Observe that practically the introduced considerations do not change other values of α and β .

In order to reduce the system (1) into one equation of the first order we introduce the operator

$$A_0 = \begin{pmatrix} 0 & E & 0 \\ -L^2 & 0 & -LC \\ 0 & C^*L & M \end{pmatrix},$$

acting in the space $\tilde{H} = H \oplus H \oplus H$. Here and further E means the identity operator in the respective space. We emphasize that the commutativity of the operators C and L yields the commutativity of C^* and $L^* = L$ [5].

Lemma 1. *The operator A_0 allows of closing of A in the space \tilde{H} . The operator A^{-1} exists, is bounded and defined in the whole space \tilde{H} .*

Proof. The operational matrix

$$\begin{pmatrix} CM^{-1}C^* & -L^{-2} & -L^{-1}CM^{-1} \\ E & 0 & 0 \\ -LM^{-1}C^* & 0 & M^{-1} \end{pmatrix}$$

defines the bounded operator in \tilde{H} . A direct check shows that its product (of any order) with the operational matrix defining the operator A_0 gives the unit operational matrix. All claims of the lemma result from this. Moreover, the operator A^{-1} is defined by the last operational matrix. \square

Further on, the operator and its closure will be tagged with one symbol. If we assume $\nu = w'$, and u will denote the column

$$\begin{pmatrix} w \\ \nu \\ \theta \end{pmatrix},$$

then the homogeneous system (1) in the space \tilde{H} can be written in the following way:

$$u'(t) = Au(t). \tag{2}$$

Furthermore, we denote by \tilde{L}^{-2} the operator in \tilde{H} , defined by the equation

$$\tilde{L}^{-2} = \begin{pmatrix} L^{-2} & 0 & 0 \\ 0 & L^{-2} & 0 \\ 0 & 0 & L^{-2} \end{pmatrix}.$$

It results from the commuting properties of the operators that also \tilde{L}^{-2} and A^{-1} commute.

2. Basic assumption

In order to formulate the basic assumption we need to define the following spaces and the attached operators. The smoothness of a solution to the homogeneous equation (2) depends on that if the initial condition $u(0)$ belongs to the domains of the operators \tilde{L} and A . Observe that in applications the operator \tilde{L} is simpler than the operator A and therefore the spaces are constructed with regard to the powers of \tilde{L} . However, we can not completely omit the operator A . Observe also that one can use the powers of A only (without \tilde{L}).

We define the scalar product by the formula $(x, y)_{F^*} = (A^*x, A^*y)$ on the manifold $F^* = D(A^*)$. As the operator A^* possesses the inverse one defined everywhere, then F^* is a complete space and $\|x\|_{F^*} \geq \|x\|_{\tilde{H}}$. This is why F^* can be considered as the space with a positive norm in relation to \tilde{H} [2]. We shall denote by $\tilde{H}^{(-1)}$ the space with a negative norm, constructed using the pair F^*, \tilde{H} .

The operator A^* maps continuously and bijectively F^* into \tilde{H} . This is why the conjugated operator \tilde{A} acts from \tilde{H} to $\tilde{H}^{(-1)}$ continuously and bijectively, and it serves as the extension of A .

We shall define the scalar product on the set $\tilde{H}^{(+1)} = D(A)$ by the equation $(x, y)_{\tilde{H}^{(+1)}} = (Ax, Ay)$. $\tilde{H}^{(+1)}$ appears as the full space and the operator A maps $\tilde{H}^{(+1)}$ to \tilde{H} continuously and bijectively (one-to-one).

