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ON THE ORDER STRUCTURE OF TIME PROJECTION

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Abstract. The order structure of time projections associated with random times in a von Neumann algebra is investigated in the general setup as well as that of the CAR and CCR algebras. In the second case various additional properties (such as e.g. the upper/lower continuity) of the lattice of time projections are also discussed.

0. Introduction

In this paper we investigate the order structure of time projections associated with random times in an arbitrary non-commutative filtration of a von Neumann algebra as well as those employed in quantum stochastic theory of the canonical anticommutation relations (CAR) and the canonical commutation relations (CCR) (cf. [4], [9]). Within the context of certain quasi-free representations of the CAR and CCR, we give an answer to a question posed in [3] — the dual of Theorem 1.12 of [3] (see Theorem 2.7 and Theorem 3.5 in this paper, which can be considered as a partial answer to that question for an arbitrary non-commutative filtration).

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Section 2 contains a brief review of random times and the associated time projections. The structure of time projections is discussed within an abstract setup.

In Section 3 we specialize to the quantum stochastic theory of the CAR, although all of the results have analogues within the theory of the CCR. Various properties, of the lattice of time projections (in particular, the upper/lower continuity) are discussed in this context.

1. Notation and preliminaries

Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ — the bounded linear operators on $\mathcal{H}$, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ — a von Neumann algebra, and let $(\mathcal{A}_t)$, $t \in \mathbb{R}^+$, be an increasing, right continuous family of von Neumann subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_\infty$ is generated by the collection $\{\mathcal{A}_t \colon t \in [0, \infty)\}$. We also suppose that there is a cyclic and separating unit vector Ω for $\mathcal{A}$ in $\mathcal{H}$, and that there is a family (E_t) of normal ω-invariant conditional expectations $E_t \colon \mathcal{A} \to \mathcal{A}_t$, where ω is the vector state induced by Ω. If we denote the closure of $\mathcal{A}_t\Omega$ in $\mathcal{H}$ by $\mathcal{H}_t$, and the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_t$ by P_t, we have

$$P_t(a\Omega) = E_t(a)\,\Omega$$

for any $a \in \mathcal{A}$. Furthermore, since $\mathcal{H}_t$ is invariant under $\mathcal{A}_t$, it follows that $P_t \in \mathcal{A}_t'$ (see [1], [2] for a more detailed description).

Let us now recall some basic notions from lattice theory which will be employed in the sequel. For simplicity and since this is all we need, we restrict attention to lattices of orthogonal projections in a Hilbert space.

Definition 1.1 (cf. [11, 7.7 p. 152])**.** Let $\mathcal{L}$ be a lattice of projections acting on a Hilbert space.

(i) $\mathcal{L}$ is said to be *modular*, if for each $p, q, r \in \mathcal{L}$ such that $p \le r$ we have $(p \vee q) \wedge r = p \vee (q \wedge r)$.

(ii) $\mathcal{L}$ is said to be *upper* (resp. *lower*) *continuous*, if for each $q \in \mathcal{L}$ and each increasingly (resp. decreasingly) directed set $\{p_i \colon i \in I\} \subset \mathcal{L}$ such that $\bigvee_{i\in I} p_i \in \mathcal{L}$ (resp. $\bigwedge_{i\in I} p_i \in \mathcal{L}$) we have $\bigvee_{i\in I}(p_i \wedge q) = \left(\bigvee_{i\in I} p_i\right) \wedge q$ (resp. $\bigwedge_{i\in I}(p_i \vee q) = \left(\bigwedge_{i\in I} p_i\right) \wedge q$).

2. Random times and time projections

We recall the definition and elementary properties of a random time and its associated time projection. For more details the reader is referred to [1], [2], [3].

Definition 2.1. A *random time*, τ, is an increasing family of projections $\tau = (q_t)$, $t \in [0, \infty]$, where $q_t \in \mathcal{A}_t$, $q_0 = 0$ and $q_\infty = \mathbf{1}$. A random time $\tau = (q_t)$ is called *simple*, if it assumes only finitely many distinct values.

Let Θ denote the set of all finite partitions of $[0, \infty]$. Then, for $\theta \in \Theta$, say $\theta = \{0 = t_0 < t_1 < ... < t_n = \infty\}$, the simple random time associated with τ and θ is given by $\tau(\theta) = (q_t^\theta)$, where

$$q_t^\theta = \sum_{i=0}^{n-1} q_{t_i} \chi_{[t_i, t_{i+1})}(t)$$

for $t \in [0, \infty)$, and $q_\infty^\theta = \mathbf{1}$.

Definition 2.2. (i) Let $\tau = (q_t)$ and $\sigma = (q_t')$ be random times. We say that $\tau \leq \sigma$, if $q_t' \leq q_t$ for each $t \in \mathbb{R}^+$. We define $\tau \wedge \sigma$ and $\tau \vee \sigma$ to be the random times $\tau \wedge \sigma = (q_t \vee q_t')$ and $\tau \vee \sigma = (q_t \wedge q_t')$. In a similar fashion, for any family Λ of random times, we define $\sup \Lambda$ and $\inf \Lambda$ as the random times consisting respectively of infima and suprema of the corresponding projections.

(ii) Let $\theta = \{0 = t_0 < t_1 < ... < t_n = \infty\} \in \Theta$. We define:

$$M_{\tau(\theta)} = \sum_{i=0}^{n-1} \left(q_{t_{i+1}} - q_{t_i}\right) P_{t_{i+1}} = \sum_{i=0}^{n-1} \Delta q_{t_{i+1}} P_{t_{i+1}}.$$

$M_{\tau(\theta)}$ has the following properties (see Theorem 2.3 of [1]):

1. $M_{\tau(\theta)}$ is an orthogonal projection;
2. For $\theta, \eta \in \Theta$ with η finer than $\theta, M_{\tau(\eta)} \leq M_{\tau(\theta)}$;
3. If σ is another random time with $\tau \leq \sigma$, then $M_{\tau(\theta)} \leq M_{\sigma(\theta)}$ for each $\theta \in \Theta$.

