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# SOLUTION OF THE STIELTJES TRUNCATED MOMENT PROBLEM

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**Abstract.** The conditions of solvability and description of all solutions of the truncated Stieltjes moment problem are obtained using as the starting point earlier results on the Hamburger truncated moment problem. An algebraic algorithm for the explicit solution of both problems is proposed.

## 1. Introduction

The truncated Hamburger moment problem consists in the determination of non-decreasing functions $\sigma(t)$ on the real axis by its first $2n+1$ power moments. The additional demand: $\sigma(t) = 0$ for $t < 0$, transforms it into the truncated Stieltjes moment problem. We solve here the last problem on the basis of the results on the Hamburger problem obtained earlier [6], making clear, which additional conditions should be imposed on the given moments $(s_j)_{j=0}^{n}$ to provide the existence of the solutions $\sigma(t)$ of the Hamburger

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problem with support on the positive half-axis and how to restrict the class of parameters in the Nevanlinna formula giving description of all of the solutions of the Hamburger problem (see [6]) to single out only those with the spectrum on the positive half-axis. In this way we obtain a complete solution of the truncated Stieltjes moment problem, using, as in [1, 2, 7] the methods of the extension theory of Hermitian operators and, in particular, the results on extensions of non-negative operators and matrices. Moreover, applying methods of the extension theory we found a presumably new purely algebraic algorithm for the solution of both problems.

This paper is organized in the following way.

In Section 2 we specify the solvability criterium for the truncated Stieltjes moment problem. We do this since the classical "full" moment problems does not include the truncated problems as a special case [3], [1].

In Section 3 we describe the so-called canonical solutions of the truncated Stieljes problem, for which the sought functions $\sigma(t)$ have at most $n$ points of growth located on the interval $[0, \infty)$. First we present here the explicit expression for the unique solution of this problem in the so-called degenerate case and then describe the set of all canonical solutions in the cases, when for a given set of moments there are different solutions of the truncated Stiltjes moment problem. The algebraic algorithm proposed for the explicit construction of all of such solutions seems to be new and is valid also for the Hamburger moment problem. In parallel, we give the description of the canonical solutions in the traditional Nevanlinna form.

In Section 4 we replace the "free" number parameters in the obtained description formulas for the canonical solutions by certain Nevanlinna functions in the upper half-plane and in this way get all solutions of the truncated Stieltjes problem in the non-degenerate case.

## 2. Existence of solutions of the truncated Stieltjes moment problem

The *truncated Stieltjes problem* of moments is formulated in the following way:

*Given a set of real numbers*

$$\{s_0, s_1, s_2, \dots, s_{2n}\}, \quad n = 0, 1, 2, \dots . \tag{2.1}$$

*To find all distributions* $\sigma(t)$ *such that*

$$\int_0^{\infty} t^k \, d\sigma(t) = s_k \ , \ k = 0, 1, 2, \dots, 2n. \tag{2.2}$$

The formulation of the corresponding Hamburger problem is similar, the only difference is that the lower limit of the integral in (2.2) is replaced by $-\infty$. Evidently, any solution of the Stieltjes problem is a special solution of the Hamburger problem, for which there are no growth points of the distribution $\sigma(t)$ on the half-axis $(-\infty, 0)$. Therefore a criterium of solvability of the Hamburger problem is only a necessary condition for the solvability of the Stieltjes problem.

**Theorem 2.1.** *A system of real numbers* (2.1) *admits the representation* (2.2) *with non-decreasing* $\sigma(t)$ *if and only if*

a) *the Hankel matrix* $\Gamma_n := (s_{k+j})_{k,j=0}^{n}$ *is non-negative;*

b) *for any set of complex numbers* $\xi_0, \ldots, \xi_r$, $0 \le r \le n-1$, *the condition*

$$\sum_{j,k=0}^{r} s_{j+k}\xi_k\overline{\xi_j} = 0 \tag{2.3}$$

*implies*

$$\sum\nolimits_{j,k=0}^{r} s_{j+k+2}\xi_k\overline{\xi_j} = 0; \tag{2.4}$$

c) *the Hankel matrix* $\Gamma_{n-1}^{(1)} := (s_{k+j+1})_{k,j=0}^{n-1}$ *is non-negative and for any set* $\xi_0, ..., \xi_r \in \mathbb{C}$, $0 \le r \le n-1$, *the condition*

$$\sum_{j,k=0}^{r} s_{j+k+1}\xi_k\overline{\xi_j} = 0 \tag{2.5}$$

*implies* (2.4).

**Proof.** By [3], [7] (see also: [1, 2]) the conditions a), b) of the theorem form a criterium of solvability of the truncated Hamburger moment problem. Therefore we need only to prove that the condition c), in addition to a), b), is equivalent to the existence among the solutions of the Hamburger problem of those for which $\sigma(t) = \text{const}$ for $t < 0$.

Suppose that the relations (2.2) hold. For an arbitrary set of complex numbers $\xi_0, \xi_1, \ldots, \xi_{n-1}$ we define

$$P(t) = \xi_0 + \xi_1\, t + \xi_2\, t^2 + \ldots + \xi_r\, t^r\ . \tag{2.6}$$

By (2.2)

$$\sum_{k,j=0}^{r} s_{j+k+1}\xi_k\overline{\xi_j} = \int_0^\infty |P(t)|^2\ t d\sigma\,(t) \ge 0. \tag{2.7}$$

Hence the matrix $(s_{k+j+1})_{k,j=0}^{n-1}$ is non-negative.

If for some set $\xi_0, ..., \xi_r \in \mathbb{C}$, $0 \le r \le n-1$, (2.3) holds, then for the polynomial $P\,(t)$ defined by (2.6) we have:

$$\int_0^\infty |P(t)|^2\ t d\sigma\,(t) = 0$$

and hence,

$$\sum_{j,k=0}^{r} s_{j+k+2}\xi_k\overline{\xi_j} = \int_0^\infty |P(t)|^2 \ t^2 d\sigma(t) = 0.$$

Note that due to the conditions a) and c) of the theorem the moments $s_j$ are non-negative, $s_j \geq 0$, $j = 0, ..., 2n$. Excluding the trivial case, when the sought $\sigma(t)$ may have only one point of growth at $t = 0$, from now on we will assume that all these numbers are strictly positive, i.e. $s_j > 0$, $j = 0, ..., 2n$.