Because the operator \tilde{L}^{-1} commutes with A, A^* , and \tilde{L}^{-1} is self-adjoint and bounded in \tilde{H} , then \tilde{L}^{-1} is self-adjoint and bounded in the spaces $\tilde{H}^{(+1)}$ and $\tilde{H}^{(-1)}$ (the \tilde{L}^{-1} operator is extended due to continuity in the space \tilde{H}^{-1}). The Hilbert space scales will be denoted by the symbols $\tilde{H}_\alpha^{(+1)}, \tilde{H}_\alpha = \tilde{H}_\alpha^0, \tilde{H}_\alpha^{(-1)}$ ($-\infty < \alpha < \infty$) [2], and they are generated by the operator \tilde{L} acting in $\tilde{H}^{(+1)}, \tilde{H}$ and $\tilde{H}^{(-1)}$, correspondingly. As it is known [2] for $\alpha \geq 0$ the operator \tilde{L} maps $\tilde{H}_{\alpha+1}^{(\beta)}$ into $\tilde{H}_\alpha^{(\beta)}$ ($\beta = -1, 0, 1$) continuously and bijectively, and \tilde{L} allows for continuous and bijective extension from $\tilde{H}_{-\alpha}^{(\beta)}$ to $\tilde{H}_{-\alpha-1}^{(\beta)}$ ($\alpha \geq 0$). We shall denote this extension as \tilde{L} in order not to introduce redundant denotations.

3. Main results

The operators A, \tilde{A} map $\tilde{H}_\alpha^{(+1)}$ into \tilde{H}_α , and \tilde{H}_α into $\tilde{H}_\alpha^{(-1)}$ ($\alpha \geq 0$) continuously and bijectively, respectively. The same theorem is true for $\alpha < 0$, and for the operators $\tilde{L}^\alpha A \tilde{L}^{-\alpha}$ and $\tilde{L}^\alpha \tilde{A} \tilde{L}^{-\alpha}$, which serve as the

extension of A and \tilde{A} , respectively. These extensions are also denoted by A and \tilde{A} .

Theorem 1. *For any element $u_0 \in \tilde{H}_\alpha^{(\beta)}$ ($\beta = +1, 0; -\infty < \alpha < \infty$) there exists one function $u(t)$ continuous in the closed interval $[0, T]$ in the space $\tilde{H}_{\alpha-2}^{(\beta-1)}$, continuous in the half open from the left side interval $(0, T]$ in $\tilde{H}_{\alpha-4}^{(\beta)}$, differentiable on $(0, T]$ in $\tilde{H}_{\alpha-4}^{(\beta-1)}$ and satisfying both the equation*

$$u'(t) = \tilde{A}u(t), \tag{3}$$

and the condition $u(0) = u_0$. (The continuity is understood in the strong sense in a suitable space.)

Observe that from the formulated theorem one can conclude that, in particular, for $u_0 \in \tilde{H}_4^{(+1)}$ the function $u(t)$ is a weak solution of the equation (2) (in the sense given in reference [3]).

The proof of Theorem 1 is based on the quoted below estimates of the resolvent of the operator A in a way similar to that presented in [3], and the Laplace inverse transformation.

Let $\lambda = \sigma + i\tau$. We define the operators $\Delta_1 = \Delta_1(\lambda)$, $\Delta_2 = \Delta_2(\lambda)$ in H in the following way:

$$\begin{aligned} \Delta_1 &= (\lambda^2 E + L^2) - \lambda L^2 C(M - \lambda E)^{-1} C^*, \\ \Delta_2 &= (M - \lambda E) - \lambda L^2 (L^2 + \lambda^2 E)^{-1} C^* C. \end{aligned}$$

Lemma 2. *If $\sigma = \text{Re}\lambda > 0$ then the operators Δ_1, Δ_2 have bounded inverses:*

$$\| \Delta_1^{-1} \| \leq \frac{1}{\sigma |\lambda|}, \quad \| \Delta_2^{-1} \| \leq \frac{1}{\sigma}.$$

Proof. Let $S = (1/\lambda)\Delta_1$. Then for any $f \in D(L^2)$ we have:

$$\begin{aligned} (Sf, f) &= (\sigma + i\tau)(f, f) + \frac{1}{|\lambda|^2}(\sigma - i\tau)(L^2 f, f) \\ &\quad - ((M - \sigma E + i\tau E) | M - \lambda E |^{-1} LC^* f, | M - \lambda E |^{-1} LC^* f). \end{aligned}$$

It results from this and from the negative definition of M that $\text{Re}(Sf, f) \geq \sigma \| f \|^2$, and therefore $\| Sf \| \geq \sigma \| f \|$. Analogously we get $\| S^*g \| \geq \sigma \| g \|$. This is why the operator S^{-1} exists, is bounded, defined on the whole H , and $\| S^{-1} \| \leq 1/\sigma$.

