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HUKUHARA'S DERIVATIVE AND CONCAVE ITERATION SEMIGROUPS OF LINEAR SET-VALUED FUNCTIONS

A. SMAJDOR

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Abstract. Let K be a closed convex cone with the nonempty interior in a real Banach space and $cc(K)$ denote the family of all nonempty convex compact subsets of K. If $\{F^t\colon t \geq 0\}$ is a concave iteration semigroup of continuous linear set-valued functions $F^t\colon K \to cc(K)$ with $F^0(x) = \{x\}$ for $x \in K$, then

$$D_t F^t(x) = F^t(G(x))$$

for $x \in K$ and $t \geq 0$, where $D_t F^t(x)$ denotes the Hukuhara derivative of $F^t(x)$ with respect to t and

$$G(x) := \lim_{s \to 0+} \frac{F^s(x) - x}{s}$$

for $x \in K$.

1. Let A and B be two subsets of a real vector space X. We define the *sum* of A and B by the formula

$$A + B = \{a + b\colon a \in A,\ b \in B\}.$$

A subset K of a real vector space is called a *cone* iff

$$tK := \{tx\colon x \in K\} \subset K$$

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for all positive t. A cone is said to be *convex* iff it is a convex set.

Let X and Y be two real vector spaces and let $K \subset X$ be a convex cone. A set-valued function $F\colon K \to n(Y)$, where $n(Y)$ denotes the family of all nonempty subsets of Y, is called *additive* (*superadditive*) iff

$$F(x) + F(y) = F(x+y) \;\; (F(x) + F(y) \subset F(x+y))$$

for all $x, y \in K$. A set-valued function $F\colon K \to n(K)$ is said to be *homogeneous ($\mathbb{Q}_+$-homogeneous)* if $F(\lambda x) = \lambda F(x)$ for all $x \in K$ and $\lambda > 0$ ($\lambda \in \mathbb{Q}_+$), where $\mathbb{Q}_+$ is the set of all positive rational numbers. A set-valued function $F\colon K \to n(K)$ is *linear* if it is additive and homogeneous.

A set-valued function $F\colon [0, +\infty) \to n(Y)$ is said to be *concave* iff

$$F(\lambda x + (1-\lambda)y) \subset \lambda F(x) + (1-\lambda)F(y)$$

for all $x, y \in [0, +\infty)$ and $\lambda \in (0, 1)$.

The following property of additive set valued functions is easy to check.

Lemma 1. *Let X and Y be two real vector spaces and let K be a convex cone in X. Assume that $F\colon K \to n(Y)$ is an additive set-valued function and $A, B \in n(K)$. Then*

$$F(A) + F(B) = F(A+B),$$

where $F(A) = \bigcup\{F(x)\colon x \in A\}$.

We need the following lemma.

Lemma 2 (cf. [7]). *Let A, B and C be subsets of a real topological vector space such that*

$$A + B \subset C + B.$$

If C is convex closed and B is non-empty bounded, then

$$A \subset C.$$

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers. All vector spaces are supposed to be real. If X is a topological vector space, then $c(X)$ denotes the set of all compact members of $n(X)$ and $cc(X)$ stands for the set of all convex sets of $c(X)$. By $B(X)$ we denote the set of all bounded members of $n(X)$.

Let $A, B \in cc(X)$. By Lemma 2 there exists at most one set $C \in cc(X)$ fulfilling the equality

$$A = B + C.$$

M. Hukuhara denoted such set C by $A - B$ and called the difference between A and B (see [3]).

A set valued function $F\colon X \to n(Y)$, where X and Y are two topological spaces, is said to be upper semicontinuous (lower semicontinuous) iff the set

$$F^+(U) = \{x \in X\colon F(x) \subset U\} \qquad (F^-(U) = \{x \in X\colon F(x) \cap U \neq \emptyset\})$$

is open for every open subset U of Y. A set-valued function is continuous iff it is upper semicontinuous and lower semicontinuous.

Observe that Lemma 1 implies the following.

Lemma 3. *Let X and Y be two topological vector spaces and let K be a closed convex cone in X. Assume that $F\colon K \to cc(K)$ is a continuous additive set-valued function and $A, B \in cc(K)$. If there exists the difference $A - B$, then there exists $F(A) - F(B)$ and $F(A) - F(B) = F(A - B)$.*

Let X be a metric space. For $A, A_n \in c(X)$, $n \in \mathbb{N}$, the symbol $\lim_{n\to\infty} A_n = A$ means that $\lim_{n\to\infty} d(A_n, A) = 0$, where d denotes the Hausdorff metric derived by the metric in X. Let X and Y be two metric spaces. A set-valued function $F\colon X \to c(Y)$ is continuous if and only if it is continuous as a single-valued function from X into $c(Y)$ with the Hausdorff metric in $c(Y)$ derived by the metric in Y (see [1], §6 of Chapter VI).