Suppose now that a)–c) hold. In this case for the given set of real numbers $s_0, ..., s_{2n}$ by the conditions a), b) the corresponding truncated Hamburger moment problem has at least one solution [7] (and also [1]). Let $\sigma(t)$, $-\infty < t < \infty$, be such a solution, i.e.

$$\int_{-\infty}^{\infty} t^k \ d\sigma(t) = s_k \ , \ k = 0, 1, 2, \ldots, 2n. \tag{2.8}$$

Consider the set of continuous functions $f(t)$, $-\infty < t < \infty$, with values in $\mathbb{C}$, for which

$$\int_{-\infty}^{\infty} |f(t)|^2 \ d\sigma(t) < \infty. \tag{2.9}$$

Construct a pre-Hilbert space $\mathcal{L}$ of such functions taking the bilinear functional

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)} \ d\sigma(t) \tag{2.10}$$

as a scalar product. Note that by (2.8) the polynomials

$$f(t) = \xi_0 + \xi_1 t + \ldots + \xi_r t^r, \quad \xi_0, \ldots, \xi_r \in \mathbb{C}, \tag{2.11}$$

of degree $r \leq n$ belong to $\mathcal{L}$. We will denote the linear subset of these polynomials by $\mathcal{P}_n$.

Let $\mathcal{L}_0$ be the subspace of $\mathcal{L}$ consisting of all functions $f$ such that $\|f\| := \sqrt{\langle f, f\rangle} = 0$. If $g = f + f_0$, where $f \in \mathcal{L}$, $f_0 \in \mathcal{L}_0$, than, due to the Schwarz inequality $\langle f, f_0\rangle = 0$ and hence $\|g\| = \|f\|$. Let us denote by $\tilde{\mathcal{L}}$ the factor — space $\mathcal{L}\backslash\mathcal{L}_0$. For any class of elements $\hat{g} = f + \mathcal{L}_0$ of this factor space we set $\|\hat{g}\|_{\tilde{\mathcal{L}}} = \|f\|$. Taking the closure of $\tilde{\mathcal{L}}$ with respect to this norm, we obtain the Hilbert space $\mathrm{L}^2_\sigma$. We keep the same symbol $\langle ., .\rangle$ for the scalar product in $\mathrm{L}^2_\sigma$. Let $\mathrm{L}_n$ be the subspace of $\mathrm{L}^2_\sigma$ generated by the subset of polynomials $\mathcal{P}_n$. By (2.8) and (2.10) for $f, g \in \mathcal{P}_n$,

$$f(t) = \sum_{r=0}^{n} \xi_r t^r, \ g(t) = \sum_{r=0}^{n} \eta_r t^r, \quad \xi_0, \ldots, \eta_n \in \mathbb{C},$$

we have

$$\langle f, g\rangle = \sum_{j,k=0}^{n} s_{j+k}\xi_k\overline{\eta_j}. \tag{2.12}$$

Therefore for all distributions $\sigma(t)$ satisfying (2.8), the restrictions onto $\mathrm{L}_n$ of the scalar products defined in the corresponding spaces $\mathrm{L}^2_\sigma$ must coincide. Among non-decreasing functions $\sigma(t)$ satisfying (2.8), the ones for which $\mathrm{L}^2_\sigma = \mathrm{L}_n$ are called *canonical*. It was proven in [7] that the set of canonical solutions of the truncated Hamburger moment problem is non-empty whenever the latter is solvable, i.e. whenever the conditions a), b) of the theorem hold. By (2.12) a canonical $\sigma(t)$ is a non-decreasing function having only a finite number of growth points $\le n$.

Take some canonical solution $\tilde{\sigma}(t)$ of the truncated Hamburger moment problem for the given set of moments and consider the selfadjoint operator $\widetilde{A}$ of multiplication by the independent variable $t$ in the related space $\mathrm{L}^2_{\tilde{\sigma}} = \mathrm{L}_n$. Take the class $\widehat{e}_0 \subset \mathrm{L}_n$ containing the polynomial $\widehat{e}_0(t) \equiv 1$ and the classes containing the polynomials $\widehat{e}_k(t) \equiv t^k$, $0 \le k \le n$. According to the definition of $\widetilde{A}$ we have the representation

$$\widehat{e}_k = \widetilde{A}^k\widehat{e}_0, \quad 0 \le k \le n. \tag{2.13}$$

For the unity decomposition $\widetilde{E}_t$, $-\infty < t < \infty$, of $\widetilde{A}$ let us introduce a non-decreasing function $\widetilde{\sigma}(t)$, $-\infty < t < \infty$ of bounded variation

$$\widetilde{\sigma}(t) := \left\langle \widetilde{E}_t\widehat{e}_0, \widehat{e}_0\right\rangle_{\mathrm{L}_n}. \tag{2.14}$$

By (2.13), (2.14), and (2.8)

$$\begin{aligned} s_{j+k} &= \langle \widehat{e}_k, \widehat{e}_j\rangle_{\mathrm{L}_n} = \left\langle \widetilde{A}^k\widehat{e}_0, \widetilde{A}^j\widehat{e}_0\right\rangle_{\mathrm{L}_n} \\ &= \int_{-\infty}^{\infty} t^{j+k} d\left\langle \widetilde{E}_t\widehat{e}_0, \widehat{e}_0\right\rangle_{\mathrm{L}_n} = \int_{-\infty}^{\infty} t^{j+k} d\widetilde{\sigma}(t), \quad 0 \le j, k \le n. \end{aligned}$$