These properties and the fact that Θ is a directed set ordered by inclusion, imply that $\{M_{\tau(\theta)} \colon \theta \in \Theta\}$ is a decreasing net of orthogonal projections. Hence there exists a unique orthogonal projection

$$M_\tau = \bigwedge_{\theta \in \Theta} M_{\tau(\theta)};$$

moreover,

$$M_{\tau(\theta)} \searrow M_\tau$$

in the strong operator topology as θ refines. We shall call M_τ the *time projection* for the random time τ (Definition 2.4 of [1]). The next result summarises what is known about the order structure of random times.

Let τ, σ be random times. For $\theta, \eta \in \Theta$ we have

$$M_{\tau(\theta)} \vee M_{\sigma(\eta)} = M_{\tau(\theta) \vee \sigma(\eta)} \quad \text{and} \quad M_{\tau(\theta)} \wedge M_{\sigma(\eta)} = M_{\tau(\theta) \wedge \sigma(\eta)}.$$

Also

$$M_\tau \wedge M_\sigma = M_{\tau\wedge\sigma},$$

so that, in particular, if $\sigma \leq \tau$ then $M_\sigma \leq M_\tau$ (*Optional Stopping Theorem*). The complete proofs of these relations can be found in [2], [3]. It is not clear whether the corresponding result for suprema ($M_\tau \vee M_\sigma = M_{\tau\vee\sigma}$) holds in general. As seen in [2], this relation is true in the quasi-free representations of the CAR and the CCR, where one has integral formulae for the various time projections. In this section we observe that it is true when $M_\tau^\perp$ and $M_\sigma^\perp$ are finite projections in the time algebra $\mathcal{T} = \{M_\tau \colon \tau \text{ is a random time}\}''$. This will show that it is not necessary to assume that the time algebra is finite, as in Theorem 3.18 of [5], in order to prove that $M_\tau \vee M_\sigma = M_{\tau\vee\sigma}$.

Proposition 2.3. *For any random times τ, σ with $M_\tau^\perp$ and $M_\sigma^\perp$ finite, we have*

$$M_\tau \vee M_\sigma = M_{\tau\vee\sigma}.$$

Proof. We first note that

$$M_\tau \vee M_\sigma = (\bigwedge_\theta M_{\tau(\theta)}) \vee M_\sigma = \left[(\bigvee_\theta M_{\tau(\theta)}^\perp) \wedge M_\sigma^\perp\right]^\perp,$$

and since $M_\tau^\perp = \bigvee_\theta M_{\tau(\theta)}^\perp$ is finite, $\{M_{\tau(\theta)}^\perp : \theta \in \Theta\}$ is an increasing net of finite projections in the time algebra $\mathcal{T}$. Then, by Corollary 7.6 of [11],

$$(\bigvee_\theta M_{\tau(\theta)}^\perp) \wedge M_\sigma^\perp = \bigvee_\theta (M_{\tau(\theta)}^\perp \wedge M_\sigma^\perp),$$

and hence

$$M_\tau \vee M_\sigma = \left[\bigvee_\theta \left(M_{\tau(\theta)}^\perp \wedge M_\sigma^\perp\right)\right]^\perp = \bigwedge_\theta \left(M_{\tau(\theta)} \vee M_\sigma\right).$$

Similarly,

$$\bigwedge_\theta \left(M_{\tau(\theta)} \vee M_\sigma\right) = \bigwedge_\theta \bigwedge_\eta \left(M_{\tau(\theta)} \vee M_{\sigma(\eta)}\right).$$

Thus

$$M_\tau \vee M_\sigma = \bigwedge_\theta \bigwedge_\eta \left(M_{\tau(\theta)} \vee M_{\sigma(\eta)}\right),$$

and since $M_{\tau\vee\sigma} \leq M_{\tau(\theta)\vee\sigma(\eta)} = M_{\tau(\theta)} \vee M_{\sigma(\eta)}$ for $\theta, \eta \in \Theta$, we have

$$M_{\tau\vee\sigma} \leq \bigwedge_\theta \bigwedge_\eta \left(M_{\tau(\theta)} \vee M_{\sigma(\eta)}\right) = M_\tau \vee M_\sigma.$$

But we know that $M_\tau \vee M_\sigma \leq M_{\tau\vee\sigma}$ because $M_\tau \leq M_{\tau\vee\sigma}$ and $M_\sigma \leq M_{\tau\vee\sigma}$. Combining both inequalities, we get $M_\tau \vee M_\sigma = M_{\tau\vee\sigma}$, as required. □

We now turn to the order structure of the family of the time projections and give a partial answer to the question: is the map $\tau \mapsto M_\tau$ an order continuous lattice morphism, that is for any family $\mathcal{F} = \{M_\tau \colon \tau \in \Lambda\}$ of time projections, is $\bigvee_\tau M_\tau = M_{\sup \Lambda}$? As we shall see in the next section, this is true in the quasi-free representations of the CAR and CCR. According to Theorem 1.12 of [3], the corresponding result for infima is true in general; in this section we shall give a simple proof of this fact.

Let $\{\tau_\alpha\} = \{(q_t^{(\alpha)})\}$ be a net of random times, and let $\tau = (q_t)$ be a random time. τ_α is said to *converge strongly* to τ, if $q_t^{(\alpha)} \to q_t$ strongly for each $t \in \mathbb{R}^+$.

