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FURTHER RESULTS ON OSCILLATION OF HYPERBOLIC DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Abstract. In this paper, we investigate a class of hyperbolic differential equations of neutral type

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}[u + c(t)u(x, t-\tau)] = a_0(t)\triangle u + a_1(t)\triangle u(x, t-\rho)\\ &-\int_a^b q(x, t, \xi)u[x, g(t, \xi)]d\mu(\xi), \quad (x, t) \in \Omega \times \mathbb{R}_+ \equiv G,\end{aligned} \tag{E}$$

and obtain some new sufficient conditions of the oscillation for such equations satisfying boundary condition

$$\frac{\partial u}{\partial N} + \nu(x, t)u = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+. \tag{B}$$

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1. Introduction

In this paper, we consider the following hyperbolic differential equations of neutral type

$$\frac{\partial^2}{\partial t^2}[u+c(t)u(x,t-\tau)] = a_0(t)\triangle u + a_1(t)\triangle u(x,t-\rho) - \int_a^b q(x,t,\xi)u[x,g(t,\xi)]d\mu(\xi), \quad (x,t)\in\Omega\times\mathbb{R}_+\equiv G, \tag{E}$$

and satisfying the boundary condition of the following type

$$\frac{\partial u}{\partial N} + \nu(x,t)u = 0, \quad (x,t)\in\partial\Omega\times\mathbb{R}_+, \tag{B}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$, $\mathbb{R}_+=[0,\infty)$, $u=u(x,t)$, $\triangle$ is the Laplacian operator in $\mathbb{R}^n$, $\tau>0$ and $\rho>0$ are constants, N is the unit exterior normal vector to $\partial\Omega$. $\nu(x,t)$ is a nonnegative continuous function on $\partial\Omega\times\mathbb{R}_+$.

The oscillation theory for partial differential equations has an intensive development in the last decades. For the study of hyperbolic differential equations of neutral type, we mention here the literatures by Bainov and Mishev [1], Kreith, Kusano and Yoshida [3], Mishev and Bainov [7], Yoshida [11], Lalli, Yu and Cui [4]–[5], Bainov, Cui and Minchev [2], Liu and Fu [6], Wang and Yu [8], Wang [9]–[10] and references cited therein. The objective of this paper is to obtain some general oscillatory criteria of solutions of boundary value problem Equations (E), (B) by introducing general weighted function $H(t,s)$, and by choosing different $H(t,s)$, we can obtain various corollaries, namely various conditions under each of which boundary value problem (E), (B) has oscillatory solution. The results generalize some known ones in the literatures.

We assume throughout this paper that the following conditions (H) hold.

(H_1) $c(t)$, $a_0(t)$, $a_1(t)\in C(\mathbb{R}_+,\mathbb{R}_+)$;

(H_2) $q(x,t,\xi)\in C(\overline{G}\times[a,b],\mathbb{R}_+)$;

(H_3) $g(t,\xi)\in C(\mathbb{R}_+\times[a,b],\mathbb{R})$ is nondecreasing with respect to t and ξ respectively, $g(t,\xi)\le t$ for $\xi\in[a,b]$, and $\liminf_{t\to\infty,\,\xi\in[a,b]}\{g(t,\xi)\}=\infty$;

(H_4) $\mu(\xi)\in([a,b],\mathbb{R})$ is nondecreasing, integral of of equation (E) is a Stieltjes one.

Definition 1. A function $u(x,t)\in C^2(\Omega\times[t_{-1},\infty),\mathbb{R})\cap C^1(\overline{\Omega}\times[t_{-1},\infty),\mathbb{R})$ is called a solution of the boundary value problem (E), (B), if it satisfies equation (E) in the domain G and boundary condition (B) on $\partial\Omega \times \mathbb{R}_+$. Where $t_{-1} = \min\{-\tau, -\rho, g(0,a)\}$.

Definition 2. A solution $u(x,t)$ of the boundary value problem (E), (B) is called oscillatory in the domain G if for each positive number t_μ there exists a point $(x_0, t_0) \in \Omega \times [t_\mu, \infty)$ such that the condition $u(x_0, t_0) = 0$ holds.

2. Main results

Next, we give the main results of this paper.