Let us denote by $\mathrm{L}_{n-1}$ the subspace of $\mathrm{L}_n$ generated by polynomials of a degree $\le n-1$. By definition of the operator $\widetilde{A}$ its restriction $A_0$ to the subspace $\mathrm{L}_{n-1}$ is a symmetric operator which actually does not depend on the choice of a canonical solution of the truncated Hamburger moment problem. Therefore each canonical solution $\widetilde{\sigma}(t)$ of this problem generates some selfadjoint extension $\widetilde{A}$ of $A_0$ in $\mathrm{L}_n$. On the other hand, each canonical selfadjoint extension $\widetilde{A}$ of $A_0$ in $\mathrm{L}_n$ generates a certain solution $\widetilde{\sigma}(t)$ of the truncated Hamburger moment problem. By the above formulas such a solution is at the same time a solution of the Stieltjes problem if and only if the corresponding spectral function $\widetilde{E}_t$ has no points of growth on the half-axis $(-\infty, 0)$, i.e. if and only if $\widetilde{A}$ is a non-negative extension of $A_0$. Such an extension of $A_0$ may exist only if the operator $A_0$ is itself non-negative, i.e. the quadratic form of $A_0$ is non-negative. But this is the case, since by our assumptions for a class $\hat{f} \in \mathrm{L}_{n-1}$ containing a polynomial

$$f(t) = \sum_{r=0}^{n-1} \xi_r t^r,$$

we have

$$\left\langle A_0 \hat{f}, \hat{f} \right\rangle_{\mathrm{L}_n} = \sum_{j,k=0}^{n-1} s_{j+k+1} \xi_k \overline{\xi_j} \geq 0. \tag{2.15}$$

If $\mathrm{L}_n = \mathrm{L}_{n-1}$, i.e. if $\det \Gamma_n = 0$, then $A_0$ is a selfadjoint operator and in this case the truncated Hamburger problem has a unique solution $\sigma_0(t)$. This solution is generated according to (2.14) by the spectral function $E_t^0$ of $A_0$. Since $A_0 \geq 0$, then $\sigma_0(t)$ is also the unique solution of the truncated Stieltjes problem.

If $\mathrm{L}_n \neq \mathrm{L}_{n-1}$, i.e. if $\det \Gamma_n > 0$, then put $\mathcal{N} = \mathrm{L}_n \ominus \mathrm{L}_{n-1}$, $\dim \mathcal{N} = 1$. Observe that in this case the matrix $\Gamma_n$ is positive definite and, hence, invertible. Moreover, by the invertibility of $\Gamma_n$ and condition c) of the theorem the sign "=" in (2.15) can be dropped.

Indeed, if the quadratic form in (2.15) would be equal to zero for some set of complex numbers $\xi_0, ..., \xi_{n-1}$, $\max_{0 \leq k \leq n-1} |\xi_k| > 0$, then by condition c) of the theorem the matrix $\Gamma_{n-1}^{(2)} = (s_{j+k+2})_{j,k=0}^{n-1}$ is not invertible. But $\Gamma_{n-1}^{(2)}$ is a diagonal block of positive definite matrix $\Gamma_n$, a contradiction.

With respect to the representation of $\mathrm{L}_n$ as the orthogonal sum $\mathrm{L}_{n-1} \oplus \mathcal{N}$, we can represent a self-adjoint extension $\widetilde{A}$ of $A_0$ as a $2 \times 2$ block operator matrix

$$\widetilde{A} = \begin{pmatrix} A_{00} & G^* \\ G & \widetilde{H} \end{pmatrix},$$

where $A_{00}$ is a symmetric operator in $\mathrm{L}_{n-1}$, the quadratic form of which coincides with that of $A_0$, $G = P_{\mathcal{N}} A_{0|\mathrm{L}_{n-1}}$, where $P_{\mathcal{N}}$ is the orthogonal projector onto the one-dimensional subspace $\mathcal{N}$ in $\mathrm{L}_n$, and $\widetilde{H}$ is a self-adjoint operator in $\mathcal{N}$, which defines the extension $\widetilde{A}$. By (2.15) $A_{00}$ is a positive definite operator. Using the Schur-Frobenius factorization we can represent $\widetilde{A}$ in the form

$$\widetilde{A} = \begin{pmatrix} I & 0 \\ GA_{00}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{00} & 0 \\ 0 & \widetilde{H} - GA_{00}^{-1}G^* \end{pmatrix} \begin{pmatrix} I & A_{00}^{-1}G^* \\ 0 & I \end{pmatrix}.$$

By this representation the extension $\widetilde{A} \geq 0$ if and only if $\widetilde{H} \geq GA_{00}^{-1}G^*$. Since in the role of $\widetilde{H}$ there can appear here any selfadjoint operator in $\mathcal{N}$ of the form

$$\widetilde{H} = GA_{00}^{-1}G^* + \tau I_{\mathcal{N}},$$

where $\tau \geq 0$ and $I_{\mathcal{N}}$ is the unity operator in $\mathcal{N}$, we conclude that the condition c) of Theorem 2.1 provides the existence of non-negative extensions $\widetilde{A}$ of $A_0$. □

## 3. Canonical solutions

We will call *canonical* the solutions of the truncated matrix Stieltjes problem given by the expression (2.14), where $\widetilde{E}_t$ is the spectral function of some non-negative selfadjoint extensions $\widetilde{A}$ of $A_0$ in $\mathrm{L}_n$. The established correspondence between the set of such extensions of $A_0$ and the set of the canonical solutions of the Stieltjes problem makes it possible to find under conditions of Theorem 2.1 an explicit algebraic formulas for the description of the sought canonical solutions using as a starting point (2.14) and the relation

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\widetilde{\sigma}(t)}{t-z} =& \int_{-\infty}^{\infty} \frac{1}{t-z} d\left\langle \widetilde{E}_t \widehat{e}_0, \widehat{e}_0 \right\rangle \\ =& \langle (\widetilde{A}-z)^{-1} \widehat{e}_0, \widehat{e}_0 \rangle.\end{aligned} \tag{3.1}$$

