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STABILITY OF THE INTEGRAL CONVOLUTION OF k-UNIFORMLY CONVEX AND k-STARLIKE FUNCTIONS

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Abstract. For a constant $k \in [0,\infty)$ a normalized function f, analytic in the unit disk, is said to be k-uniformly convex if $\operatorname{Re}(1 + zf''(z)/f'(z)) > k|zf''(z)/f'(z)|$ at any point in the unit disk. The class of k-uniformly convex functions is denoted k-$\mathcal{UCV}$ (cf. [4]). The function g is said to be k-starlike if $g(z) = zf'(z)$ and $f \in k$-$\mathcal{UCV}$.

For analytic functions f, g, where $f(z) = z + a_2z^2 + \cdots$ and $g(z) = z + b_2z^2 + \cdots$, the integral convolution is defined as follows:

$$(f \otimes g)(z) = z + \sum_{n=2}^{\infty} \frac{a_n b_n}{n} z^n.$$

In this note a problem of stability of the integral convolution of k-uniformly convex and k-starlike functions is investigated.

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1. Introduction and definitions

Let $\mathcal{H}$ denote the class of functions f analytic in the unit disk $\mathcal{U}$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

and let $\mathcal{S}$ denote the subclass of functions in $\mathcal{H}$ which are *univalent* in $\mathcal{U}$. Further, for $k \in [0, \infty)$ let k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ be the subclasses of $\mathcal{S}$ consisting, respectively, of functions which are *k-uniformly convex* and *k-starlike* in $\mathcal{U}$ defined, respectively, as follows:

$$k\text{-}\mathcal{UCV} := \left\{ f \in \mathcal{S} : \ \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \mathcal{U} \right\}, \tag{1.2}$$

$$k\text{-}\mathcal{ST} \ := \ \left\{ f \in \mathcal{S} : \ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathcal{U} \right\}. \tag{1.3}$$

Observe that the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ are related by the classical Alexander theorem, that is known also as an equivalence between usual class of convex functions $\mathcal{CV}$ and the class of starlike functions $\mathcal{ST}$. Also note, that the class k-$\mathcal{UCV}$ was defined pure geometrically as a subclass of univalent functions that map each circular arc contained in the unit disc $\mathcal{U}$ with a center ζ, $|\zeta| \le k$ ($0 \le k < \infty$), onto a convex arc. Classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ were introduced by Kanas and Wiśniowska ([4], [6]) and studied in a series of papers ([4], [5], [6], [7]). Some of properties of k-$\mathcal{UCV}$ and k-$\mathcal{ST}$, in particular, concerning the stability of the Hadamard product, were studied by Bednarz and Kanas in [1].

The *convolution, or Hadamard product,* of two functions f and g of power series $f(z) = z + a_2 z^2 + \cdots$, and $g(z) = z + b_2 z^2 + \cdots$, convergent in $\mathcal{U}$, is the function $h = f * g$ with the power series

$$h(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}. \tag{1.4}$$

Convolution has the algebraic properties of ordinary multiplication, the geometric series $K(z) = z + z^2 + \cdots = \frac{z}{1-z}$ acts as the identity element under convolution: $(f * K)(z) = f(z)$, for any $f \in \mathcal{H}, z \in \mathcal{U}$.

The *integral convolution* $H = f \otimes g$ is defined by

$$H(z) = (f \otimes g)(z) = z + \sum_{n=2}^{\infty} \frac{a_n b_n}{n} z^n = \int_0^z \frac{h(\zeta)}{\zeta} d\zeta, \quad z \in \mathcal{U}. \tag{1.5}$$

Note that, if I denotes $I(z) \equiv z$ then

$$f * I = I \quad \text{and} \quad f \otimes I = I.$$

Various properties of Hadamard product and integral convolution were studied by several authors, e.g. Robertson ([14]), Pólya and Schoenberg ([9]), Ruscheweyh and Sheil-Small ([13]), Hayman ([3]), Bshouty ([2]). The most famous is Pólya and Schoenberg conjecture ([9]), that the class of starlike functions is preserved by the integral convolution. This conjecture was proved by Ruscheweyh and Sheil-Small ([13]). However as was shown by Hayman ([3]) and Bshouty ([2]) independently, the univalence is not preserved by integral convolution.