Lemma 2.4. *Let $\{\tau_\alpha\}$ be a net of random times converging strongly to a random time τ. Then $M_{\tau_\alpha(\theta)}$ converges strongly to $M_{\tau(\theta)}$ for each $\theta \in \Theta$.*

Proof. Suppose that $\tau_\alpha = (q_t^{(\alpha)})$ and $\tau = (q_t)$. By assumption, $q_t^{(\alpha)} \to q_t$ strongly for each $t \in \mathbb{R}^+$. Let $\theta \in \Theta$. Then

$$M_{\tau_\alpha(\theta)} = \sum_{i=0}^{n-1} \left(q_{t_{i+1}}^{(\alpha)} - q_{t_i}^{(\alpha)} \right) P_{t_{i+1}} \to \sum_{i=0}^{n-1} \left(q_{t_{i+1}} - q_{t_i} \right) P_{t_{i+1}} = M_{\tau(\theta)}.$$

□

Proposition 2.5. *Let $\{\tau_\alpha \colon \alpha \in \Lambda\}$ be a family of random times, and let τ be the supremum of this family. Then for each $\theta \in \Theta$,*

$$M_{\tau(\theta)} = \bigvee_\alpha M_{\tau_\alpha(\theta)}.$$

Proof. We have

$$\bigvee_\alpha M_{\tau_\alpha(\theta)} = \bigvee_J M_J^\theta = \lim_J M_J^\theta,$$

where $J = \{\alpha_1, ..., \alpha_n\}$ is a finite subset of Λ, and

$$M_J^\theta = M_{\tau_{\alpha_1(\theta)}} \vee ... \vee M_{\tau_{\alpha_n(\theta)}}.$$

Moreover,

$$M_J^\theta = M_{\tau_{\alpha_1}(\theta) \vee ... \vee \tau_{\alpha_n}(\theta)} = M_{\tau_J(\theta)},$$

where

$$\tau_J(\theta) = \tau_{\alpha_1(\theta)} \vee ... \vee \tau_{\alpha_n(\theta)}.$$

Since

$$\lim_J \tau_J(\theta) = \tau(\theta),$$

we get on account of Lemma 2.4

$$M_{\tau(\theta)} = \lim_J M_{\tau_J(\theta)} = \lim_J M_J^\theta,$$

which finishes the proof. □

Note that the corresponding result for infima is also true, i.e. $M_{\tau(\theta)} = \bigwedge_\alpha M_{\tau_\alpha(\theta)}$, where $\{\tau_\alpha\}$ is a family of random times and $\tau = \inf_\alpha \tau_\alpha$. Using this fact, we give below a simple proof of Theorem 1.12 of [3].

Theorem 2.6. *Let $\{\tau_\alpha\}$ be a family of random times. Let τ be the infimum of this family. Then*

$$M_\tau = \bigwedge_\alpha M_{\tau_\alpha}.$$

Proof. We have

$$M_\tau = \bigwedge_\theta M_{\tau(\theta)} = \bigwedge_\theta \bigwedge_\alpha M_{\tau_\alpha(\theta)} = \bigwedge_\alpha \bigwedge_\theta M_{\tau_\alpha(\theta)} = \bigwedge_\alpha M_{\tau_\alpha}.$$

□

It is an interesting open problem whether the corresponding result for suprema holds true. Below we show it under some additional assumptions.

Theorem 2.7. *Suppose that the lattice of time projections in $\mathcal{T}$ satisfies $\bigvee_n M_{\tau_n} = M_{\sup_n \tau_n}$ for any countable family of time projections, and assume that $\mathcal{H}$ is a separable Hilbert space. Let $\{\tau_\alpha\}$ be a family of random times, and let τ be the supremum of this family. Then*

$$M_\tau = \bigvee_\alpha M_{\tau_\alpha}.$$

Proof. Since $\mathcal{H}$ is separable, the strong operator topology on the closed unit ball of $\mathcal{B}(\mathcal{H})$ is metrizable (see e.g. [12, Proposition 2.7 p. 71]). Hence there is an increasing sequence $\{M_{\tau_n}\}$ in $\{M_{\tau_\alpha}\}$ converging strongly to $\bigvee_\alpha M_{\tau_\alpha}$. This means that,

$$\bigvee_\alpha M_{\tau_\alpha} = \bigvee_n M_{\tau_n} = M_{\sup_n \tau_n} \quad \text{and} \quad M_{\tau_\alpha} \leq M_{\sup_n \tau_n} \quad \text{for each } \alpha.$$

Hence for each $\theta \in \Theta$ and each α, $M_{\tau_\alpha(\theta)} \leq M_{(\sup_n \tau_n)(\theta)}$. So $\bigvee_\alpha M_{\tau_\alpha(\theta)} \leq M_{(\sup_n \tau_n)(\theta)}$. From Proposition 2.5, $M_{\tau(\theta)} = \bigvee_\alpha M_{\tau_\alpha(\theta)}$ for each $\theta \in \Theta$. Thus

$$\begin{aligned} M_\tau &= \bigwedge_\theta M_{\tau(\theta)} = \bigwedge_\theta \bigvee_\alpha M_{\tau_\alpha(\theta)} \leq \bigwedge_\theta M_{(\sup_n \tau_n)(\theta)} = M_{\sup_n \tau_n} = \bigvee_n M_{\tau_n} \\ &= \bigvee_\alpha M_{\tau_\alpha}, \end{aligned}$$

so $M_\tau \leq \bigvee_\alpha M_{\tau_\alpha}$, but on the other hand $\bigvee_\alpha M_{\tau_\alpha} \leq M_\tau$. Therefore $M_\tau = \bigvee_\alpha M_{\tau_\alpha}$, as required. □

Hereafter we consider a filtration of a finite von Neumann algebra. We want to investigate the lattice of time projections in greater detail. For $\theta \in \Theta$, let $\mathcal{T}^\theta$ denote the von Neumann algebra generated by $\{M_{\tau(\theta)}\colon\ \tau$ is a random time$\}$.