Let us consider first the degenerate case $\det \Gamma_n = 0$. Then by the above arguments the truncated Hamburger moment problem has a unique solution, which is at the same time the unique solution of the Stieltjes problem. To obtain an explicit expression for $\langle (\widetilde{A}-z)^{-1} \widehat{e}_0, \widehat{e}_0 \rangle$ in the degenerate case without loss of generality we can assume that

$$\operatorname{rank} \Gamma_n = n. \tag{3.2}$$

Otherwise we might ignore some last moments and consider instead of $\Gamma_n$ a Hankel matrix $\Gamma_{n'} = (s_{j+k})_{j,k=0}^{n'}$ with $n' < n$, for which

$$\operatorname{rank} \Gamma_{n'} = n'.$$

By (3.2) there is only one (up to some numerical factors) set of numbers $\xi_{00}, \dots, \xi_{0n}$ such that

$$\sum_{j,k=0}^{n} s_{j+k} \xi_{0k} \overline{\xi_{0j}} = 0.$$

Condition b) of Theorem 2.1 together with 3.2 provide that for this vector $\xi_{0n} \neq 0$. As follows, for this vector the polynomial

$$d(t) := \sum_{r=0}^{n} \xi_{0r} t^r$$

has exactly the degree $n$. Set

$$e(z) = \int_{-\infty}^{\infty} \frac{d(t) - d(z)}{t - z} d\widetilde{\sigma}(t),$$

where $\widetilde{\sigma}(t)$ is the unique solution of the truncated Hamburger (Stieltjes) moment problem for the degenerate case. According to [7] in this case $\widetilde{\sigma}(t)$ can be calculated immediately by the poles and corresponding residues at them of the rational function

$$\langle (\widetilde{A} - z)^{-1} \widehat{e}_0, \widehat{e}_0 \rangle = \int_{-\infty}^{\infty} \frac{d\widetilde{\sigma}(t)}{t - z} = \frac{e(z)}{d(z)}. \tag{3.3}$$

From now on we will assume that $\det \Gamma_n > 0$, i.e. we will consider the *non-degenerate* case of the above problems.

Let $\mathbb{C}_n$ denote the $(n+1)$-dimensional linear space of column vectors

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_n \end{pmatrix}, \quad \xi_0, \dots, \xi_n \in \mathbb{C}, \tag{3.4}$$

and the scalar product

$$(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{j=0}^{n} \xi_j \overline{\eta_j}.$$

The same linear vector space but with the scalar product

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = (\Gamma_n \boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{j,k=0}^{n} s_{j+k} \xi_k \overline{\eta_j},$$

which was considered above as the space of polynomials, we will denote it as before by $\mathrm{L}_n$.

Let $\mathbb{C}_{n-1}$ be the subspace of $\mathbb{C}_n$ consisting of vectors (3.4) with $\xi_n = 0$ and let $\mathfrak{N} = \mathbb{C}_n \ominus \mathbb{C}_{n-1}$. We denote by $P_{\mathfrak{N}}$ the orthogonal projector in $\mathbb{C}_n$ onto $\mathfrak{N}$. Evidently, in the natural basis of subspaces of $\mathbb{C}_n$ the projector is given as the $(n+1) \times (n+1)$ matrix

$$P_{\mathfrak{N}} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$

Let us consider the linear operator $T$ given as the $(n+1)\times(n+1)$ block operator matrix

$$T=\begin{pmatrix}0&0&0&\cdots&0\\1&0&0&\cdots&0\\0&1&0&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&\cdots&1&0\end{pmatrix}.$$

The introduced above symmetric operator $A_0$ in $\mathrm{L}_n$ is the restriction of $T$ to $\mathbb{C}_{n-1}$. Let $\widetilde{\Gamma}^{(1)}_{n-1}$ be the $(n+1)\times(n+1)$ block operator matrix

$$\widetilde{\Gamma}^{(1)}_{n-1}=\begin{pmatrix}\Gamma^{(1)}_{n-1}&0_{n,1}\\0_{1,n}&0_{1,1}\end{pmatrix},$$

where $0_{n,m}$ are the $n\times m$ null-matrices. Note that for $\boldsymbol{\xi}\in\mathbb{C}_{n-1}$ and any $\boldsymbol{\eta}\in\mathbb{C}_n$ we have

$$\begin{aligned}\langle A_0\boldsymbol{\xi},\boldsymbol{\eta}\rangle=&\langle T\boldsymbol{\xi},\boldsymbol{\eta}\rangle=\left(\widetilde{\Gamma}^{(1)}_{n-1}\boldsymbol{\xi},\boldsymbol{\eta}\right)+(P_{\mathfrak{N}}\Gamma_nT\boldsymbol{\xi},\boldsymbol{\eta})\\=&\langle\Gamma_n^{-1}\widetilde{\Gamma}^{(1)}_{n-1}\boldsymbol{\xi},\boldsymbol{\eta}\rangle+\langle\Gamma_n^{-1}P_{\mathfrak{N}}\Gamma_nT\boldsymbol{\xi},\boldsymbol{\eta}\rangle.\end{aligned}$$

Hence

$$A_{0|\mathbb{C}_{n-1}}=\Gamma_n^{-1}\widetilde{\Gamma}^{(1)}_{n-1|\mathbb{C}_{n-1}}+\Gamma_n^{-1}P_{\mathfrak{N}}\Gamma_nT_{|\mathbb{C}_{n-1}}.\tag{3.5}$$

Put $P_{\mathfrak{N}}^{\perp}=I-P_{\mathfrak{N}}$. By (3.5) any selfadjoint extension $\widetilde{A}$ of $A$ in $\mathrm{L}_n$ has the form

$$\begin{aligned}\widetilde{A}=&\Gamma_n^{-1}\widetilde{\Gamma}^{(1)}_{n-1}P_{\mathfrak{N}}^{\perp}+\Gamma_n^{-1}P_{\mathfrak{N}}\Gamma_nTP_{\mathfrak{N}}^{\perp}\\&+\Gamma_n^{-1}P_{\mathfrak{N}}^{\perp}T^*\Gamma_nP_{\mathfrak{N}}+\Gamma_n^{-1}\widetilde{H},\end{aligned}\tag{3.6}$$

where

$$\widetilde{H}=\begin{pmatrix}0&0\\0&H\end{pmatrix},$$

and $H$ is some real number, which defines the extension $\widetilde{A}$. In a more detailed form,

$$\widetilde{A}=\Gamma_n^{-1}\begin{pmatrix}&&&s_{n+1}\\&\Gamma^{(1)}_{n-1}&&\vdots\\&&&s_{2n}\\s_{n+1}&\cdots&s_{2n}&H\end{pmatrix}\tag{3.7}$$

$$=T+\Gamma_n^{-1}\begin{pmatrix}&&&s_{n+1}\\&0_{n,n}&&\vdots\\&&&s_{2n}\\0&\cdots&0&H\end{pmatrix}\tag{3.8}$$