The equation

$$-\lambda L^2 (L^2 + \lambda^2 E)^{-1} = -L^2 (\lambda L^2 + | \lambda |^2 \lambda E) | L^2 + \lambda^2 E |^{-2}$$

yields $\operatorname{Re}(-\lambda L^2(L^2 + \lambda^2 E)^{-1} C^* C g, g) \leq 0$ ($g \in H$). From the above and taking into account the negative definition of the operator M we get: $\operatorname{Re}(\Delta_2 f, f) \leq -\sigma(f, f)$ ($f \in D(\Delta_2)$). This is why $|\operatorname{Re}(\Delta_2 f, f)| \geq \sigma \|f\|^2$, which yields $\|\Delta_2 f\| \geq \sigma \|f\|$. Analogously, we get $\|\Delta_2^* f\| \geq \sigma \|f\|$. Two last inequalities prove the lemma on the operator Δ_2 . \square

Lemma 3. *If $\operatorname{Re}\lambda \geq 0$ then the operator A possesses the resolvent $R(\lambda)$ which is defined by the matrix*

$$R(\lambda) = \begin{pmatrix} -\lambda \Delta_1^{-1} + L^2 \Delta_1^{-1} C(M - \lambda E)^{-1} C^* & -\Delta_1^{-1} & -\Delta_1^{-1} LC(M - \lambda E)^{-1} \\ L^2 \Delta_1^{-1} & -\lambda \Delta_1^{-1} & -\lambda \Delta_1^{-1} LC(M - \lambda E)^{-1} \\ -L^3(L^2 + \lambda^2 E)^{-1} \Delta_2^{-1} C^* & \lambda L(L^2 + \lambda^2 E)^{-1} \Delta_2^{-1} C^* & \Delta_2^{-1} \end{pmatrix}.$$

Proof. By direct check and taking into account the commutativity of L and M^{-1} one can verify that the expression $R(\lambda)(A_0 - \lambda E)$ gives a unit operator matrix. Besides, the space of values of the operator $A_0 - \lambda E$ is dense in \tilde{H} . Indeed, let such the element $x = (x_1, x_2, x_3) \in \tilde{H}$ occurs, that $(A_0 - \lambda E)^* x = 0$. This equation is equivalent to the following three:

$$-\bar{\lambda}x_1 - L^2x_2 = 0, \quad x_1 - \bar{\lambda}x_2 + CLx_3 = 0, \quad -C^*Lx_2 + (M - \bar{\lambda}E)x_3 = 0.$$

Finding x_1 from the first equation, x_3 from the last one and by substituting in the second one, we get $\Delta_1(\bar{\lambda})x_2 = 0$. It results from Lemma 2 that $x_2 = 0$, and therefore $x_1 = x_3 = 0$. This is why $x = 0$. \square

Lemma 4. *If $\sigma = \operatorname{Re}\lambda > 0$ are large enough the following estimations hold*

$$\|R(\lambda)\| \leq k \frac{|\lambda|}{\sigma}, \quad (4)$$

$$\|R(\lambda)\tilde{L}^{-2}\| \leq k \frac{1}{|\lambda|}. \quad (5)$$

Here and further k denotes constant (independent of λ), but different, generally speaking, for different inequalities.

Proof. First we are going to prove the inequality

$$\|L^2\Delta_1^{-1}\| \leq \frac{|\lambda|}{\sigma}. \quad (6)$$

As in the proof of Lemma 2, if $g \in H$, the following holds

$$\begin{aligned} (SL^{-2}g, g) &= (\sigma + i\tau)(L^{-2}g, g) + \frac{1}{|\lambda|^2}(\sigma - i\tau)(g, g) \\ &\quad - ((M - \sigma E - i\tau E) | M - \lambda E |^{-1} C^*g, | M - \lambda E |^{-1} C^*g). \end{aligned}$$

Then $\operatorname{Re}(SL^{-2}g, g) \geq \sigma/|\lambda|^2$. Thus (as in Lemma 2) we see that $\|L^2S^{-1}\| \leq |\lambda|^2/\sigma$ and this is why (6) is satisfied.