Proposition 2.8. *The lattice of time projections in $\mathcal{T}^\theta$ is both upper and lower continuous.*

Proof. Let $\{M_{\tau_\alpha(\theta)}\}$ be an increasingly directed family of time projections in $\mathcal{T}^\theta$, and let $M_{\sigma(\theta)}$ be any time projection in $\mathcal{T}^\theta$. Then

$$\begin{aligned}
\bigvee_\alpha \big(M_{\tau_\alpha(\theta)} \wedge M_{\sigma(\theta)}\big) &= \bigvee_\alpha M_{\tau_\alpha(\theta)\wedge\sigma(\theta)} \qquad = \bigvee_\alpha M_{(\tau_\alpha\wedge\sigma)(\theta)} \\
&= M_{\sup_\alpha(\tau_\alpha\wedge\sigma)(\theta)} \qquad \text{(by Proposition 2.5)} \\
&= M_{(\sup_\alpha \tau_\alpha)(\theta)\wedge\sigma(\theta)} \qquad \text{(by Theorem 1.9 of [3])} \\
&= M_{(\sup_\alpha \tau_\alpha)(\theta)} \wedge M_{\sigma(\theta)} \\
&= \big(\bigvee_\alpha M_{\tau_\alpha(\theta)}\big) \wedge M_{\sigma(\theta)} \qquad \text{(by Proposition 2.5).}
\end{aligned}$$

To show lower continuity, suppose $\{M_{\tau_\alpha(\theta)}\}$ is a decreasingly directed family of time projections in $\mathcal{T}^\theta$. Then

$$\begin{aligned}
\bigwedge_\alpha \big(M_{\tau_\alpha(\theta)} \vee M_{\sigma(\theta)}\big) &= \bigwedge_\alpha M_{\tau_\alpha(\theta)\vee\sigma(\theta)} \qquad = \bigwedge_\alpha M_{(\tau_\alpha\vee\sigma)(\theta)} \\
&= M_{\inf_\alpha(\tau_\alpha\vee\sigma)(\theta)} \qquad \text{(by Proposition 2.5 for infima)} \\
&= M_{(\inf_\alpha \tau_\alpha(\theta))\vee\sigma(\theta)} \qquad \text{(by Theorem 1.9 of [3])} \\
&= M_{(\inf_\alpha \tau_\alpha)(\theta)\vee\sigma(\theta)} \qquad = M_{\inf_\alpha \tau_\alpha(\theta)} \vee M_{\sigma(\theta)} \\
&= \big(\bigwedge_\alpha M_{\tau_\alpha(\theta)}\big) \vee M_{\sigma(\theta)} \qquad \text{(by Proposition 2.5 for infima),}
\end{aligned}$$

as required. □

Proposition 2.9. *Let τ and σ be random times. Then*

$$M_{\tau\vee\sigma} = \bigwedge_\eta \bigwedge_\theta \big(M_{\tau(\theta)} \vee M_{\sigma(\eta)}\big)\,.$$

Proof. Let $\eta \in \Theta$. Then

$$\begin{aligned}
\bigwedge_{\theta} \left(M_{\tau(\theta)} \vee M_{\sigma(\eta)}\right) &= \bigwedge_{\theta} M_{\tau(\theta)\vee\sigma(\eta)} \\
&= M_{\inf_\theta(\tau(\theta)\vee\sigma(\eta))} \quad \text{(by Theorem 2.6)} \\
&= M_{(\inf_\theta \tau(\theta))\vee\sigma(\eta)} \quad \text{(by Theorem 1.9 of [3])} \\
&= M_{\tau\vee\sigma(\eta)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\bigwedge_{\eta} M_{\tau\vee\sigma(\eta)} &= M_{\inf_\eta(\tau\vee\sigma(\eta))} \quad \text{(by Theorem 2.6)} \\
&= M_{\tau\vee(\inf_\eta \sigma(\eta))} \quad \text{(by Theorem 1.9 of [3])} \\
&= M_{\tau\vee\sigma}.
\end{aligned}$$

So

$$M_{\tau\vee\sigma} = \bigwedge_{\eta}\bigwedge_{\theta} \left(M_{\tau(\theta)} \vee M_{\sigma(\eta)}\right).$$

□

For general time projections we have the following result.

Proposition 2.10. *The lattice of time projections is lower continuous if and only if* $M_\tau \vee M_\sigma = M_{\tau\vee\sigma}$ *for any two random times* τ *and* σ*.*

Proof. Suppose that $M_\tau \vee M_\sigma = M_{\tau\vee\sigma}$ for any two random times. Let $\{M_{\tau_\alpha}\}$ be a decreasingly directed family of time projections, and let M_σ be the time projection associated with random time σ. Then

$$\bigwedge_{\alpha} (M_{\tau_\alpha} \vee M_\sigma) = \bigwedge_{\alpha} M_{\tau_\alpha\vee\sigma}.$$

We may assume that $\{\tau_\alpha\}$ is a decreasingly directed family of random times with $\tau = \inf_\alpha \tau_\alpha$. The family $\{\tau_\alpha \vee \sigma\}$ is also decreasingly directed with $\inf_\alpha (\tau_\alpha \vee \sigma) = (\inf_\alpha \tau_\alpha) \vee \sigma$ (Theorem 1.9 of [3]). Thus

$$\begin{aligned}
\bigwedge_{\alpha} (M_{\tau_\alpha} \vee M_\sigma) = \bigwedge_{\alpha} M_{\tau_\alpha\vee\sigma} &= M_{\inf_\alpha(\tau_\alpha\vee\sigma)} \quad \text{(by Theorem 2.6)} \\
&= M_{(\inf_\alpha \tau_\alpha)\vee\sigma} \quad \text{(by Theorem 1.9 of [3])} \\
&= M_{\inf_\alpha \tau_\alpha} \vee M_\sigma \quad \text{(by assumption)} \\
&= (\bigwedge_{\alpha} M_{\tau_\alpha}) \vee M_\sigma \quad \text{(by Theorem 2.6).}
\end{aligned}$$

Conversely, suppose that the family of time projections is lower continuous. Let τ and σ be any two random times. Then

$$M_\tau \vee M_\sigma = (\bigwedge_\theta M_{\tau(\theta)}) \vee M_\sigma = \bigwedge_\theta \left(M_{\tau(\theta)} \vee M_\sigma\right) = \bigwedge_\theta \left(M_{\tau(\theta)} \vee (\bigwedge_\eta M_{\sigma(\eta)})\right)$$

$$= \bigwedge_\theta \bigwedge_\eta \left(M_{\tau(\theta)} \vee M_{\sigma(\eta)}\right) = M_{\tau\vee\sigma} \quad \text{(by Proposition 2.9).}$$