Observe, as before, that the invertibility of $\Gamma_n$ and condition c) of Theorem 2.1 provide the invertibility of the matrix $\Gamma^{(1)}_{n-1}$. Write $\Gamma^{(1)-1}_{n-1} = (b_{jk})^{n-1}_{j,k=0}$ and put

$$\left(\widetilde{\Gamma}^{(1)}_{n-1}\right)^{-1}_{\text{cond}} = \begin{pmatrix} \Gamma^{(1)-1}_{n-1} & 0_{n,1} \\ 0_{1,n} & 0_{1,1} \end{pmatrix}.$$

By the above argument the operator defined by the block matrix (3.7) is non-negative if and only if

$$\widetilde{H} - P_{\mathfrak{N}}\Gamma_n T P^{\perp}_{\mathfrak{N}} \left(\widetilde{\Gamma}^{(1)}_{n-1}\right)^{-1}_{\text{cond}} P^{\perp}_{\mathfrak{N}} T^{*}\Gamma_n P_{\mathfrak{N}} \geq 0,$$

or, equivalently, if and only if

$$H - \sum_{j,k=0}^{n-1} s_{n+j+1} b_{jk} s_{n+k+1} \geq 0. \tag{3.9}$$

Since

$$Q := \sum_{j,k=0}^{n-1} s_{n+j+1} b_{jk} s_{n+k+1}$$

is positive, all numbers $H$ generating non-negative extensions $\widetilde{A}$ and hence solutions of the Stieltjes problem must be positive definite and, moreover, satisfy the inequality $H \geq Q$. Note that the requirement $\widetilde{A} \gg 0$ excludes the equality in (3.9).

Put now

$$\Theta_H(z) := \Gamma_n \left(\Gamma^{(1)}_{H;n} - z\Gamma_n\right)^{-1} \Gamma_n, \tag{3.10}$$

where

$$\Gamma^{(1)}_{H;n} := \begin{pmatrix} & & & s_{n+1} \\ & \Gamma^{(1)}_{n-1} & & \vdots \\ & & & s_{2n} \\ s_{n+1} & \cdots & s_{2n} & H \end{pmatrix} \tag{3.11}$$

and let $\Delta_H(z)$, $\operatorname{Im} z > 0$, be the upper left element of $\Theta_H(z)$,

$$\Delta_H(z) := \Theta_{H;00}(z). \tag{3.12}$$

The following theorem is an evident combination of the above arguments.

**Theorem 3.1.** *Let conditions of Theorem* 2.1 *hold and* $\det \Gamma_n > 0$. *Then the relation*

$$\int_{-\infty}^{\infty} \frac{d\sigma_H(t)}{t-z} = \Delta_H(z), \quad \operatorname{Im} z > 0,$$

*establishes the one-to-one correspondence between the set of all canonical solutions of the truncated matrix Stieltjes moment problem with the given moments* $s_0, ..., s_{2n}$*, and the set of positive real numbers* $H$ *satisfying the inequality*

$$H - \sum_{j,k=0}^{n-1} s_{n+j+1} b_{jk} s_{n+k+1} \geq 0. \tag{3.13}$$

Actually Theorem 3.1 with (3.10), (3.11) describes in the non-degenerate case perhaps a new algebraic algorithm permitting to obtain canonical solutions of the truncated Stieltjes moment problem and with omission of (3.13) also the algorithm for getting of those for the Hamburger problem.

Compare this algorithm with that described, in particular, in [7] for the Hamburger problem. To this end set

$$\Pi_r = (\underbrace{0, ..., 0}_{r}, 1)^T, \ \Upsilon_r(t) = (1, t, ..., t^r), \quad r = n-1, n,$$

the symbol $T$ denotes the operation of transposition. Since $\Gamma_n$ is positive definite and invertible, the same is true for all $\Gamma_r := (s_{k+j})_{k,j=0}^r$, $0 \leq r \leq n-1$. Let us introduce polynomials

$$D_r(t) = \Upsilon_r(t) \Gamma_r^{-1} \Pi_r, \quad r = n-1, n \tag{3.14}$$

and the corresponding conjugate polynomials

$$E_r(z) := \int_{-\infty}^{\infty} \frac{D_r(t) - D_r(z)}{t - z} d\sigma(t). \tag{3.15}$$

Let $\mathbb{N}$ be the Nevanlinna class of holomorphic in the upper half-plane functions with non-negative imaginary parts and let

$$\mathbb{N}_0 = \left\{ \Omega \in \mathbb{N} \mid \lim_{y \uparrow \infty} \frac{1}{y} \Omega(iy) = 0 \right\}.$$

By [6], [4], [7] under all above assumptions the Nevanlinna formula

$$\varphi(z) = \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t - z} = -\frac{E_n(z)(R(z) + z) - E_{n-1}(z)}{D_n(z)(R(z) + z) - D_{n-1}(z)},$$

$$R(z) = \left(\Gamma_n^{-1}\right)_{nn}^{-1} \Omega(z), \quad \operatorname{Im} z > 0, \tag{3.16}$$

which in the context of operator theory follows directly from the M. G. Krein resolvent formula [5], establishes the one-to-one correspondence between the set of all distributions $\sigma(t)$, $-\infty < t < \infty$, satisfying (2.8), and the set of Nevanlinna functions $\Omega \in \mathbb{N}_0$.