Let us denote the elements of operator matrix defining $R(\lambda)$ by a_{ij} . Then:

$$\begin{aligned} \|a_{11}\| &\leq \frac{k}{\sigma}, & \|a_{12}\| &\leq \frac{1}{|\lambda|\sigma}, & \|a_{13}\| &\leq \frac{k}{|\lambda|\sigma}, \\ \|a_{21}\| &\leq \frac{|\lambda|}{\sigma}, & \|a_{22}\| &\leq \frac{1}{\sigma}, & \|a_{23}\| &\leq \frac{k}{\sigma}, \\ \|a_{11}L^{-2}\| &\leq \frac{k}{|\lambda|}, & \|a_{12}L^{-2}\| &\leq \frac{k}{|\lambda|\sigma}, & \|a_{13}L^{-2}\| &\leq \frac{k}{|\lambda|^2\sigma}, \\ \|a_{21}L^{-2}\| &\leq \frac{1}{|\lambda|\sigma}, & \|a_{22}L^{-2}\| &\leq \frac{k}{|\lambda|}, & \|a_{23}L^{-2}\| &\leq \frac{k}{|\lambda|\sigma}. \end{aligned}$$

The estimations of elements of the first two rows of the matrix $R(\lambda)$ immediately result from (6) of Lemma 2, the inequalities $\|L(M - \lambda E)^{-1}\| \leq k$, and

$$\|(M - \lambda E)^{-1}\| \leq \frac{1}{|\lambda|}, \tag{7}$$

and also from the following two expressions

$$a_{11} = -\frac{1}{\lambda}E - \frac{1}{\lambda}L^2\Delta_1^{-1}, \quad a_{22}L^{-2} = -\frac{1}{\lambda}L^{-2} + \frac{1}{\lambda}\Delta_1^{-1} + \Delta_1^{-1}C(M - \lambda E)^{-1}C^*,$$

which can be verified using the elementary transformations.

In order to evaluate the elements of the third row of the matrix $R(\lambda)$ we shall add some supporting inequalities.

Because for any $f \in D(L^2)$ we have $\operatorname{Re}((L^2/\lambda + \lambda E)f, f) \geq \sigma(f, f)$, then $\|(L^2/\lambda + \lambda E)^{-1}\| \leq 1/\sigma$. This yields

$$\|(L^2 + \lambda^2 E)^{-1}\| \leq \frac{1}{|\lambda|\sigma}. \tag{8}$$

From this inequality (using the elementary transformations) we get

$$\|L^2(L^2 + \lambda^2 E)^{-1}\| \leq \frac{|\lambda|}{\sigma}. \tag{9}$$

We shall prove that

$$\|L\Delta_2^{-1}\| \leq m, \tag{10}$$

where m does not depend on λ . For any $f \in D(\Delta_2L^{-1})$ we have

$$\begin{aligned} (\Delta_2L^{-1}f, f) &= (ML^{-1}f, f) - (\sigma + i\tau)(L^{-1}f, f) - (\sigma - i\tau)|\lambda|^2(Lg, g) \\ &\quad - (\sigma + i\tau)(L^3g, g), \end{aligned}$$

where $g = \sqrt{|L^2 + \lambda^2 E|^{-2} C^* C} f$. It results from the negative definition of L that

$$\operatorname{Re}(\Delta_2L^{-1}f, f) \geq \operatorname{Re}(ML^{-1}f, f) \geq k \|f\|^2.$$