Let $Q(t,\xi) = \min_{x\in\overline{\Omega}}\{q(x,t,\xi)\}$. With each solution $u(x,t)$ of the boundary value problem (E), (B), we associate a function $U(t)$ defined by

$$U(t) = \int_\Omega u(x,t)dx, \quad t > 0. \tag{1}$$

Theorem 1. *Assume that $0 \le c(t) \le 1$, and there exist $\frac{d}{dt}g(t,a)$ and function $H(t,s) \in C^1(D;\mathbb{R})$, $h(t,s) \in C(D;\mathbb{R})$, in which $D = \{(t,s)|t \ge s \ge t_0 > 0\}$ satisfying*

(H_5) *$H(t,t) = 0$, $t \ge t_0 > 0$; $H(t,s) > 0$, $t > s \ge t_0 > 0$;*

(H_6) *$H'_t(t,s) \ge 0$, $H'_s(t,s) \le 0$, and $-H'_s(t,s) = h(t,s)\sqrt{H(t,s)}$, $(t,s) \in D$.*

If there exists a function $\rho(t) \in C^1(\mathbb{R}_+, (0,\infty))$, for $t_0 > 0$, satisfying

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\left[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)\right]^2}{\rho(s)g'(s,a)} ds < \infty, \tag{2}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s) \int_a^b Q(s,\xi)\{1 - c[g(s,\xi)]\} d\mu(\xi) ds = \infty, \tag{3}$$

then each solution $u(x,t)$ of the boundary value problem (E), (B) is oscillatory in the domain G.

Proof. Assume that the boundary value problem (E), (B) has a nonoscillatory solution $u(x,t)$. Without loss of generality, assume that $u(x,t) > 0$, $(x,t) \in \Omega \times \mathbb{R}_+$ ($u(x,t) < 0$ can be considered in same method). From (H_3), there exists a $t_1 \ge 0$ such that $u(x, t-\tau) > 0$, $u[x, g(t,\xi)] > 0$, $u(x, t-\rho) > 0$

for $t \geq t_1$ and $\xi \in [a, b]$. Integrating with respect to x over the domain Ω, for $t \geq t_1$, we obtain

$$\frac{d^2}{dt^2}\left[\int_\Omega u dx + c(t)\int_\Omega u(x, t-\tau)dx\right] + \int_\Omega \int_a^b q(x, t, \xi)u[x, g(t, \xi)]d\mu(\xi)dx$$
$$= a_0(t)\int_\Omega \triangle u dx + a_1(t)\int_\Omega \triangle u(x, t-\rho)dx, \tag{4}$$

It is clear that

$$\int_\Omega \int_a^b q(x, t, \xi)u[x, g(t, \xi)]d\mu(\xi)dx = \int_a^b \int_\Omega q(x, t, \xi)u[x, g(t, \xi)]dxd\mu(\xi)$$
$$\geq \int_a^b Q(t, \xi)\int_\Omega u[x, g(t, \xi)]dxd\mu(\xi). \tag{5}$$

From Green's formula and boundary condition, we have

$$\int_\Omega \triangle u dx = \int_{\partial\Omega} \frac{\partial u}{\partial N} d\omega = -\int_{\partial\Omega} \nu u d\omega \leq 0, \tag{6}$$

and

$$\int_\Omega \triangle u(x, t-\rho)dx = -\int_{\partial\Omega} \nu(x, t-\rho)u(x, t-\rho)d\omega \leq 0, \tag{7}$$

where $d\omega$ is the surface integral element on $\partial\Omega$. Combining (1), (5)–(7), for $t \geq t_1$, it follows that from (4)

$$\frac{d^2}{dt^2}[U(t) + c(t)U(t-\tau)] + \int_a^b Q(t, \xi)U[g(t, \xi)]d\mu(\xi) \leq 0, \quad t \geq t_1. \tag{8}$$

Set

$$Y(t) = U(t) + c(t)U(t-\tau), \tag{9}$$

then, $Y(t) \geq U(t) > 0$, $Y''(t) \leq 0$, $t \geq t_1$, and we can claim that $Y'(t) \geq 0$, $t \geq t_1$. In fact, assume the contrary, that there exists a $t_2 \geq t_1$ such that $Y'(t_1) < 0$. In view of $Y'(t)$ is monotone decreasing, there exists a $t_3 \geq t_2$ such that $Y'(t_3) < 0$ and $Y'(t) \leq Y'(t_3) < 0$, $t \geq t_3$. Integrating from t_3 to t, we have $Y(t) \leq Y(t_3) + Y'(t_3)(t - t_3)$, which implies $\lim_{t\to\infty} Y(t) = -\infty$, this contradicts $Y(t) > 0$. From (8) and (9), we have

$$0 \geq Y''(t) + \int_a^b Q(t, \xi)U[g(t, \xi)]d\mu(\xi)$$
$$= Y''(t) + \int_a^b Q(t, \xi)\{Y[g(t, \xi)] - c[g(t, \xi)]U[g(t, \xi) - \tau]\}d\mu(\xi). \tag{10}$$