The same formula with $\Omega(z)$ replaced by any real constant $\widehat{H}$ establishes the one-to-one correspondence between the set of all canonical measures $\sigma_{\widehat{H}}(t)$, satisfying (2.8), and the set of all real numbers $\widehat{H}$. For a canonical solution $\sigma_{\widehat{H}}(t)$ of the truncated Hamburger problem for the given moments the expression in the right hand side of (3.16) is a rational function of the Nevanlinna class $\mathbb{N}_0$. The poles of this function are the roots of the polynomial

$$\mathfrak{P}_{\widehat{H}}(z) := \left(D_n(z)\left(\left(\Gamma_n^{-1}\right)_{nn}^{-1}\widehat{H}+z\right)-D_{n-1}(z)\right) \tag{3.17}$$

By [7] $\mathfrak{P}_{\widehat{H}}(z)$ has only real roots. These roots are unique points of growth of $\sigma_{\widehat{H}}(t)$. Therefore a canonical solution $\sigma_{\widehat{H}}(t)$ of the Hamburger problem is at the same time a solution of the Stieltjes problem for the same set of moments, if and only if $\mathfrak{P}_{\widehat{H}}(z)$ for the corresponding number $\widehat{H}$ has no roots on the half-axis $(-\infty,0)$. By [7] the real numbers $H$ in (3.7), (3.8) defining a canonical solution $\sigma_H(t)$ of the Stieltjes or Hamburger problem through the selfadjoint extensions $\widetilde{A}$ of $A_0$ given by (3.6) and the number $\widehat{H}$ replacing $\Omega(z)$ in (3.16) to obtain the same solution $\sigma_H(t)$, are connected by the relation

$$\begin{pmatrix}0 & 0\\ 0 & \widehat{H}\end{pmatrix} = P_{\mathfrak{N}}\widetilde{A}\Gamma_n^{-1}P_{\mathfrak{N}}.$$

Hence

$$\begin{aligned}\widehat{H} =&(\Gamma_n^{-1})_{n-1,n}+\sum_{j=0}^{n-1}(\Gamma_n^{-1})_{nj}s_{j+1}(\Gamma_n^{-1})_{nn}+(\Gamma_n^{-1})_{nn}H(\Gamma_n^{-1})_{nn}\\ :=&\Lambda_n+(\Gamma_n^{-1})_{nn}H(\Gamma_n^{-1})_{nn}\end{aligned} \tag{3.18}$$

and

$$H=(\Gamma_n^{-1})_{nn}^{-1}\left(\widehat{H}-\Lambda_n\right)(\Gamma_n^{-1})_{nn}^{-1}. \tag{3.19}$$

We see that *the formula*

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{d\sigma_H(t)}{t-z}=-\frac{E_n(z)(R_H+z)-E_{n-1}(z)}{D_n(z)(R_H+z)-D_{n-1}(z)},\\ &R_H=\left(\Gamma_n^{-1}\right)_{nn}^{-1}\Lambda_n-H(\Gamma_n^{-1})_{nn},\quad \operatorname{Im}z>0,\end{aligned} \tag{3.20}$$

*establishes in the non-degenerate case the one-to-one correspondence between the set of all canonical solutions $\sigma_H(t)$ of the truncated matrix Stieltjes problem and the set of positive numbers $H$ satisfying* (3.9).

Comparing the latter result with the assertion of Theorem 3.1 we conclude that

$$\left(\Gamma_n\left(\Gamma^{(1)}_{H;n}-z\Gamma_n\right)^{-1}\Gamma_n\right)_{00}$$
$$=-\frac{E_n(z)(R_H+z)-E_{n-1}(z)}{D_n(z)(R_H+z)-D_{n-1}(z)}, \tag{3.21}$$
$$R_H=\left(\Gamma_n^{-1}\right)^{-1}_{nn}\Lambda_n-H(\Gamma_n^{-1})_{nn},\quad \operatorname{Im}z\neq 0. \tag{3.22}$$

Observe that for $R_H$ given by (3.22) with $H>0$ satisfying (3.9) all roots of polynomial $D_n(z)(R_H+z)-D_{n-1}(z)$ are located on the interval $[0,\infty)$.

## 4. Description of all solutions of the truncated Stieltjes problem in the non-degenerate case

Due to (3.1), the description of all solutions of the Stieltjes problem is reduced to the construction of an appropriate formula for the upper left element of the resolvent $\left((\widetilde{A}-z)^{-1}\right)$ of non-negative selfadjoint extensions of $A_0$ with going out of $\mathrm{L}_n$. Since each solution of the Stieltjes problem is at the same time a solution of the Hamburger problem for the same set of moments, then we can use (3.16) as the sought description formula specifying only how to restrict the set of the "parameters" $\Omega(z)$ to get by (3.16) all measures $\sigma(t)$ corresponding to the non-negative extensions, and only them.

To this end let us consider a non-negative self-adjoint extension $A$ of $A_0$ with going out of $\mathrm{L}_n$ into a Hilbert space $\mathcal{H}=\mathrm{L}_n\oplus\mathcal{H}'$, $\dim\mathcal{H}'\leq\infty$. In general, $A$ is an unbounded operator, but since $A$ is an extension of $A_0$, then $\mathrm{L}_{n-1}\subset\mathcal{D}_A$. Suppose first that $\mathrm{L}_n(=\mathrm{L}_{n-1}\oplus\mathcal{N})\subset\mathcal{D}_A$. Then with respect to the splitting $\mathcal{H}=\mathrm{L}_{n-1}\oplus\mathcal{N}\oplus\mathcal{H}'$ we can represent $A$ in the form

$$A=\begin{pmatrix}A_{00} & G^* & 0\\ G & H_A & G_1^*\\ 0 & G_1 & A_{11}\end{pmatrix}, \tag{4.1}$$

where $A_{00}$, $G$ are defined as above, $H_A$ is a non-negative operator in $\mathcal{N}$, $G_1$ is a bounded operator from $\mathcal{N}$ into $\mathcal{H}'$ and $A_{11}$ is a non-negative selfadjoint operator in $\mathcal{H}'$. Taking any $\lambda<0$ and applying the Schur-Frobenius factorization yields