We will use the following six lemmas.

Lemma 4 (Theorem 3 in [10], see also Lemma 4 in [9])**.** *Let X and Y be two real normed spaces and let K be a convex cone in X. Suppose that $\{F_i\colon i \in I\}$ is a family of superadditive lower semicontinuous in K and $\mathbb{Q}_+$-homogeneous set-valued functions $F_i\colon K \to n(Y)$. If K is of the second category in K and $\bigcup_{i\in I} F_i(x) \in B(Y)$ for $x \in K$, then there exists a positive constant M such that*

$$\|F_i(x)\| := \sup\{\|y\|\colon y \in F_i(x)\} \leq M\|x\|$$

for every $i \in I$ and $x \in K$.

Corollary . *If X, Y and K are such as in Lemma 4, then the functional*

$$F \mapsto \|F\| := \sup\left\{\frac{\|F(x)\|}{\|x\|}\colon x \in K,\ x \neq 0\right\}$$

is finite for every $\mathbb{Q}_+$-homogeneous superadditive lower semicontinuous set-valued function $F\colon K \to B(Y)$.

Lemma 5 (Lemma 5 in [9])**.** *Let X and Y be two real normed spaces and let d be the Hausdorff distance derived from the norm in Y. Suppose that K is a convex cone in X with the nonempty interior. Then there exists a positive constant M_0 such that for every linear continuous set-valued function $F\colon K \to c(Y)$ the inequality*

$$d(F(x), F(y)) \leq M_0\|F\|\|x - y\|$$

holds for every $x, y \in K$.

Lemma 6 (Theorem 2 in [4]). *Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces and let d_X and d_Y be the Hausdorff metric derived from ρ_X and ρ_Y, respectively. If $F\colon X \to n(Y)$ is a set-valued function and M is a positive constant such that*

$$d_Y(F(x), F(y)) \leq M\rho_X(x, y)$$

for every $x, y \in X$, then

$$d_Y(F(A), F(B)) \leq M d_X(A, B)$$

for every nonempty subsets A, B of X.

Lemma 7 (see e.g. Proposition 2.4.7 in [2]). *Let X be a normed space. If (A_n) is a sequence of elements of the set $c(X)$ such that $A_{n+1} \subset A_n$ for $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

Lemma 8 (Lemma 3 in [8]). *Let K be a closed convex cone such that $\operatorname{int} K \neq \emptyset$ in Banach space X and let Y be a normed space. If (F_n) is a sequence of continuous additive set-valued functions $F_n\colon K \to cc(Y)$ such that $F_{n+1}(x) \subset F_n(x)$ for all $x \in K$ and $n \in \mathbb{N}$, then the formula*

$$F_0(x) := \bigcap_{n=1}^{\infty} F_n(x), \quad x \in K$$

defines a continuous additive set-valued function $F_0\colon K \to cc(Y)$. Moreover,

$$\lim_{n\to\infty} F_n(x) = F_0(x), \quad x \in K \tag{1}$$

and the convergence in (1) is uniform on every nonempty compact subset of K.

Lemma 9 (Lemma 4 in [8]). *Let D be a nonempty set and Y be a normed space. Suppose that $F_0, F_n\colon D \to c(Y)$ are set-valued functions. If the sequence (F_n) uniformly converges to F_0 on D, then*

$$\lim_{n\to\infty} F_n(D) = F_0(D).$$

Let X be a normed space and let $\phi\colon [0,\infty) \to cc(X)$ be a set-valued function such that the Hukuhara differences $\phi(t+s) - \phi(t)$ exist for nonnegative t and s and the Hukuhara diferences $\phi(t) - \phi(t-s)$ exist for positive t and

$s \in (0, t)$. Let $t > 0$. The Hukuhara derivative of ϕ at t is defined by the formula

$$D\phi(t) = \lim_{s \to 0+} \frac{\phi(t+s) - \phi(t)}{s} = \lim_{s \to 0+} \frac{\phi(t) - \phi(t-s)}{s},$$

whenever both these limits exist (see [3]). Moreover,

$$D\phi(0) = \lim_{s \to 0+} \frac{\phi(s) - \phi(0)}{s}.$$

2. Let K be a nonempty set. A family $\{F^t : t \geq 0\}$ of set-valued functions $F^t : K \to n(K)$ is said to be *an iteration semigroup* iff

$$F^t \circ F^s(x) := F^t[F^s(x)] = F^{s+t}(x)$$

for all $x \in K$ and $t, s \geq 0$.