In view of $Y'(t) \geq 0$, and $Y(t) \geq U(t)$, $t \geq t_1$, we have $Y[g(t,\xi)] \geq Y[g(t,\xi)-\tau] \geq U[g(t,\xi)-\tau]$, thus

$$Y''(t) + \int_a^b Q(t,\xi)\{1 - c[g(t,\xi)]\}Y[g(t,\xi)]d\mu(\xi) \leq 0, \quad t \geq t_1. \tag{11}$$

Furthermore, in view of $g(t,\xi)$ is nondecreasing with respect to ξ, we have $Y[g(t,a)] \leq Y[g(t,\xi)]$, thus

$$Y''(t) + Y[g(t,a)]\int_a^b Q(t,\xi)\{1 - c[g(t,\xi)]\}d\mu(\xi) \leq 0, \quad t \geq t_1. \tag{12}$$

Let

$$Z(t) = \frac{Y'(t)}{Y[g(t,a)]}, \tag{13}$$

then $Z(t) \geq 0$. In view of $\frac{d}{dt}g(t,a)$ exists, we have

$$Y'[g(t,a)] = \frac{dY}{dg}\frac{d}{dt}g(t,a),$$

and in view of $g(t,\xi)$ is nondecreasing with respect to ξ, $g(t,\xi) \leq t$ for $\xi \in [a,b]$, we obtain $Y'(t) \leq Y'[g(t,a)]$. Thus

$$\begin{aligned} Z'(t) &= \frac{Y''(t)}{Y[g(t,a)]} - \frac{Y'(t)Y'[g(t,a)]g'(t,a)}{Y^2[g(t,a)]} \\ &\leq -\int_a^b Q(t,\xi)\{1-c[g(t,\xi)]\}d\mu(\xi) - g'(t,a)Z^2(t), \quad t \geq t_1. \end{aligned} \tag{14}$$

Integrating by parts for any $t > T \geq t_1$, and using the conditions (H_5) and (H_6), we have

$$\begin{aligned} &\int_{t_1}^t H(t,s)\rho(s)\int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi)ds \\ &\leq -\int_{t_1}^t H(t,s)\rho(s)Z'(s)ds - \int_{t_1}^t H(t,s)\rho(s)g'(s,a)Z^2(s)ds \\ &= H(t,t_1)\rho(t_1)Z(t_1) - \int_{t_1}^t \sqrt{H(t,s)}[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]Z(s)ds \\ &- \int_{t_1}^t H(t,s)\rho(s)g'(s,a)Z^2(s)ds. \end{aligned} \tag{15}$$

Furthermore, we have

$$\begin{aligned} &\int_{t_1}^t H(t,s)\rho(s)\int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi)ds \\ &\leq H(t,t_1)\rho(t_1)Z(t_1) \end{aligned}$$

$$-\int_{t_1}^{t}\left\{\sqrt{\rho(s)g'(s,a)H(t,s)}Z(s)+\frac{\left[h(t,s)\rho(s)-\sqrt{H(t,s)}\rho'(s)\right]^2}{2\sqrt{\rho(s)g'(s,a)}}\right\}ds$$
$$+\frac{1}{4}\int_{t_1}^{t}\frac{\left[h(t,s)\rho(s)-\sqrt{H(t,s)}\rho'(s)\right]^2}{\rho(s)g'(s,a)}ds$$
$$\le H(t,t_1)\rho(t_1)Z(t_1)+\frac{1}{4}\int_{t_1}^{t}\frac{[h(t,s)\rho(s)-\sqrt{H(t,s)}\rho'(s)]^2}{\rho(s)g'(s,a)}ds. \tag{16}$$