In accordance to Ruscheweyh ([11]), let $\mathcal{V}^*$ denote the *dual set* of $\mathcal{V} \subset \mathcal{H}$. Then

$$\mathcal{V}^* = \left\{ g \in \mathcal{H} : \frac{(f * g)(z)}{z} \neq 0,\ \forall f \in \mathcal{V},\ \forall z \in \mathcal{U} \right\}, \tag{1.6}$$

and $\mathcal{V}^{**} = (\mathcal{V}^*)^*$ denote the *second dual*, which is the smallest of all dual classes containing $\mathcal{V}$. The *duality principle* states that for compact and complete $\mathcal{V}$ the closed convex hull of $\mathcal{V}$ is the same as the closed convex hull of $\mathcal{V}^{**}$, so that under fairly weak conditions on $\mathcal{V}$, many extremal problems in $\mathcal{V}$ are solved in $\mathcal{V}^{**}$ and can be obtained by unified approach.

Dual sets for the classes k-$\mathcal{ST}$ and k-$\mathcal{UCV}$ were found by Kanas, Wiśniowska (cf. [4], [7]). Let us denote the dual set for k-$\mathcal{ST}$ by $\mathcal{B}$ and for k-$\mathcal{UCV}$ by $\mathcal{G}$.

Then

$$f \in k\text{-}\mathcal{ST} \Longleftrightarrow \frac{(f * h)(z)}{z} \neq 0,\ \forall h \in \mathcal{B},\ \forall z \in \mathcal{U}, \tag{1.7}$$

and

$$f \in k\text{-}\mathcal{UCV} \Longleftrightarrow \frac{(f * h)(z)}{z} \neq 0,\ \forall h \in \mathcal{G},\ \forall z \in \mathcal{U}, \tag{1.8}$$

respectively (cf. [7]). For $h(z) = z + c_2 z^2 + \ldots,\ h \in \mathcal{B}$ we have the following estimates (cf. [7]),

$$|c_n| \leq n + (n-1)k, \quad n \geq 2, \tag{1.9}$$

and for $h \in \mathcal{G}$

$$|c_n| \leq n[n + (n-1)k], \quad n \geq 2. \tag{1.10}$$

Further, sufficient conditions to be in k-$\mathcal{ST}$ and k-$\mathcal{UCV}$ are (cf. [7])

$$\sum_{n=2}^{\infty} [n + (n-1)k]|a_n| \leq 1 \quad \Longrightarrow \quad f \in k\text{-}\mathcal{ST}, \tag{1.11}$$

$$\sum_{n=2}^{\infty} n[n + (n-1)k]|a_n| \leq 1 \quad \Longrightarrow \quad f \in k\text{-}\mathcal{UCV}, \tag{1.12}$$

respectively.

For $\delta \geq 0$ Ruscheweyh ([12]) defined N_δ *neighbourhood of a function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ by

$$N_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{H} : \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \right\}. \qquad (1.13)$$

By $N_\delta(\mathcal{A})$, $\mathcal{A} \subset \mathcal{H}$, we denote the union of all neighbourhoods $N_\delta(f)$ with f ranging over the class $\mathcal{A}$. The quantity $\sum_{n=2}^{\infty} n|a_n - b_n|$ can be regarded as the distance between two functions f and g in a some subclass of $\mathcal{H}$, equipped with the pre-norm of functions $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ defined as $||F|| = \sum_{n=2}^{\infty} n|A_n|$. Ruscheweyh proved certain inclusions for the mentioned above neighbourhoods, in particular that $N_{1/4}(f) \subset \mathcal{ST}$ holds for all $f \in \mathcal{CV}$. Problem of neighbourhoods in various classes of functions was also studied in papers [1], [7], [10], [15], [16].