The last inequality results from the positive definition of the limited operator LM^{-1} , inverted to ML^{-1} . Therefore, the inequality (10) has been satisfied. The equation

$$\Delta_2^{-1}L^{-1} = (M - \lambda E)^{-1}L^{-1} + (L\Delta_2^{-1})(L^2 + \lambda^2 E)^{-1}C^*C\lambda(M - \lambda E)^{-1}$$

and the inequalities (7), (8), (10) yield

$$\|\Delta_2^{-1}L^{-1}\| \leq \frac{k}{|\lambda|}. \quad (11)$$

Now the inequalities (8), (9), (10), (11) and Lemma 2 allow to obtain the estimation of the elements of the third row of the matrix $R(\lambda)$:

$$\begin{aligned} \|a_{31}\| &\leq k \frac{|\lambda|}{\sigma}, & \|a_{32}\| &\leq \frac{k}{\sigma}, & \|a_{33}\| &\leq \frac{1}{\sigma}, \\ \|a_{31}L^{-2}\| &\leq \frac{k}{|\lambda|\sigma}, & \|a_{32}L^{-2}\| &\leq \frac{k}{|\lambda|\sigma}, & \|a_{33}L^{-2}\| &\leq \frac{k}{|\lambda|}. \end{aligned}$$

The proof of Lemma results from the obtained a_{ij} estimates.

Let us notice that some evaluations can be even more improved, for instance $\|a_{12}L^{-2}\| \leq k/|\lambda|^2$. However, they will be not used further.

Now we prove directly the theorem. Following reference [3] we are looking for the solution of the equation (2) in the form

$$u(t) = -\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} R(\lambda) u(0) d\lambda,$$

where the integration takes place along the straight line, parallel to imaginary axis and passing through the point $(\sigma, 0)$, where $\sigma > 0$ is sufficiently large.

We assume first that $u_0 \in \tilde{H}_4^{(+1)}$, i.e. $u_0 = R(0)\tilde{L}^{-4}z_0$ ($z_0 \in \tilde{H}$), and we consider the function

$$\begin{aligned} u(t) &= -\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} R(\lambda) R(0)\tilde{L}^{-4}z_0 d\lambda \\ &= -\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} \left(\frac{1}{\lambda} R(\lambda)\tilde{L}^{-4}z_0 - \frac{1}{\lambda} R(0)\tilde{L}^{-4}z_0 \right) d\lambda. \end{aligned} \quad (12)$$

Here, while passing from the first integral to the second one, the resolvent identity has been used. The existence of the integral, the continuity of $u(t)$ on $[0, T]$ in the space \tilde{H}_{+2} (and correspondingly, in \tilde{H}) and the identity $u(0) = R(0)\tilde{L}^{-4}z_0$ result directly from the Lemma 4.

Applying the integration by parts to the first integral in (12) and taking into consideration Lemma 4 we get

$$u(t) = \frac{1}{2\pi i} \frac{1}{t} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} R^2(\lambda) R(0) \tilde{L}^{-4} z_0 d\lambda.$$

It results from the above that $u(t)$ is continuous on $(0, T]$ in the space $\tilde{H}^{(+1)}$ and it possesses the continuous derivative on $(0, T]$ in \tilde{H}

$$\begin{aligned} u'(t) = & -\frac{1}{2\pi i} \frac{1}{t^2} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} R^2(\lambda) R(0) \tilde{L}^{-4} z_0 d\lambda \\ & + \frac{1}{2\pi i} \frac{1}{t} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} \lambda R^2(\lambda) R(0) \tilde{L}^{-4} z_0 d\lambda, \end{aligned}$$

and satisfies the equation (2).

The uniqueness of the solution of the Cauchy problem for the equation (2) follows from the estimation (4) and the results presented in reference [3].