□

As seen from the second part of the above proof, lower continuity of the family of time projections yields the relation $M_\tau \vee M_\sigma = M_{\tau\vee\sigma}$ without the assumption of finiteness of the algebra. Indeed, the equality

$$M_\tau \vee M_\sigma = \bigwedge_\theta \bigwedge_\eta \left(M_{\tau(\theta)} \vee M_{\sigma(\eta)}\right)$$

follows solely from the lower continuity of the lattice of time projections; furthermore we have

$$M_{\tau(\theta)} \vee M_{\sigma(\eta)} = M_{\tau(\theta)\vee\sigma(\eta)} \geq M_{\tau\vee\sigma},$$

since $\tau \vee \sigma \leq \tau(\theta) \vee \sigma(\eta)$. Hence

$$M_\tau \vee M_\sigma \geq M_{\tau\vee\sigma},$$

while the converse inequality is obvious.

3. Times with respect to a filtration of the CAR algebra

In this section we restrict our attention to the CAR theory as developed in [4]. Let ω denote the gauge-invariant quasi-free state of the CAR algebra over $L^2(\mathbb{R}^+)$ with two-point function

$$\omega\left(b^*(f)b(g)\right) = \int_0^\infty f(s)\overline{g(s)}\rho(s)ds$$

where $0 < \rho < 1$ almost everywhere, ds denotes Lebesgue measure, and $f, g \in L^2(\mathbb{R}^+)$.

Let $\mathcal{A}$ denote the von Neumann algebra generated by the quasi-free representation of the CAR algebra on $\mathcal{H}$ — the tensor product of two copies of the anti-symmetric Fock space over $L^2(\mathbb{R}^+)$, given in [4] (see also [6]). Thus ω is the vector state $\omega(\cdot) = \langle\cdot\Omega, \Omega\rangle$, with $\Omega = \Omega_\circ \otimes \Omega_\circ$, where $\Omega_\circ$ is the Fock vacuum vector. The conditions on ρ imply that Ω is cyclic and separating for $\mathcal{A}$. For $t \in [0, \infty]$, let $\mathcal{A}_t$ denote the von Neumann subalgebra of $\mathcal{A}_\infty = \mathcal{A}$ generated by the operators $\{b^*(f)\colon \mathrm{supp} f \subset [0, t]\}$. Then there exist ω-invariant normal conditional expectations $E_t\colon \mathcal{A} \to \mathcal{A}_t$ satisfying $E_t\left(b(f)\right) = b(f\chi_{[0,t]})$, [7] has the details. Let $b_t = b(\chi_{[0,t]})$. Then

b_t and b_t^* are $\mathcal{A}$-valued martingales and one can define quantum stochastic integrals $\int_0^\infty db_s^* \xi(s)$ and $\int_0^\infty db_s \eta(s)$ as elements of $\mathcal{H}$, for suitable adapted $\mathcal{H}$-valued processes ξ and η. These two stochastic integrals are orthogonal in $\mathcal{H}$, and satisfy

$$P_t\left(\int_0^\infty db_s^* \xi(s)\right) = \int_0^t db_s^* \xi(s) \quad \text{and} \quad P_t\left(\int_0^\infty db_s \eta(s)\right) = \int_0^t db_s \eta(s),$$

consequently, the stochastic integrals are orthogonal to Ω and obey isometry relations (for details of these results see [4]).

The stochastic integral representation of elements of $\mathcal{H}$ states that for any $\zeta \in \mathcal{H}$, there are a unique $\alpha \in \mathbb{C}$ and processes $\xi \in L^2(\mathbb{R}^+, (1-\rho(s))ds, \mathcal{H})$, $\eta \in L^2(\mathbb{R}^+, \rho(s)ds, \mathcal{H})$ such that

$$\zeta = \alpha\Omega + \int_0^\infty db_s^* \xi(s) + \int_0^\infty db_s \eta(s).$$

This representation theorem, which may be found in [13], allows a concrete representation for the action of the time projections M_τ, namely,

$$M_\tau \zeta = \alpha\Omega + \int_0^\infty db_s^* \beta\left(q_s^\perp\right) \xi(s) + \int_0^\infty db_s \beta\left(q_s^\perp\right) \eta(s),$$

where $\tau = (q_t)$ and β is the parity automorphism of $\mathcal{A}$ which is spatial: $\beta(a) = \theta a \theta$, where $\theta = \theta^* = \theta^{-1}$ and $\theta\Omega = \Omega$ (for details of these results see [2]).

We shall use the formula above for the action of the time projection to discuss the relationship between random times, the associated time projections and properties of the lattice of the time projections.

Let us start with the following result which can be considered as a converse of Corollary 3.4 of [2].

Proposition 3.1. *Let $\tau = (q_t)$ be a random time, and let $\tau_n = \left(q_t^{(n)}\right)$ be a sequence of random times with $M_{\tau_n} \to M_\tau$ strongly. Then there is a subsequence (τ_{n_k}) such that $\tau_{n_k} \to \tau$ Lebesgue almost everywhere.*

Proof. Let $\zeta \in \mathcal{H}$. There are $\alpha \in \mathbb{C}$ and processes $\xi \in L^2(\mathbb{R}^+, (1-\rho(s))ds, \mathcal{H})$, $\eta \in L^2(\mathbb{R}^+, \rho(s)ds, \mathcal{H})$ such that

$$M_{\tau_n} \zeta = \alpha\Omega + \int_0^\infty db_s^* \beta\left(q_s^{(n)\perp}\right) \xi(s) + \int_0^\infty db_s \beta\left(q_s^{(n)\perp}\right) \eta(s),$$

and

$$M_\tau \zeta = \alpha\Omega + \int_0^\infty db_s^* \beta\left(q_s^\perp\right) \xi(s) + \int_0^\infty db_s \beta\left(q_s^\perp\right) \eta(s).$$