Assume that A, B are subclasses of the class $\mathcal{H}$. Then the set of all function $f * g$ and $f \otimes g$, where $f \in A$ and $g \in B$, will be denoted by $A * B$ and $A \otimes B$, respectively (cf. eg. [13], [15]). Let $A * B \subset C$, the convolution (1.4) is called *C-stable on the pair of classes* (A, B) if there exists $\delta > 0$ such that $N_\delta(A) * N_\delta(B) \subset C$ and *C-unstable* otherwise (cf. [8]). Stability of inclusions for integral convolution is defined in a similar way. The constant δ which characterizes the stability of Hadamard or integral convolutions is defined as

$$\delta(A * B, C) = \sup\{\delta : N_\delta(A) * N_\delta(B) \subset C\}, \qquad (1.14)$$

and

$$\delta(A \otimes B, C) = \sup\{\delta : N_\delta(A) \otimes N_\delta(B) \subset C\}, \qquad (1.15)$$

respectively. If the related value (1.14) or (1.15) is positive then there exists a neighborhood of the class A and B mapped by the convolution or the integral convolution into C.

The stability of the Hadamard or integral convolution can be regarded as a problem of preserving a product of the topology of neighbourhoods in A and B onto the product $A * B$, and $A \otimes B$, respectively.

Problem of the stability of the inclusion for the Hadamard product as well as the integral convolution in the classes of univalent, starlike and convex functions was considered by Nezhmetdinov ([8]). Among other he proved that $\delta(\{I\} * \{I\}, \mathcal{CV}) = 1$ and $\delta(\{I\} * \{I\}, \mathcal{S}) = \sqrt{2}$. Numerous results concerning the stability of the Hadamard product were obtained by Kanas and Bednarz ([1], [7]). In particular in [7] Kanas proved that

$$N_{1/[4(k+1)]}(f) \subset k\text{-}\mathcal{ST} \quad \text{for} \quad f \in k\text{-}\mathcal{UCV}, \qquad (1.16)$$

that is a k-uniform version of the Ruscheweyh result. Also, we should mention some inequalities about stability of the Hadamard product for functions from the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$, below:

$$\delta(\{I\} * \{I\}, k\text{-}\mathcal{UCV}) \geq \frac{1}{\sqrt{k+1}} \tag{1.17}$$

$$\delta(\{I\} * \{I\}, k\text{-}\mathcal{ST}) \geq \sqrt{\frac{2}{k+1}} \tag{1.18}$$

$$\delta(k\text{-}\mathcal{UCV} * \mathcal{CV}, k\text{-}\mathcal{ST}) \geq \sqrt{4 + \frac{1}{2(k+1)^2}} - 2. \tag{1.19}$$

Let $Q[f]$ denote the integral transformation:

$$Q[f](z) = \int_0^z \frac{f(t)}{t} dt, \quad z \in \mathcal{U}, \tag{1.20}$$

where $f \in \mathcal{A} \subset \mathcal{H}$. Also, denote by $Q[A]$ the image of a subclass A under the mapping Q. It is well known that $Q[\mathcal{ST}] = \mathcal{S}$, however $Q[\mathcal{S}] \not\subset \mathcal{S}$ (cf. [13], [2], [3]). If the inclusion $Q[A] \subset B$ holds, then we can define the constant

$$\delta(Q[A], B) = \sup\{\delta : Q[N_\delta(A)] \subset B\}$$

representing a quantitative characteristic of the stability of the inclusion.

In the present article we consider a problem of a stability of the integral convolution over the class k-$\mathcal{UCV}$ and k-$\mathcal{ST}$. We also study a stability of geometric properties of a function f under the integral transformation $Q[f]$ when f is in k-$\mathcal{UCV}$ or k-$\mathcal{ST}$.

2. Stability of the integral convolution

In this section we obtain some results concerning the stability of the integral convolution in the class k-$\mathcal{UCV}$ and k-$\mathcal{ST}$.