Now, let $u_0 \in \tilde{H}_\alpha^{(\beta)}$ ($\beta = +1, 0; -\infty < \alpha < \infty$). Then $x_0 = A^{\beta-1} \tilde{L}^{\alpha-4} u_0 \in \tilde{H}_4^{(+1)}$ (here A^0 denotes E). Because the operators $R(\lambda)$ and \tilde{L}^α commute and from the previous considerations we get the function

$$u(t) = -\frac{1}{2\pi i} \tilde{A}^{1-\beta} \tilde{L}^{4-\alpha} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} R(\lambda) x_0 d\lambda$$

with all properties mentioned in the formulated theorem. The uniqueness of such function results from the fact that the function $A^{\beta-1} \tilde{L}^{\alpha-4} u(t)$ is the solution of the equation (2). Theorem 1 has been proved. □

Remark 1. It results from the above proof that in order to solve the equations (2) and (3), the estimation $\| u(t) \|_{\tilde{H}_\alpha^{(\beta-1)}} \leq k \| u(0) \|_{\tilde{H}_{\alpha+2}^{(\beta)}}$ ($\beta = +1, 0; -\infty < \alpha < \infty$) holds, where k does not depend on t . It defines the continuous dependence of solution on the initial data in the corresponding spaces.

Let us pass to the consideration of the non-homogeneous equations

$$u'(t) = \tilde{A}u(t) + f(t). \tag{13}$$

By $U(t)$ we denote the operator

$$U(t)x = -\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} R(\lambda) x d\lambda, \quad 0 \leq t \leq T,$$

defined for all x , for which the last integral exists. The Lemma 4 and the equation

$$U(t)R(0)\tilde{L}^{-2}z = -\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} \frac{1}{\lambda} (R(\lambda) - R(0))\tilde{L}^{-2}z d\lambda$$

imply that for every fixed t , the operator $U(t): \tilde{H}_{\alpha}^{(\beta)} \rightarrow \tilde{H}_{\alpha-2}^{(\beta-1)}$ ($\beta = +1, 0; -\infty < \alpha < \infty$) is bounded and the operator function $U(t)$ is strongly continuous on $[0, T]$. For any $u_0 \in \tilde{H}_{\alpha}^{(\beta)}$, the function $u(t) = U(t)u_0$ has the properties given in Theorem 1.

Theorem 2. *Let for any $t \in [0, T]$, $f(t)$ belongs to the domain of the operator $\tilde{L}^{\alpha}A^{1+\beta}$ ($\beta = +1, 0; -\infty < \alpha < \infty$) and let the function $\tilde{L}^{\alpha}A^{1+\beta}f(t)$ be continuous in \tilde{H} . Then the function*

$$u(t) = \int_0^t U(t-s)f(s)ds \tag{14}$$

is continuous on $[0, T]$ in the space $\tilde{H}_{\alpha-2}^{(\beta)}$, is differentiable on $[0, T]$ in the space $\tilde{H}_{\alpha-2}^{(\beta-1)}$ and satisfies the equation (13).

Proof. Let $g(t) = \tilde{L}^{\alpha}A^{1+\beta}f(t)$. Following the steps given in the reference [6] one can conclude that the function (14) and its derivative have the form

$$\begin{aligned} u(t) &= \int_0^t U(t-s)A^{-1-\beta}\tilde{L}^{-\alpha}g(s)ds, \\ u'(t) &= A^{-1-\beta}\tilde{L}^{-\alpha}g(t) + \int_0^t U'_t(t-s)A^{-1-\beta}\tilde{L}^{-\alpha}g(s)ds \\ &= f(t) + \int_0^t \tilde{A}U(t-s)A^{-1-\beta}\tilde{L}^{-\alpha}g(s)ds = f(t) + \tilde{A}u(t). \end{aligned}$$

The existence of integrals in these equations results from the continuity of g and the properties of $U(t)$. In what follows $u(t)$ is the solution of the equation (13). The theorem has been proved. \square

Remark 2. The continuity condition of the function $\tilde{L}^{\alpha}A^{1+\beta}$ can be weakened if the function f is smooth enough.

Finally, we acknowledge the recent developments of a similar problem considered in our work pointed out by one of the reviewers and presented in references [6], [4] (see also the extended bibliography therein). However, in the mentioned works there is no non-local term C but there appears a mixed fourth order term accounting for the rotational forces. In addition, in reference [6] the stability properties are considered using the boundary feedback modifications, which are not considered in our work. In the works of Lasiecka (see [4] and the cited literature therein) the asymptotic behaviour governed by von Kármán equations is analysed which is out of the content of our paper.

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