Let K be a convex cone in a normed space. An iteration semigroup $\{F^t : t \geq 0\}$ of set-valued functions $F^t : K \to cc(K)$ is said to be *differentiable* iff all set-valued functions $t \mapsto F^t(x)$, $(x \in K)$ have Hukuhara's derivative on $[0, +\infty)$. An iteration semigroup $\{F^t : t \geq 0\}$ of set-valued functions $F^t : K \to n(K)$ is said to be *concave* iff the set-valued function $t \mapsto F^t(x)$ is concave for every $x \in K$.

Concave iteration semigroups $\{F^t : t \geq 0\}$ of set-valued functions $F^t : K \to cc(K)$ was introduced in the paper [5] in which also the following lemma was proved.

Lemma 10. *Assume that K is a closed convex cone with the nonempty interior in a real Banach space X. Let $\{F^t : t \geq 0\}$ be a concave iteration semigroup of continuous linear set-valued functions $F^t : K \to c(K)$ with $F^0(x) = \{x\}$ for $x \in K$. Then there exists a set-valued function $G : K \to cc(K)$ such that the family $\{(1/t)(F^t - F^0) : t > 0\}$ uniformly converges to G on every compact subset of K, when t tends to zero. Moreover, G is linear continuous and*

$$G(x) = \bigcap_{t>0} \frac{A^t(x) - x}{t}$$

for every $x \in K$.

Some examples of concave iteration semigroups of continuous linear set-valued functions can be found in [5]. We add the following two.

Example 1. Let $F^t : [0, +\infty)^2 \to cc([0, +\infty)^2)$ for $t \geq 0$ be set-valued functions defined by

$$F^t((x, y)) = [x, x \cdot \cosh t + y \cdot \sinh t] \times [y, x \cdot \sinh t + y \cdot \cosh t]$$

for $(x,y) \in [0,+\infty)^2$. Then the family $\{F^t : t \geq 0\}$ of set-valued functions F^t is a concave iteration semigroup of continuous linear set-valued functions. Moreover, $G((x,y)) = [0,y] \times [0,x]$ for $(x,y) \in [0,+\infty)^2$.

Example 2. Let X be the linear space of all functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$, where $a, b \in \mathbb{R}$. The space X is a Banach space with the norm

$$\|f\| = |b| + |a|.$$

Let

$$K := \{f \in X : a \geq 0\}.$$

The set K is a closed convex cone in X and $f_0 \in \operatorname{int} K$, where $f_0(x) = x + 1$ for $x \in \mathbb{R}$. Now, we define set-valued functions F^t as follows

$$F^t(f) := \{g \in X : \exists_{u \in [0,t]} \forall_{x \in \mathbb{R}} g(x) = f(x+u)\}$$

for $f \in K$. It is easy to check that sets $F^t(f)$ are nonempty, compact and convex subsets of K. Moreover F^t are linear continuous multifunctions and the family $\{F^t : t \geq 0\}$ is a concave iteration semigroup with $G(f) = \{g \in X : g(x) \equiv d,\ 0 \leq d \leq a\}$.

Lemma 11. *Assume that K is a closed convex cone with nonempty interior in a real Banach space X. Let $\{F^t : t \geq 0\}$ be a concave iteration semigroup of continuous linear set-valued functions $F^t : K \to c(K)$ with $F^0(x) = \{x\}$ for $x \in K$. Then $F^t(y)$ converges to y uniformly on every nonempty compact subset C of K, when $t \to 0$.*

Proof. Fix $\varepsilon > 0$ and a set $C \in c(K)$. According to Lemma 10 there exists $\delta \in \big(0, \varepsilon/(1 + \|G(C)\|)\big)$ such that

$$d(\frac{F^s(y) - y}{s}, G(y)) < 1$$

for every $y \in C$ and $s \in (0,\delta)$. This implies that

$$\frac{F^s(y) - y}{s} \subset G(y) + S,$$

where S is the closed unit ball, and

$$G(y) \subset \frac{F^s(y) - y}{s} + S$$

for $y \in C$ and $s \in (0,\delta)$. Therefore

$$F^s(y) \subset [y + sG(y)] + sS$$

and

$$[sG(y) + y] \subset F^s(y) + sS$$

for the same y and s. Thus

$$d(F^s(y), sG(y) + y) \le s$$

for $y \in C$ and $0 < s < \delta$. Hence we have for the same y and s

$$\begin{aligned} d(F^s(y), \{y\}) &\le d(F^s(y), y + sG(y)) + d(y + sG(y), \{y\}) \\ &= d(F^s(y), y + sG(y)) + s\|G(y)\| < s + s\|G(C)\| \\ &= s(1 + \|G(C)\|) < \varepsilon. \end{aligned}$$