From (16), for $t > t_1 \ge t_0$, we have

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)\int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi)ds$$
$$=\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t_1}+\int_{t_1}^{t}\right]\left[H(t,s)\rho(s)\int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}\right]d\mu(\xi)ds$$
$$\le\frac{1}{H(t,t_0)}\int_{t_0}^{t_1}H(t,s)\rho(s)\int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi)ds$$
$$+\frac{H(t,t_1)}{H(t,t_0)}\rho(t_1)Z(t_1)+\frac{1}{4H(t,t_0)}\int_{t_1}^{t}\frac{\left[h(t,s)\rho(s)-\sqrt{H(t,s)}\rho'(s)\right]^2}{\rho(s)g'(s,a)}ds,$$

in view of $H'_s(t,s)\le 0$, we have

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)\int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi)ds$$
$$\le L+\frac{1}{4}\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\left[h(t,s)\rho(s)-\sqrt{H(t,s)}\rho'(s)\right]^2}{\rho(s)g'(s,a)}ds, \tag{17}$$

where $L=\rho(t_1)Z(t_1)+\int_{t_0}^{t_1}\rho(s)\int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi)ds$. Thus, in view of condition (2), we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)\int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi)ds<\infty,$$

which contradicts (3). Therefore, the proof of Theorem 1 is completed. □

In Theorem 1, by choosing $\rho(s)\equiv 1$, we have the following corollary.

Corollary 1. *Assume that the conditions of Theorem* 1 *hold, and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi)ds=\infty, \tag{18}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{h^2(t,s)}{g'(s,a)} ds < \infty, \tag{19}$$

then each solution $u(x,t)$ of the boundary value problem (E), (B) is oscillatory in the domain G.

From Theorem 1 and Corollary 1, we can obtain various oscillatory criteria by means of the choices of weighted function $H(t,s)$. For example, Choosing $H(t,s) = (t-s)^{m-1}$, $t \geq s \geq t_0$, in which $m > 2$ is an integer, then $h(t,s) = (m-1)(t-s)^{(m-3)/2}$, $t \geq s \geq t_0$. From Corollary 1, we have

Corollary 2. *If there exists a $\frac{d}{dt}g(t,a)$ and an integer $m > 2$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\} d\mu(\xi) ds = \infty, \tag{20}$$

$$\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \frac{(m-1)^2(t-s)^{m-3}}{g'(s,a)} ds < \infty, \tag{21}$$

then each solution $u(x,t)$ of the boundary value problem (E), (B) is oscillatory in the domain G.

When

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s) \int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\} d\mu(\xi) ds < \infty, \tag{22}$$

we have the following result.

Theorem 2. *Assume that the conditions of Theorem 1 and (22) hold. If $H_t'(t,s)$ is nondecreasing, and there exists a function $\varphi(t) \in C([t_0,\infty),\mathbb{R})$ satisfying*

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{u}^{t} \left[H(t,s)\rho(s) \int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\} d\mu(\xi) - \frac{\left[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)\right]^2}{4\rho(s)g'(s,a)} \right] ds \geq \varphi(u), \quad u \geq t_0, \tag{23}$$

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{H(t,s)g'(s,a)\varphi_+^2(s)}{\rho(s)} ds = \infty, \quad \varphi_+(s) = \max_{s\geq t_0}\{\varphi(s),0\}, \tag{24}$$

then each solution $u(x,t)$ of the boundary value problem (E), (B) is oscillatory in the domain G.

Proof. Assume that the boundary value problem (E), (B) has a nonoscillatory solution $u(x,t)$. Without loss of generality, assume that $u(x,t) > 0$, $(x,t) \in \Omega \times \mathbb{R}_+$ ($u(x,t) < 0$ can be considered in same method). Then proceeding as in Theorem 1, there exists a $t_1 > u \geq t_0$ such that

$$\int_u^t H(t,s)\rho(s) \int_a^b Q(s,\xi)\{1 - c[g(s,\xi)]\} d\mu(\xi) ds$$
$$\leq H(t,u)\rho(u)Z(u) + \frac{1}{4}\int_u^t \frac{\left[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)\right]^2}{\rho(s)g'(s,a)} ds. \tag{16}$$

Furthermore, for $t > u \geq t_0$, we have

$$\frac{1}{H(t,t_0)} \int_u^t \left[H(t,s)\rho(s) \int_a^b Q(s,\xi)\{1 - c[g(s,\xi)]\} d\mu(\xi) - \frac{\left[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)\right]^2}{4\rho(s)g'(s,a)}\right] ds \tag{25}$$
$$\leq \frac{H(t,u)}{H(t,t_0)}\rho(u)Z(u).$$