So by the isometry and orthogonality relations we obtain

$$\begin{aligned}
&\|(M_{\tau_n} - M_\tau)\,\zeta\|^2 \\
&= \left\| \int_0^\infty db^*_s \beta\left(q_s^{(n)\perp} - q_s^\perp\right)\xi(s) + \int_0^\infty db_s \beta\left(q_s^{(n)\perp} - q_s^\perp\right)\eta(s)\right\|^2 \\
&= \left\| \int_0^\infty db^*_s \beta\left(q_s^{(n)\perp} - q_s^\perp\right)\xi(s)\right\|^2 + \left\| \int_0^\infty db_s \beta\left(q_s^{(n)\perp} - q_s^\perp\right)\eta(s)\right\|^2 \\
&= \int_0^\infty \left\|\beta\left(q_s^{(n)\perp} - q_s^\perp\right)\xi(s)\right\|^2 (1-\rho(s))\,ds \\
&+ \int_0^\infty \left\|\beta\left(q_s^{(n)\perp} - q_s^\perp\right)\eta(s)\right\|^2 \rho(s)ds.
\end{aligned}$$

When $n \to \infty$, $\|(M_{\tau_n} - M_\tau)\,\zeta\|^2 \to 0$, thus both the integrals

$$\int_0^\infty \left\|\beta\left(q_s^{(n)\perp} - q_s^\perp\right)\xi(s)\right\|^2 (1-\rho(s))\,ds$$

and

$$\int_0^\infty \left\|\beta\left(q_s^{(n)\perp} - q_s^\perp\right)\eta(s)\right\|^2 \rho(s)ds$$

converge to 0 and hence $\left\|\beta\left(q_s^{(n)\perp} - q_s^\perp\right)\right\|^2$ converges to 0 in measure $(1-\rho(s))ds$, and thus in Lebesgue measure. So, there is a subsequence $\left\|\beta\left(q_s^{(n_k)\perp} - q_s^\perp\right)\xi(s)\right\|^2$ converging to 0 Lebesgue almost everywhere. Now by taking $\xi(t) = e^{-t}\Omega \in L^2(\mathbb{R}^+, (1-\rho(s))ds, \mathcal{H})$, we deduce that $\left\|\beta\left(q_s^{(n_k)\perp} - q_s^\perp\right)\Omega\right\|^2$ converges to 0 almost everywhere. Since Ω is separating for $\mathcal{A}$, we get that $\left\|\beta\left(q_s^{(n_k)\perp} - q_s^\perp\right)\zeta\right\|$ converges to 0, Lebesgue almost everywhere for each $\zeta \in \mathcal{H}$, and hence $\beta\left(q_s^{(n_k)\perp} - q_s^\perp\right)$ converges strongly to 0 Lebesgue almost everywhere. Since the automorphism β is spatial, $q_s^{(n_k)} \to q_s$ strongly Lebesgue almost everywhere. This means that the subsequence (τ_{n_k}) converges Lebesgue almost everywhere to τ, and the proof is complete. □

As a corollary to the above result we obtain the following one-to-one correspondence between random times and time projections.

Corollary 3.2. *If $M_\tau = M_\sigma$, then $\tau = \sigma$ Lebesgue almost everywhere. Indeed, it is enough to take $\tau_n = \tau$ and apply Proposition* 3.1.

We shall use the above corollary to examine the relation between the modularity of the lattice of random times and the lattice of associated time projections (Proposition 3.6). Now we give the following theorem, which characterises the commuting time projections.

Proposition 3.3. *Let $\tau = (q_t)$ and $\sigma = (q'_t)$ be random times. Then $M_\tau M_\sigma = M_\sigma M_\tau$ if and only if $q_t q'_t = q'_t q_t$ Lebesgue almost everywhere.*

Proof. Let $\zeta \in \mathcal{H}$. There are $\alpha \in \mathbb{C}$ and processes $\xi \in L^2(\mathbb{R}^+, (1-\rho(s))ds, \mathcal{H})$, $\eta \in L^2(\mathbb{R}^+, \rho(s)ds, \mathcal{H})$ such that

$$M_\tau \zeta = \alpha\Omega + \int_0^\infty db_s^* \beta\left(q_s^\perp\right)\xi(s) + \int_0^\infty db_s \beta\left(q_s^\perp\right)\eta(s),$$

and

$$M_\sigma \zeta = \alpha\Omega + \int_0^\infty db_s^* \beta\left(q_s'^\perp\right)\xi(s) + \int_0^\infty db_s \beta\left(q_s'^\perp\right)\eta(s).$$

Then

$$M_\tau M_\sigma \zeta = \alpha\Omega + \int_0^\infty db_s^* \beta\left(q_s^\perp q_s'^\perp\right)\xi(s) + \int_0^\infty db_s \beta\left(q_s^\perp q_s'^\perp\right)\eta(s),$$

and

$$M_\sigma M_\tau \zeta = \alpha\Omega + \int_0^\infty db_s^* \beta\left(q_s'^\perp q_s^\perp\right)\xi(s) + \int_0^\infty db_s \beta\left(q_s'^\perp q_s^\perp\right)\eta(s).$$

Thus

$$\begin{aligned}(M_\tau M_\sigma - M_\sigma M_\tau)\zeta = &\int_0^\infty db_s^* \beta\left(q_s^\perp q_s'^\perp - q_s'^\perp q_s^\perp\right)\xi(s) \\ &+ \int_0^\infty db_s \beta\left(q_s^\perp q_s'^\perp - q_s'^\perp q_s^\perp\right)\eta(s),\end{aligned}$$

so by the isometry and orthogonality relations

$$\begin{aligned}\|(M_\tau M_\sigma - M_\sigma M_\tau)\zeta\|^2 = &\int_0^\infty \left\|\beta\left(q_s^\perp q_s'^\perp - q_s'^\perp q_s^\perp\right)\xi(s)\right\|^2 (1-\rho(s))\, ds \\ &+ \int_0^\infty \left\|\beta\left(q_s^\perp q_s'^\perp - q_s'^\perp q_s^\perp\right)\eta(s)\right\|^2 \rho(s)\, ds. \qquad (*)\end{aligned}$$