This completes the proof. □

Under assumptions of Lemma 10,

$$G(x) := \lim_{t \to 0+} \frac{F^t(x) - x}{t}$$

for $x \in K$. Therefore G is the infinitesimal generator of semigroup $\{F^t : t \ge 0\}$ and its domain $D(G)$ is equal to K. It is well known that if $\{f^t : t \ge 0\}$ is a strongly continuous semigroup of bounded linear operators on X and g is its infinitesimal generator, then the function $t \mapsto f^t(x)$ is differentiable for every $x \in D(g)$ and the equality

$$\frac{d}{dt} f^t(x) = f^t(g(x)), \; x \in D(g)$$

holds true (see [6]). A similar result for concave iteration semigroup of linear continuous set-valued functions is contained in the following theorem.

Theorem . *Let X be a Banach space and let K be a closed convex cone with the nonempty interior. Suppose that $\{F^t : t \ge 0\}$ is a concave iteration semigroup of linear continuous set-valued functions $F^t : K \to cc(K)$ with $F^0(x) = \{x\}$. Then this iteration semigroup is differentiable and*

$$D_t F^t(x) = F^t(G(x))$$

for $x \in K$, $t \ge 0$, where D_t denotes the Hukuhara derivative of $F^t(x)$ with respect to t and G is given by Lemma 10.

Proof. It is obvious that there exist differences

$$F^s(x) - x$$

for $s > 0$ and $x \in K$ so according to Lemma 3 there exist differences

$$F^{t+s}(x) - F^t(x) = F^t[F^s(x)] - F^t(x) = F^t(F^s(x) - x)$$

and

$$F^t(x) - F^{t-s}(x) = F^{t-s}[F^s(x)] - F^{t-s}(x) = F^{t-s}(F^s(x) - x)$$

whenever $t > 0$, $s \in (0, t)$ and $x \in K$.

Lemmas 5 and 6 imply that

$$d\left(\frac{F^{t+s}(x)-F^t(x)}{s}, F^t(G(x))\right) = d\left(F^t\left(\frac{F^s(x)-x}{s}\right), F^t(G(x))\right)$$
$$\leq M_0\|F^t\| d\left(\frac{F^s(x)-x}{s}, G(x)\right)$$

for $x \in K$, $t > 0$, $s \in (0,t)$. Therefore, in view of Lemma 10

$$\lim_{s\to 0+} \frac{F^{t+s}(x)-F^t(x)}{s} = F^t(G(x))$$

for $t > 0$ and $x \in K$.

Similarly we have

$$d\left(\frac{F^t(x)-F^{t-s}(x)}{s}, F^t(G(x))\right) = d\left(F^{t-s}\left(\frac{F^s(x)-x}{s}\right), F^{t-s}(F^s(G(x)))\right)$$
$$\leq M_0\|F^{t-s}\| d\left(\frac{F^s(x)-x}{s}, F^s(G(x))\right) \tag{2}$$

for $t > 0$, $s \in (0,t)$ and $x \in K$.

Fix $x \in K$ and $t > 0$. Since $F^t(x) \in c(K)$ and

$$\|F^{t-s}(x)\| \leq \|\frac{t-s}{t}F^t(x) + \frac{s}{t}\{x\}\|$$
$$\leq \frac{t-s}{t}\|F^t(x)\| + \frac{s}{t}\|x\| \leq \max\{\|F^t(c)\|, \|x\|\} < \infty.$$

Thus the set $\bigcup_{0\leq s\leq t} F^{t-s}(x)$ is bounded. By Lemma 4 there exists a positive constant M such that

$$\|F^{t-s}\| \leq M \tag{3}$$

for $s \in [0,t]$. According to (2) and (3) we have

$$d\left(\frac{F^t(x)-F^{t-s}(x)}{s}, F^t(G(x))\right) \leq M_0 M d\left(\frac{F^s(x)-x}{s}, F^s(G(x))\right)$$
$$\leq M_0 M d\left(\frac{F^s(x)-x}{s}, G(x)\right) + d(G(x), F^s(G(x))).$$

According to Lemmas 10, 11 and 9, the right part of the last inequality has the limit zero when $s \to 0+$. Thus

$$D_t F^t(x) = F^t(G(x)).$$

This ends the proof. □

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Andrzej Smajdor
Pedagogical University
Podchorążych 2
30-084 Kraków
Poland