In view of (23) and (H_6), we have

$$\varphi(u) \leq \frac{1}{H(t,t_0)} \int_u^t \left[H(t,s)\rho(s) \int_a^b Q(s,\xi)\{1 - c[g(s,\xi)]\} d\mu(\xi) - \frac{\left[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)\right]^2}{4\rho(s)g'(s,a)}\right] ds \tag{26}$$
$$\leq \frac{H(t,u)}{H(t,t_0)}\rho(u)Z(u) \leq \rho(u)Z(u),$$

which implies that

$$\varphi_+^2(u) \leq \rho^2(u)Z^2(u). \tag{27}$$

Let

$$v(t) = \frac{1}{H(t,t_0)} \int_{t_1}^t \sqrt{H(t,s)}\left[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)\right] Z(s) ds$$
$$w(t) = \frac{1}{H(t,t_0)} \int_{t_1}^t H(t,s)\rho(s)g'(s,a)Z^2(s) ds,$$

then, in view of (15), we have

$$v(t) + w(t) \leq \frac{H(t,t_1)}{H(t,t_0)}\rho(t_1)Z(t_1) \tag{28}$$

$$- \frac{1}{H(t,t_0)} \int_{t_1}^{t} H(t,s)\rho(s) \int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\} d\mu(\xi) ds,$$

from (23), we have

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_u^t H(t,s)\rho(s) \int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\} d\mu(\xi) ds \geq \varphi(u),$$

furthermore, we obtain

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^{t} H(t,s)\rho(s) \int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\} d\mu(\xi) ds$$

$$- \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^{t} \frac{\left[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)\right]^2}{4\rho(s)g'(s,a)} ds \geq \varphi(t_1). \qquad (29)$$

In view of (22) and (29), we have

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^{t} \frac{\left[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)\right]^2}{4\rho(s)g'(s,a)} ds < \infty.$$

Thus, there exists a sequence $\{t_n\}_1^\infty$ in $[t_1,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and satisfying

$$\lim_{n\to\infty} \frac{1}{H(t_n,t_0)} \int_{t_1}^{t_n} \frac{\left[h(t_n,s)\rho(s) - \sqrt{H(t_n,s)}\rho'(s)\right]^2}{4\rho(s)g'(s,a)} ds \quad < \quad \infty, \qquad (30)$$

which implies that

$$\limsup_{t\to\infty}\{v(t)+w(t)\} \leq \rho(t_1)Z(t_1)$$

$$- \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^{t} H(t,s)\rho(s) \int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\} d\mu(\xi) ds$$

$$\leq \rho(t_1)Z(t_1) - \varphi(t_1) \triangleq M. \qquad (31)$$

Then, for any sufficiently large n, we have

$$u(t_n) + v(t_n) < M_1, \qquad (32)$$

where $M_1 > M$, M and M_1 are constants. In view of definition of $w(t)$, we have

$$w'(t) = \int_{t_1}^{t} \frac{(H_t'(t,s)H(t,t_0) - H_t'(t,t_0)H(t,s))}{H^2(t,t_0)} \rho(s)g'(s,a)Z^2(s) ds,$$

from $H_t'(t,s)$ is nondecreasing, and (H_6), we have $w'(t) \geq 0$, thus, $w(t)$ is increasing, and $\lim_{t\to\infty} w(t) = l$ exists, where l is finite or infinite. In the case of $l = \infty$, then $\lim_{n\to\infty} w(t_n) = \infty$, which implies that from (32)

$$\lim_{n\to\infty} v(t_n) = -\infty, \qquad (33)$$

and

$$\frac{v(t_n)}{w(t_n)} + 1 < \frac{M_1}{w(t_n)},$$

thus, for any $0 < \varepsilon < 1$ and sufficiently large n, we have

$$\frac{v(t_n)}{w(t_n)} < \varepsilon - 1 < 0. \tag{34}$$

On the other hand, by using the Schwartz inequality, for $t \geq t_1$, we obtain

$$\begin{aligned}
0 \leq v^2(t_n) &= \frac{1}{H^2(t_n, t_0)} \left\{ \int_{t_1}^{t_n} \sqrt{H(t_n, s)} \Big[h(t_n, s)\rho(s) \right. \\
&\qquad \left. - \sqrt{H(t_n, s)}\rho'(s) \Big] Z(s) ds \right\}^2 \\
&\leq \left\{ \frac{1}{H(t_n, t_0)} \int_{t_1}^{t_n} H(t_n, s)\rho(s) g'(s, a) Z^2(s) ds \right\} \\
&\qquad \times \left\{ \frac{1}{H(t_n, t_0)} \int_{t_1}^{t_n} \frac{\Big[h(t_n, s)\rho(s) - \sqrt{H(t_n, s)}\rho'(s) \Big]^2}{\rho(s) g'(s, a)} ds \right\} \\
&= w(t_n) \frac{1}{H(t_n, t_0)} \int_{t_1}^{t_n} \frac{\Big[h(t_n, s)\rho(s) - \sqrt{H(t_n, s)}\rho'(s) \Big]^2}{\rho(s) g'(s, a)} ds.
\end{aligned}$$