If $M_\tau M_\sigma = M_\sigma M_\tau$, then taking $\xi(t) = e^{-\,t}\Omega \in L^2(\mathbb{R}^+, (1-\rho(s))\, ds, \mathcal{H})$, we obtain

$$\left\|\beta\left(q_s^\perp q_s'^\perp - q_s'^\perp q_s^\perp\right)\Omega\right\|^2 = 0$$

for Lebesgue almost all s, since $0 < \rho < 1$ Lebesgue almost everywhere. Since Ω is a separating vector for $\mathcal{A}$ and β is an automorphism of $\mathcal{A}$, $q_s q'_s = q'_s q_s$ Lebesgue almost everywhere.

The converse is immediate from $(*)$. □

Now we shall prove a stronger version of the *Optional Stopping Theorem*, namely

Theorem 3.4. *Let $\tau = (q_t)$ and $\sigma = (q'_t)$ be random times such that $\sigma \leq \tau$ Lebesgue almost everywhere, i.e. $q_t \leq q'_t$ for almost all $t \in [0, \infty]$. Then $M_\sigma \leq M_\tau$.*

Proof. Let $\zeta \in \mathcal{H}$. There are $\alpha \in \mathbb{C}$ and processes $\xi \in L^2(\mathbb{R}^+, (1-\rho(s))ds, \mathcal{H})$, $\eta \in L^2(\mathbb{R}^+, \rho(s)ds, \mathcal{H})$ such that

$$M_\tau M_\sigma \zeta = \alpha\Omega + \int_0^\infty db_s^* \beta\left(q_s^\perp q_s'^\perp\right)\xi(s) + \int_0^\infty db_s \beta\left(q_s^\perp q_s'^\perp\right)\eta(s).$$

Since $q_s^\perp q_s'^\perp = q_s'^\perp$ for almost all $s \in [0, \infty]$, we get

$$M_\tau M_\sigma \zeta = \alpha\Omega + \int_0^\infty db_s^* \beta\left(q_s'^\perp\right)\xi(s) + \int_0^\infty db_s \beta\left(q_s'^\perp\right)\eta(s) = M_\sigma \zeta,$$

which means that $M_\sigma \leq M_\tau$. □

The next result is the ascending version of a descending random time martingale convergence theorem (Theorem 1.12 of [3]), and gives an answer to a question raised in [3].

Theorem 3.5. *Let $\{\tau_\alpha \colon \alpha \in \Lambda\}$ be a family of random times. Let τ be the supremum of this family. Then*

$$M_\tau = \bigvee_\alpha M_{\tau_\alpha}.$$

Proof. Since $\mathcal{H}$ is separable, thus by virtue of Theorem 2.7 it is enough to prove the above equality for a countable family of random times. Assume therefore that Λ is countable. By Corollary 3.6 of [2] the lattice of time projections is complete, so $\bigvee_\alpha M_{\tau_\alpha}$ is a time projection, $\bigvee_\alpha M_{\tau_\alpha} = M_\sigma$ for some random time σ. As $\tau_\alpha \leq \tau$, we have $M_{\tau_\alpha} \leq M_\tau$ for all $\alpha \in \Lambda$, and consequently $M_\sigma \leq M_\tau$. Furthermore, $M_{\tau_\alpha} \leq M_\sigma$ for each $\alpha \in \Lambda$, and according to [2; Remark p. 435], $\tau_\alpha \leq \sigma$ Lebesgue almost everywhere. Since Λ is countable, we obtain $\tau = \sup_\alpha \tau_\alpha \leq \sigma$ Lebesgue almost everywhere, and by Theorem 3.4, $M_\tau \leq M_\sigma$, which finishes the proof. □

In particular, for any two random times τ and σ, we have $M_\tau \vee M_\sigma = M_{\tau \vee \sigma}$ as proved in [2] Theorem 3.5.

Proposition 3.6. *The lattice of time projections is modular if and only if the lattice of random times is modular Lebesgue almost everywhere.*

Proof. Suppose that the lattice of time projections is modular and let τ, ρ and σ be random times with $\tau \leq \rho$. By the *Optional Stopping Theorem* (Theorem 3.4), $M_\tau \leq M_\rho$. Thus

$$\begin{aligned} M_{\tau\vee(\sigma\wedge\rho)} &= M_\tau \vee M_{(\sigma\wedge\rho)} = M_\tau \vee (M_\sigma \wedge M_\rho) = (M_\tau \vee M_\sigma) \wedge M_\rho \\ &= M_{\tau\vee\sigma} \wedge M_\rho = M_{(\tau\vee\sigma)\wedge\rho}. \end{aligned}$$

Using Corollary 3.2, we get $\tau \vee (\sigma \wedge \rho) = (\tau \vee \sigma) \wedge \rho$ Lebesgue almost everywhere.