In the sequel the following notation will be used: $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $f_0(z) = z + \sum_{n=2}^{\infty} a_{0n} z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $g_0(z) = z + \sum_{n=2}^{\infty} b_{0n} z^n$. First of all notice, that by (1.4), (1.5) and (1.20)

$$(f \otimes g)(z) = \int_0^z \frac{(f * g)(t)}{t} dt = Q[f * g](z), \tag{2.1}$$

and it is easily verified that $Q[k\text{-}\mathcal{ST}] = k\text{-}\mathcal{UCV}$, therefore

$$(f \otimes g)(z) \in k\text{-}\mathcal{UCV} \Longleftrightarrow (f * g)(z) \in k\text{-}\mathcal{ST}. \tag{2.2}$$

By the fact $\delta(k\text{-}\mathcal{ST} * \mathcal{CV}, k\text{-}\mathcal{ST}) = 0$, (cf. [7]) and by (2.2), it can be immediately seen that $\delta(k\text{-}\mathcal{ST} \otimes \mathcal{CV}, k\text{-}\mathcal{UCV}) = 0$. Below, we present other stability results over k-$\mathcal{UCV}$ and k-$\mathcal{ST}$, however the authors do not know if they are sharp.

Theorem 2.1. *For the integral convolution* (1.5) *the following inequalities concerning stability in the class k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ are satisfied:*

$$\delta(\{I\}\otimes\{I\}, k\text{-}\mathcal{ST}) \geq \frac{2}{\sqrt{k+1}}, \tag{2.3}$$

$$\delta(\{I\}\otimes\{I\}, k\text{-}\mathcal{UCV}) \geq \sqrt{\frac{2}{k+1}}, \tag{2.4}$$

$$\delta(k\text{-}\mathcal{UCV}\otimes\{I\}, k\text{-}\mathcal{ST}) \geq \sqrt{1+\frac{4}{k+1}} - 1, \tag{2.5}$$

$$\delta(k\text{-}\mathcal{ST}\otimes\{I\}, k\text{-}\mathcal{ST}) \geq \sqrt{4+\frac{4}{k+1}} - 2, \tag{2.6}$$

$$\delta(k\text{-}\mathcal{UCV}\otimes\{I\}, k\text{-}\mathcal{UCV}) \geq \sqrt{1+\frac{2}{k+1}} - 1, \tag{2.7}$$

$$\delta(k\text{-}\mathcal{UCV}\otimes\mathcal{CV}, k\text{-}\mathcal{UCV}) \geq \sqrt{4+\frac{1}{2(k+1)^2}} - 2, \tag{2.8}$$

$$\delta(k\text{-}\mathcal{ST}\otimes\mathcal{CV}, k\text{-}\mathcal{ST}) \geq \sqrt{9+\frac{1}{(k+1)^2}} - 3. \tag{2.9}$$

Proof. Making use of (2.2) and inequalities (1.18), (1.19) the relations (2.4), (2.8) follow immediately.

Further, for any $f,\ g,\ f_0,\ g_0\ \in\mathcal{H}$ and $h\in\mathcal{B}$ we have

$$\begin{aligned}\left|\frac{(f\otimes g*h)(z)}{z}\right| \geq \left|\frac{(f_0\otimes g_0*h)(z)}{z}\right| - \left|\frac{(f_0\otimes(g-g_0)*h)(z)}{z}\right| \\ - \left|\frac{((f-f_0)\otimes g_0*h)(z)}{z}\right| - \left|\frac{((f-f_0)\otimes(g-g_0)*h)(z)}{z}\right|.\end{aligned} \tag{2.10}$$

Case (2.3). Assume $f_0=g_0=I$ and let $f,g\in N_\delta(I)$, with δ satisfying (2.3). We shall show that $(f\otimes g*h)(z)/z\neq 0$ for $h\in\mathcal{B}$, or equivalently $f\otimes g\in k$-$\mathcal{ST}$ (in view of (1.7)). Observe that, by properties of Hadamard product and integral convolution, we have $(f_0\otimes(g-g_0)*h)(z)=0$, $((f-f_0)\otimes g_0*h)(z)=0$ and $(f_0\otimes g_0*h)(z)=I(z)$. Moreover, by the fact that $f,g\in N_\delta(I)$ we obtain $\sum_{n=2}^\infty n|a_n|\leq\delta$ and $n|b_n|\leq\delta$, therefore $|b_n|/n\leq \delta/n^2\leq\delta/4$ for $n\geq 2$. Hence, in view of (1.9), the inequality (2.10) becomes:

$$\left|\frac{(f\otimes g*h)(z)}{z}\right| > 1-\sum_{n=2}^\infty\frac{|a_n||b_n||c_n|}{n} \geq 1-\frac{\delta}{4}\sum_{n=2}^\infty[n+(n-1)k]|a_n|$$

$$\geq 1-(k+1)\frac{\delta}{4}\sum_{n=2}^{\infty} n|a_n| \geq 1-\frac{\delta^2}{4}(k+1),$$

that is nonnegative. It means $|(f\otimes g*h)(z)/z|>0$, so that $f\otimes g\ \in k\text{-}\mathcal{ST}$, which completes the proof.

Case (2.5). Assume that $f_0\in k\text{-}\mathcal{UCV}$ and $g_0=I$. Also, let $f\in N_\delta(f_0)$, $g\in N_\delta(g_0)$, with δ satisfying (2.5). Then, applying (2.10), we obtain

$$\left|\frac{(f\otimes g*h)(z)}{z}\right| > 1-\sum_{n=2}^{\infty}\frac{|a_{0n}||b_n||c_n|}{n}-\sum_{n=2}^{\infty}\frac{|a_n-a_{0n}||b_n||c_n|}{n}. \tag{2.11}$$

Since $f_0\in k\text{-}\mathcal{UCV}\subset\mathcal{CV}$ then $|a_{0n}|\leq 1$. By the fact that $g\in N_\delta(g_0)$ we have $\sum_{n=2}^{\infty} n|b_n|\leq\delta$ whence $\sum_{n=2}^{\infty}|b_n|=\sum_{n=2}^{\infty} n|b_n|/n\leq\delta/2$, for $n\geq 2$. Similarly, since $f\in N_\delta(f_0)$ we have $\sum_{n=2}^{\infty}|a_n-a_{0n}|\leq\delta/2$. Now, applying above inequalities to (2.11), we obtain for all $z\in\mathcal{U}$

$$\left|\frac{(f\otimes g*h)(z)}{z}\right|\geq 1-(k+1)\frac{\delta}{2}-(k+1)\frac{\delta^2}{4}$$

that is nonnegative when assuming (2.5), so that $f\otimes g\in k\text{-}\mathcal{ST}$.

Case (2.6). Assume $f_0\in k\text{-}\mathcal{ST}$ and $g_0=I$. Similarly as in the previous cases, and in view of (1.7), it suffices to show that for δ satisfying (2.6) and $h\in\mathcal{B}$ the inequality $|(f\otimes g*h)(z)/z|>0$ holds. Since $f_0\in k\text{-}\mathcal{ST}\subset\mathcal{ST}$, then $|a_{0n}|\leq n$ and, by virtue of (1.9), the inequality (2.10) takes the form

$$\begin{aligned}\left|\frac{(f\otimes g*h)(z)}{z}\right| &> 1-\sum_{n=2}^{\infty}\frac{|a_{0n}||b_n||c_n|}{n}-\sum_{n=2}^{\infty}\frac{|a_n-a_{0n}||b_n||c_n|}{n}\\ &\geq 1-(k+1)\sum_{n=2}^{\infty} n|b_n|-(k+1)\sum_{n=2}^{\infty}|a_n-a_{0n}||b_n|\\ &\geq 1-(k+1)\delta-(k+1)\frac{\delta^2}{4},\end{aligned}$$

which is nonnegative since δ satisfies (2.6).