Then

$$0 \leq \frac{v^2(t_n)}{w(t_n)} \leq \frac{1}{H(t_n, t_0)} \int_{t_1}^{t_n} \frac{\Big[h(t_n, s)\rho(s) - \sqrt{H(t_n, s)}\rho'(s) \Big]^2}{\rho(s) g'(s, a)} ds. \tag{35}$$

It follows that from (30)

$$0 \leq \lim_{n \to \infty} \frac{v^2(t_n)}{w(t_n)} < \infty. \tag{36}$$

In view of (34), we have

$$\lim_{n \to \infty} \frac{v(t_n)}{w(t_n)} = \lim_{n \to \infty} \frac{v'(t_n)}{w'(t_n)} \leq \varepsilon - 1 < 0,$$

then

$$\lim_{n \to \infty} \frac{v^2(t_n)}{w(t_n)} = \lim_{n \to \infty} \frac{2v(t_n)v'(t_n)}{w'(t_n)} \geq 2 \lim_{n \to \infty} v(t_n)(\varepsilon - 1) = \infty,$$

which contradicts (36). Thus, we have $\lim_{t \to \infty} w(t) = l < \infty$. Furthermore, in view of (27), we have

$$\lim_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_1}^{t} \frac{H(t, s) g'(s, a) \varphi_+^2(s)}{\rho(s)} ds$$

$$\le \lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^{t} H(t,s)\rho(s)g'(s,a)Z^2(s)ds = \lim_{t\to\infty} w(t) < \infty, \qquad (37)$$

which implies that

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{H(t,s)g'(s,a)\varphi_+^2(s)}{\rho(s)}ds \\
&= \lim_{t\to\infty} \frac{1}{H(t,t_0)} \left[\int_{t_0}^{t_1} + \int_{t_1}^{t}\right] \frac{H(t,s)g'(s,a)\varphi_+^2(s)}{\rho(s)}ds \\
&\le \int_{t_0}^{t_1} \frac{g'(s,a)\varphi_+^2(s)}{\rho(s)}ds + \lim_{t\to\infty} w(t) < \infty,
\end{aligned}$$

which contradicts (24). Therefore, the proof of Theorem 2 is completed. □

In Theorem 2, by choosing $\rho(t) \equiv 1$, we have the following result.

Corollary 3. *Assume that the conditions of Theorem* 1 *and* (22) *hold. If* $H'_t(t,s)$ *is nondecreasing, and there exists a function* $\varphi(t) \in C([t_0,\infty),\mathbb{R})$ *satisfying*

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{u}^{t} \left[H(t,s)\int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi) - \frac{h^2(t,s)}{4g'(s,a)}\right] ds$$
$$\ge \varphi(u), \qquad (38)$$

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)g'(s,a)\varphi_+^2(s)ds = \infty,$$
$$\varphi_+(s) = \max_{s\ge t_0}\{\varphi(s),0\}, \qquad (39)$$

then each solution $u(x,t)$ *of the boundary value problem* (E)*,* (B) *is oscillatory in the domain* G*.*

Similarly as for Corollary 2, we can obtain the following corollary from Corollary 3.

Corollary 4. *Assume that the conditions of Theorem* 1 *hold,* $\frac{d}{dt}g(t,a)$ *exists, and*

$$\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi)ds < \infty. \qquad (40)$$

If there exists an integer $m > 2$ *and function* $\varphi(t) \in C([t_0,\infty),\mathbb{R})$ *satisfying*

$$\liminf_{t\to\infty} \frac{1}{t^{m-1}} \int_{u}^{t} \left[(t-s)^{m-1} \int_a^b Q(s,\xi)\{1-c[g(s,\xi)]\}d\mu(\xi)\right.$$

$$-\frac{(m-1)}{4}\frac{(t-s)^{m-3}}{g'(s,a)}\Bigg]\,ds \geq \varphi(u), \tag{41}$$

$$\lim_{t\to\infty}\frac{1}{t^{m-1}}\int_{t_0}^{t}(t-s)^{m-1}g'(s,a)\varphi_+^2(s)ds = \infty,\qquad \varphi_+(s) = \max_{s\geq t_0}\{\varphi(s),0\}, \tag{42}$$

then each solution $u(x,t)$ of the boundary value problem (E), (B) is oscillatory in the domain G.

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