$$A-\lambda=\begin{pmatrix}1 & 0 & 0\\ G(A_{00}-\lambda)^{-1} & 1 & G_1^*(A_{11}-\lambda)^{-1}\\ 0 & 0 & 1\end{pmatrix} \tag{4.2}$$

$$\times\begin{pmatrix} A_{00}-\lambda & 0 & 0\\ 0 & H_A-\lambda-G\left(A_{00}-\lambda\right)^{-1}G^*-G_1^*\left(A_{11}-\lambda\right)^{-1}G_1 & 0\\ 0 & 0 & A_{11}-\lambda\end{pmatrix}$$

$$\times\begin{pmatrix} 1 & \left(A_{00}-\lambda\right)^{-1}G^* & 0\\ 0 & 1 & 0\\ 0 & (A_{11}-\lambda)^{-1}G_1 & 1\end{pmatrix}.$$

By (4.2) the assumption $A\geq 0$ is equivalent to the conditions:

$$A_{00}-\lambda\gg 0,\quad A_{11}-\lambda\gg 0,$$
$$H_A-G\left(A_{00}-\lambda\right)^{-1}G^*-G_1^*\left(A_{11}-\lambda\right)^{-1}G_1\geq 0 \tag{4.3}$$

for any $\lambda<0$. Writing $A-z$, $\operatorname{Im} z>0$, with respect to the representation $\mathcal{H}=\mathrm{L}_n\oplus\mathcal{H}'$ in the $LDU$ form, we have:

$$A-z=\begin{pmatrix} I & U\\ 0 & 1\end{pmatrix}\begin{pmatrix} W(z) & 0\\ 0 & A_{11}-z\end{pmatrix}\begin{pmatrix} I & 0\\ U^* & 1\end{pmatrix}, \tag{4.4}$$

where $I=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$ is the unit $2\times 2$ block matrix,

$$U=\begin{pmatrix} 0\\ G_1^*(A_{11}-z)^{-1}\end{pmatrix},$$

and

$$W(z)=\begin{pmatrix} A_{00}-z & G^*\\ G & H_A-z-G_1^*(A_{11}-z)^{-1}G_1\end{pmatrix}. \tag{4.5}$$

By (4.4) and (3.1) the solution $\sigma_A(t)$ of the truncated Stieltjes problem generated by the extension $A$ is defined by the expression

$$\int_{-\infty}^{\infty}\frac{d\sigma_A(t)}{t-z}=\left\langle (A-z)^{-1}\widehat{e}_0,\widehat{e}_0\right\rangle=\left\langle W(z)^{-1}\widehat{e}_0,\widehat{e}_0\right\rangle. \tag{4.6}$$

Put

$$\Theta_A(z):=\Gamma_n\left(\Gamma_{A;n}^{(1)}(z)-z\Gamma_n\right)^{-1}\Gamma_n, \tag{4.7}$$

where

$$\Gamma_{A;n}^{(1)}(z):=\begin{pmatrix} & \Gamma_{n-1}^{(1)} & & \begin{matrix} s_{n+1}\\ \vdots\\ s_{2n}\end{matrix}\\ s_{n+1} & \cdots & s_{2n} & H_A-G_1^*(A_{11}-z)^{-1}G_1\end{pmatrix} \tag{4.8}$$

and let $\Delta_{A;00}(z)$, $\operatorname{Im} z>0$, be the upper left element of $\Theta_A(z)$. The reasoning similar to the one which resulted in the proof of Theorem 3.1 shows that

$$\left\langle W(z)^{-1}\widehat{e}_0,\widehat{e}_0\right\rangle=\Delta_{A;00}(z). \tag{4.9}$$

Put now

$$\Omega_A(z) = G_1^*(A_{11} - z)^{-1} G_1.$$

Comparing expressions (4.7)–(4.9) and (3.10)–(3.12), we conclude that the replacement of the number $H$ by the function $H_A - \Omega_A(z)$ on both sides of (3.21) cannot violate this equality at least for $z \in (-\infty, 0)$. Therefore

$$\int_{-\infty}^{\infty} \frac{d\sigma_A(t)}{t - z} = -\frac{E_n(z)\,(R_A(z) + z) - E_{n-1}(z)}{D_n(z)\,(R_A(z) + z) - D_{n-1}(z)},$$
$$R_A = \left(\Gamma_n^{-1}\right)_{nn}^{-1} \Lambda_n + (\Omega_A(z) - H_A)\,(\Gamma_n^{-1})_{nn}, \quad \operatorname{Im} z \neq 0.$$

If $\mathcal{N} \subsetneq \mathcal{D}_A$, then the representation (4.1) is not valid anymore. However, with respect to the representation $\mathcal{H} = \mathrm{L}_{n-1} \oplus \mathcal{H}''$, $\mathcal{H}'' = \mathcal{N} \oplus \mathcal{H}'$ we can write

$$A = \begin{pmatrix} A_{00} & G^* \\ G & A'_{11} \end{pmatrix}, \tag{4.10}$$

where $A_{00}$ and $G$ are defined as before and $A'_{11}$ is some non-negative self-adjoint operator in $\mathcal{H}'$. Applying again the Schur-Frobenius factorization with account of (4.10) we obtain the representation

$$(A - z)^{-1} = \begin{pmatrix} W_{A;00}(z)^{-1} & W_A(z)^{-1} G^* (A'_{11} - z)^{-1} \\ (A'_{11} - z)^{-1} G W_A(z)^{-1} & W_{A;11}(z)^{-1} \end{pmatrix}, \tag{4.11}$$

where

$$W_{A;00}(z) = A_{00} - z - G^*(A'_{11} - z)^{-1} G, \tag{4.12}$$
$$W_{A;11}(z)^{-1} = (A'_{11} - z)^{-1} + (A'_{11} - z)^{-1} G W_{A;00}(z)^{-1} G^* (A'_{11} - z)^{-1}. \tag{4.13}$$