Case (2.7). In view of (2.2) and (1.7) it suffices to show that $|(f*g*h)(z)/z|>0$ for $f_0\in k\text{-}\mathcal{UCV}$, $f\in N_\delta(f_0)$, $g_0=I$, $g\in N_\delta(g_0)$ and $h\in\mathcal{B}$. We will proceed as in (2.10). Then

$$\begin{aligned}\left|\frac{(f*g*h)(z)}{z}\right| &\geq \left|\frac{(f_0*g_0*h)(z)}{z}\right|-\left|\frac{(f_0*(g-g_0)*h)(z)}{z}\right| \\ &\quad -\left|\frac{((f-f_0)*g_0*h)(z)}{z}\right|-\left|\frac{((f-f_0)*(g-g_0)*h)(z)}{z}\right|.\end{aligned} \tag{2.12}$$

Since $f_0 \in k\text{-}\mathcal{UCV} \subset \mathcal{CV}$, then $|a_{0n}| \le 1$ and

$$\left|\frac{(f_0 * (g - g_0) * h)(z)}{z}\right| < \sum_{n=2}^{\infty} |a_{0n}||b_n|[n + (n-1)k] \tag{2.13}$$

$$\le (k+1)\sum_{n=2}^{\infty} n|a_{0n}||b_n| \le (k+1)\delta.$$

The function $g \in N_\delta(g_0)$ so that $n|b_n| \le \delta$ for all $n \ge 2$, whence $|b_n| \le \delta/2$, and therefore

$$\left|\frac{((f - f_0) * (g - g_0) * h)(z)}{z}\right| < \sum_{n=2}^{\infty} |b_n||a_n - a_{0n}|[n + (n-1)k]$$

$$\le (k+1)\frac{\delta^2}{2}. \tag{2.14}$$

Applying (2.13) and (2.14) to (2.12) one obtains

$$\left|\frac{(f * g * h)(z)}{z}\right| \ge 1 - (k+1)\delta - (k+1)\frac{\delta^2}{2}$$

which is nonnegative if δ satisfies (2.6), so that $f * g \in k\text{-}\mathcal{ST}$, and therefore $f \otimes g \in k\text{-}\mathcal{UCV}$.

Case (2.9). Let $f_0 \in k\text{-}\mathcal{ST}, g_0 \in \mathcal{CV}$ and $f \in N_\delta(f_0)$, $g \in N_\delta(g_0)$ and $h \in \mathcal{B}$. Since $f_0 \in k\text{-}\mathcal{ST}$, $g_0 \in \mathcal{CV}$ we have $f_0 * g_0 \in k\text{-}\mathcal{ST}$ (cf. [5]) or, by (2.2), $f_0 \otimes g_0 \in k\text{-}\mathcal{UCV}$. Thus, in view of (1.16) $|(f_0 \otimes g_0 * h)(z)/z| > 1/(4(k+1))$. By the identity $f \otimes g * h = f * g \otimes h$, and the above, the relations (2.10) becomes

$$\left|\frac{(f \otimes g * h)(z)}{z}\right| > \frac{1}{4(k+1)} - \sum_{n=2}^{\infty} \frac{|a_{0n}||b_n - b_{0n}||c_n|}{n} \tag{2.15}$$

$$- \sum_{n=2}^{\infty} \frac{|b_{0n}||a_n - a_{0n}||c_n|}{n} - \sum_{n=2}^{\infty} \frac{|a_n - a_{0n}||b_n - b_{0n}||c_n|}{n}.$$

The coefficients of $f_0 \in k\text{-}\mathcal{ST} \subset \mathcal{ST}$ satisfy inequality $|a_{0n}| \le n$ for $n \ge 2$ then, by (1.9) we have:

$$\sum_{n=2}^{\infty} \frac{|a_{0n}||b_n - b_{0n}||c_n|}{n} \le \sum_{n=2}^{\infty} |b_n - b_{0n}|[n + (n-1)k] \tag{2.16}$$

$$\le (k+1)\sum_{n=2}^{\infty} n|b_n - b_{0n}| \le (k+1)\delta.$$

Similarly, since $g_0 \in \mathcal{CV}$ then $|b_{0n}| \leq 1$, whereas $f \in N_\delta(f_0)$ gives $\sum_{n=2}^\infty n|a_n - a_{0n}| \leq \delta$, so that $\sum_{n=2}^\infty |a_n - a_{0n}| \leq \delta/2$. Hence

$$\begin{aligned}\sum_{n=2}^\infty \frac{|b_{0n}||a_n - a_{0n}||c_n|}{n} \leq & \sum_{n=2}^\infty \frac{|a_n - a_{0n}|[n + (n-1)k]}{n} \\ \leq & \frac{1}{2}(k+1)\delta. \end{aligned} \tag{2.17}$$