For the converse, suppose that the lattice of random times is modular Lebesgue almost everywhere and let M_τ, M_σ and M_ρ be time projections with $M_\tau \leq M_\rho$. Using [2; Remark p. 435], we infer that $\tau \leq \rho$ Lebesgue almost everywhere. Thus

$$M_\tau \vee (M_\sigma \wedge M_\rho) = M_\tau \vee M_{\sigma\wedge\rho} = M_{\tau\vee(\sigma\wedge\rho)}.$$

But $\tau \vee (\sigma \wedge \rho) = (\tau \vee \sigma) \wedge \rho$ Lebesgue almost everywhere, so

$$M_{\tau\vee(\sigma\wedge\rho)} = M_{(\tau\vee\sigma)\wedge\rho} = M_{(\tau\vee\sigma)} \wedge M_\rho = (M_\tau \vee M_\sigma) \wedge M_\rho.$$

Hence

$$M_\tau \vee (M_\sigma \wedge M_\rho) = (M_\tau \vee M_\sigma) \wedge M_\rho.$$

□

Hereafter we examine the lattice of time projections for a filtration of a finite CAR von Neumann algebra. In particular, we shall show that the lattice is upper and lower continuous. This situation arises if we take ρ to be the constant function equal to 1/2, so that ω is a tracial vector state with two-point function

$$\omega\left(b^*(f)b(g)\right) = \frac{1}{2}\int_0^\infty f(s)\overline{g(s)}ds.$$

Let us begin with

Lemma 3.7. *Let $\mathcal{F} = \{M_{\tau_\alpha} : \alpha \in \Lambda\}$ be an increasingly directed family of time projections. Then for each time projection M_σ associated with random time σ,*

$$\bigvee_n (M_{\tau_n} \wedge M_\sigma) = (\bigvee_n M_{\tau_n}) \wedge M_\sigma = (\bigvee_\alpha M_{\tau_\alpha}) \wedge M_\sigma$$

for some sequence (M_{τ_n}) in $\mathcal{F}$.

Proof. From Theorem 3.5, $\sup \mathcal{F} = \bigvee_\alpha M_{\tau_\alpha}$ is a time projection, say, M_τ, where $\tau = \sup_\alpha \tau_\alpha$. Since the strong operator topology on the closed unit ball of $\mathcal{B}(\mathcal{H})$ is metrizable, there is an increasing sequence (M_{τ_n}) in $\mathcal{F}$

converging strongly to M_τ. Hence $(M_{\tau_n} \wedge M_\sigma)$ is an increasing sequence converging strongly to $M_\tau \wedge M_\sigma$. Indeed,

$$\begin{aligned}\bigvee_n (M_{\tau_n} \wedge M_\sigma) = \bigvee_n M_{\tau_n \wedge \sigma} &= M_{\sup_n(\tau_n \wedge \sigma)} && \text{(by Theorem 3.5)}\\ &= M_{(\sup_n \tau_n)\wedge\sigma} && \text{(by Theorem 1.9 of [3])}\\ &= M_{(\sup_n \tau_n)} \wedge M_\sigma = M_\tau \wedge M_\sigma,\end{aligned}$$

that is

$$\bigvee_n (M_{\tau_n} \wedge M_\sigma) = (\bigvee_n M_{\tau_n}) \wedge M_\sigma = (\bigvee_\alpha M_{\tau_\alpha}) \wedge M_\sigma.$$

□

In a similar fashion one may prove the corresponding result for infima, that is

$$\bigwedge_n (M_{\tau_n} \vee M_\sigma) = (\bigwedge_n M_{\tau_n}) \vee M_\sigma = (\bigwedge_\alpha M_{\tau_\alpha}) \vee M_\sigma$$

for some sequence (M_{τ_n}) in a decreasingly directed family $\mathcal{F} = \{M_{\tau_\alpha} : \alpha \in \Lambda\}$ of time projections.

Theorem 3.8. *The lattice of time projections is both upper and lower continuous.*

Proof. Consider an increasingly directed family $\{M_{\tau_\alpha} : \alpha \in \Lambda\}$ of time projections, and let M_σ be any time projection. By Lemma 3.7, for some sequence (M_{τ_n}) with $\bigvee_n M_{\tau_n} = \bigvee_\alpha M_{\tau_\alpha}$, we have

$$\bigvee_n (M_{\tau_n} \wedge M_\sigma) = (\bigvee_\alpha M_{\tau_\alpha}) \wedge M_\sigma.$$

It is clear that

$$M_{\tau_\alpha} \wedge M_\sigma \le (\bigvee_\alpha M_{\tau_\alpha}) \wedge M_\sigma, \text{ for each } \alpha \in \Lambda,$$

and so

$$\bigvee_\alpha (M_{\tau_\alpha} \wedge M_\sigma) \le (\bigvee_\alpha M_{\tau_\alpha}) \wedge M_\sigma.$$

Note that for each n, there exists $\alpha_n \in \Lambda$ such that $M_{\tau_n} \le M_{\tau_{\alpha_n}}$, and

$$M_{\tau_n} \wedge M_\sigma \le M_{\tau_{\alpha_n}} \wedge M_\sigma \le \bigvee_\alpha (M_{\tau_\alpha} \wedge M_\sigma).$$

Thus

$$\bigvee_n (M_{\tau_n} \wedge M_\sigma) \le \bigvee_\alpha (M_{\tau_\alpha} \wedge M_\sigma).$$

But

$$(\bigvee_\alpha M_{\tau_\alpha}) \wedge M_\sigma = \bigvee_n (M_{\tau_n} \wedge M_\sigma) \leq \bigvee_\alpha (M_{\tau_\alpha} \wedge M_\sigma)\,,$$

so

$$\bigvee_\alpha (M_{\tau_\alpha} \wedge M_\sigma) = (\bigvee_\alpha M_{\tau_\alpha}) \wedge M_\sigma.$$

Therefore, the lattice of time projections is upper continuous and a similar calculation shows that the lattice is lower continuous, i.e., for a decreasingly directed family $\{M_{\tau_\alpha} : \alpha \in \Lambda\}$ and arbitrary time projection M_σ,

$$\bigwedge_\alpha (M_{\tau_\alpha} \vee M_\sigma) = (\bigwedge_\alpha M_{\tau_\alpha}) \vee M_\sigma.$$

□

Let us notice that the lower continuity of the family of time projections also follows immediately from Theorem 2.10.

Remark 3.9. When dealing with the CCR algebra, there exists a martingale representation theorem (see [8], [10]), and one can use it to give a corresponding result for the action of the time projection and to prove the results of Section 3 of [2] in this context; all our results in this section have analogues in the CCR theory.

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