Let $P_n$ be the orthogonal projector $\mathrm{L}_n$ in $\mathcal{H}$ and let

$$\Xi_A(z) = P_{\mathcal{N}} (A'_{11} - z)^{-1} |_{\mathcal{N}}, \quad \operatorname{Im} z \neq 0. \tag{4.14}$$

By (4.11) and (4.13) the generalized resolvent $R_z(A) := P_n (A'_{11} - z)^{-1} |_{\mathrm{L}_n}$ of $A$ can be represented in the form

$$R_z(A) = \begin{pmatrix} W_{A;00}(z)^{-1} & W_{A;00}(z)^{-1} G^* \Xi_A(z) \\ \Xi_A(z) G W_{A;00}(z)^{-1} & \Xi_A(z) + \Xi_A(z) G W_{A;00}(z)^{-1} G^* \Xi_A(z) \end{pmatrix}, \quad \operatorname{Im} z \neq 0. \tag{4.15}$$

Using expressions (4.12) and (4.15), it is not difficult to verify by direct calculations that

$$R_z(A) = \begin{pmatrix} A_{00} - z & G^* \\ G & \Xi_A(z)^{-1} \end{pmatrix}^{-1}, \quad \operatorname{Im} z \neq 0. \tag{4.16}$$

Comparing (4.16) and (4.5), we conclude that in the general non-degenerate case the solution $\sigma_A(t)$ of the truncated Stieltjes problem generated by a non-negative selfadjoint extension $A$ of $A_0$ is defined as above through the upper left element of the matrix function $\Gamma_n \left(\Gamma^{(1)}_{A;n}(z) - z\Gamma_n\right)^{-1} \Gamma_n$, where

$$\Gamma^{(1)}_{A;n}(z) := \begin{pmatrix} & & & s_{n+1} \\ & \Gamma^{(1)}_{n-1} & & \vdots \\ & & & s_{2n} \\ s_{n+1} & \cdots & s_{2n} & \Xi_A(z)^{-1} + z \end{pmatrix}. \tag{4.17}$$

Let $\mathbb{S}_0$ be the subset of $\mathbb{N}$ consisting of all Nevanlinna functions $\Omega(z)$, $\operatorname{Im} z > 0$, which admit the integral representation

$$\Omega(z) = \int_0^\infty \frac{d\rho(t)}{t-z}$$

with a non-decreasing function $\rho(t)$ such that

$$\int_0^\infty d\rho(t) < \infty.$$

Evidently, the introduced above functions $\Omega_A(z)$, $\Xi_A(z) \in \mathbb{S}_0$. On the other hand, using usual constructions of the spectral theory of linear operators in Hilbert spaces one can verify that any function $\Xi(z) \in \mathbb{S}_0$ admits *the realization* (4.14), i.e. for such a function there exists a non-negative operator $A'_{11}$ in a Hilbert space $\mathcal{N} \oplus \mathcal{H}'$ such that for $\Xi$ the equality (4.14) holds. However, the functions $\Xi_A$ which are connected with non-negative extensions $A$ generating solutions of the Stieltjes problem satisfy the additional condition: they are such that for any $\lambda < 0$ the block operator $R_\lambda(A)$ defined by (4.16) is positive. Write

$$\left(\Gamma^{(1)}_{n-1} - z\right)^{-1} = (b_{jk}(z))_{j,k=0}^{n-1}, \quad z \notin [0,\infty),$$

The latter condition for $\Xi$ is equivalent to

$$\Xi(\lambda)^{-1} - \sum_{j,k=0}^{n-1} s_{n+j+1} b_{jk}(\lambda) s_{n+k+1} > 0, \quad \lambda < 0. \tag{4.18}$$

Note that the function on the left hand side of (4.18) is non-increasing on the negative half-axis. Therefore formally admitting that $\Xi(0)^{-1}$ may be $+\infty$, instead of (4.18) we can write

$$\Xi(-0)^{-1} - \sum_{j,k=0}^{n-1} s_{n+j+1} b_{jk} s_{n+k+1} \geq 0. \tag{4.19}$$

We have thus proven the following theorems.

**Theorem 4.1.** *Let conditions of Theorem* 2.1 *hold and* $\det \Gamma_n > 0$. *Then the relation*

$$\int_{-\infty}^{\infty} \frac{d\sigma_\Xi(t)}{t-z} = \Delta_\Xi(z), \quad \operatorname{Im} z > 0,$$

*where* $\Delta_\Xi(z)$ *is the upper-left element of the matrix function*

$$\Gamma_n \left(\Gamma^{(1)}_{\Xi;n}(z) - z\Gamma_n\right)^{-1} \Gamma_n$$

*with*

$$\Gamma^{(1)}_{\Xi;n}(z) := \begin{pmatrix} & & & s_{n+1} \\ & \Gamma^{(1)}_{n-1} & & \vdots \\ & & & s_{2n} \\ s_{n+1} & \cdots & s_{2n} & \Xi(z)^{-1} + z \end{pmatrix},$$

*establishes the one-to-one correspondence between the set of all solutions of the truncated Stieltjes moment problem with given moments* $s_0, ..., s_{2n}$, *and the subset of Nevanlinna functions* $\Xi$ *from* $\mathbb{S}_0$ *satisfying* (4.18).

**Theorem 4.2.** *Let conditions of Theorem* 4.1 *hold. Then the formula*

$$\int_{-\infty}^{\infty} \frac{d\sigma_\Xi(t)}{t-z} = -\frac{E_n(z)(R_\Xi(z)+z) - E_{n-1}(z)}{D_n(z)(R_\Xi(z)+z) - D_{n-1}(z)},$$
$$R_\Xi = \left(\Gamma_n^{-1}\right)_{nn}^{-1} \Lambda_n - \Xi(z)^{-1}(\Gamma_n^{-1})_{nn}, \quad \operatorname{Im} z > 0,$$

*establishes the one-to-one correspondence between the set of all solutions* $\sigma_\Xi(t)$ *of the truncated Stieltjes problem and the subset of Nevanlinna matrix functions* $\Xi$ *from* $\mathbb{S}_0$ *satisfying* (4.18).

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