Finally, we have

$$\begin{aligned}\sum_{n=2}^\infty \frac{|a_n - a_{0n}||b_n - b_{0n}||c_n|}{n} \leq & (k+1)\sum_{n=2}^\infty |a_n - a_{0n}||b_n - b_{0n}| \\ \leq & (k+1)(\delta^2/4). \end{aligned} \tag{2.18}$$

By virtue of (2.16), (2.17), and (2.18) the inequality (2.15) gives

$$\left|\frac{(f \otimes g * h)(z)}{z}\right| \geq \frac{1}{4(k+1)} - (k+1)\delta - (k+1)\delta/2 - (k+1)\frac{\delta^2}{4}$$

that is nonnegative provided that δ satisfies the inequality (2.9). From this we conclude $f \otimes g \in k\text{-}\mathcal{ST}$, that completes the proof. □

3. Stability of geometric properties of the integral transformation

In this section we provide some estimates of radii of $N_\delta(f)$ such that the integral operator (1.20) carry the neighborhood into $k\text{-}\mathcal{UCV}$ or $k\text{-}\mathcal{ST}$, however the authors do not know if the results are sharp.

Theorem 3.1. *For the integral representation* (1.18) *the following inequalities are valid:*

$$\delta(Q[\{I\}], k\text{-}\mathcal{ST}) \geq \frac{2}{k+1} \tag{3.1}$$

$$\delta(Q[\{I\}], k\text{-}\mathcal{UCV}) \geq \frac{1}{k+1} \tag{3.2}$$

$$\delta(Q[k\text{-}\mathcal{UCV}], k\text{-}\mathcal{UCV}) \geq \frac{1}{4(k+1)}. \tag{3.3}$$

Proof. By using the Hadamard product (1.4) and the integral convolution (1.5), the transformation (1.20) can be rewritten as:

$$Q[f](z) = f(z) * \left(z + \sum_{n=2}^\infty \frac{z^n}{n}\right) = f \otimes \left(z + \sum_{n=2}^\infty z^n\right) = (f \otimes K)(z).$$

Case (3.1). Suppose that $f \in N_\delta(I)$ with δ satisfying (3.1). Then it is easy to see, that for all $h \in \mathcal{B}$ and for all $z \in \mathcal{U}$ we have:

$$\begin{aligned}\left|\frac{(Q[f]*h)(z)}{z}\right| &\geq 1-\left|\frac{[(f-I)\otimes K*h](z)}{z}\right| \\ &\geq 1-\sum_{n=2}^{\infty}\frac{|a_n||c_n|}{n}|z|^{n-1} > 1-(k+1)\sum_{n=2}^{\infty}|a_n| \\ &\geq 1-(k+1)\frac{\delta}{2}\geq 0.\end{aligned} \tag{3.4}$$

The assertion and the above give that $\dfrac{(Q[f]*h)(z)}{z} \neq 0$ and, consequently, $Q[f] \in k\text{-}\mathcal{ST}$.

Case (3.2). Assuming $h \in \mathcal{G}$ and $f \in N_\delta(I)$ with δ satisfying (3.2) and applying the estimation (1.10), we have

$$\left|\frac{(Q[f]*h)(z)}{z}\right| \geq 1-\sum_{n=2}^{\infty}\frac{|a_n||c_n|}{n}|z|^{n-1} > 1-(k+1)\sum_{n=2}^{\infty}n|a_n| \geq 1-(k+1)\delta.$$

The above is nonnegative, so that $Q[N_\delta(I)] \subset k\text{-}\mathcal{UCV}$ is valid.

Case (3.3). By the relation $Q[k\text{-}\mathcal{ST}] = k\text{-}\mathcal{UCV}$ and (1.16) we immediately obtain $Q[N_{1/[4(k+1)]}(k\text{-}\mathcal{UCV})] \subset k\text{-}\mathcal{UCV}$. □

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