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EXISTENCE FOR SOME QUASILINEAR ELLIPTIC SYSTEMS WITH CRITICAL GROWTH NONLINEARITY AND L^1 DATA

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Abstract. We prove the existence of weak solutions for some quasilinear elliptic reaction-diffusion systems with Dirichlet boundary conditions and satisfying to the two main properties: the positivity of the solutions and the balance law. The nonlinearity we consider here has critical growth with respect to the gradient and the data are in L^1.

1. Introduction

The aim of this work is to study existence of weak solutions for the following quasilinear elliptic system:

$$\begin{cases} -\Delta u = -f(x,u,v,\nabla u,\nabla v) + F(x) & \text{in } \Omega \\ -\Delta v = f(x,u,v,\nabla u,\nabla v) + G(x) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. $f : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to [0, +\infty)$ is a nonlinear function which has critical growth with respect to the gradient. $F, G : \Omega \to [0, +\infty)$ are non-negative integrable functions.

We are interested in the case where the data are nonregular and the growth of the nonlinear term f with respect to the gradient is quadratic. To help understand the situation, let us mention some previous works concerning systems of the form

$$\begin{cases} -\Delta u = f(x, u, v, \nabla u, \nabla v) + F(x) & \text{in } \Omega \\ -\Delta v = g(x, u, v, \nabla u, \nabla v) + G(x) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

In general such systems have been studied under the so-called "triangular structure", namely

$$\begin{cases} f(x, u, v, p, q) + g(x, u, v, p, q) \leq L_1(u + v + 1), \\ f(x, u, v, p, q) \leq L_2(u + v + 1), \\ \text{for all } (u, v, p, q) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \text{ and a.e. } x \in \Omega,\ L_1, L_2 \geq 0. \end{cases}$$

With the boundary conditions and the first inequality in the triangular structure (which is called balance law or mass control), one can derive an L^1 estimate on the components of the solution. But in general this is not sufficient to prove existence and then additional hypotheses are required. We refer the reader to [11], [12], [16] and the references there in for the semilinear case (f and g dot not depend on the gradient). In the quasilinear case (f and g depend on the gradient), the existence has been obtained in [16] when $g = -f$, the data are not regular and the dependence of the nonlinear term with respect to the gradient is subquadratic. This result has been generalized later in [3]. The authors considered the case where f and g satisfy the triangular structure, the data are not regular, and the growth with respect to the gradient is quadratic for the first component.

It is worth to recall here that the parabolic version of these systems has been extensively studied, see [13], [17], [21] et al for the semilinear case and [8], [2] for the quasilinear case.

Let us also point out that the triangular structure played an important role in the study of such systems. It may indeed happen, as it is proved in [18] and [19], that the solutions blow up in finite time if this condition is not fulfilled.

This could justify the conditions on the nonlinear terms we are considering here. Moreover the structure of system (1) leads us to introducing the function w solution of the linear problem

$$\begin{cases} -\Delta w = F + G & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Then solving problem (1) is equivalent to solve the equation

$$\begin{cases} -\Delta u + f(x, u, w - u, \nabla u, \nabla w - \nabla u) = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

and set $v := w - u$.

Let us now make some precise statements on a model problem like

$$\begin{cases} u, v \in W_0^{1,p}(\Omega) \\ -\Delta u = -u\,|\nabla v|^p + F(x) & \text{in } \Omega \\ -\Delta v = u\,|\nabla v|^p + G(x) & \text{in } \Omega, \end{cases} \tag{4}$$

where $|\cdot|$ denotes the $\mathbb{R}^N$-euclidian norm and $p \geq 1$.

If we apply the same transformation to this problem, we obtain the equation

$$\begin{cases} u \in W_0^{1,p}(\Omega) \\ -\Delta u + u\,|\nabla w - \nabla u|^p = F & \text{in } \Omega. \end{cases} \tag{5}$$

If F and G are regular ($F, G \in W^{1,\infty}(\Omega)$), the method of sub- and supersolution can be used to prove existence in (4). For instance $u_1 = 0$ is a subsolution and $u_2 = w$ is a supersolution, then (5) has a solution $u \in W_0^{1,\infty}(\Omega) \cap W^{2,p}(\Omega)$ $\forall p < \infty$, see Lions [14].

If $F, G \in H^{-1}(\Omega)$ and $p \leq 2$ then $|\nabla w|^p \in L^1(\Omega)$. Many authors dealt with this problem and showed that (5) has a solution $u \in H_0^1(\Omega)$ see [6], [7] and the references there in, see also [4] for the case of data measure.

For our purpose, we are particularly interested in situations where F and G are not regular ($F, G \in L^1(\Omega)$) and $p \geq 1$. In this case $|\nabla w|^p$ do not belong necessarily to $L^1(\Omega)$. To overcome this difficulty, we will see how we will proceed to adapt the techniques used in [7] to the resolution of our problem.

We have organized the paper as follows. In Section 2 we give the precise setting of the problem and state the main result. In Section 3 we present an approximate problem and we give suitable estimates to prove that (3) has a solution in the case where the growth of f with respect to the gradient is quadratic.

2. Assumptions and statement of main results

Let $f : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to [0, +\infty)$ be a caratheodory function (that is $f$is measurable with respect to $x \in \Omega$, and f is continuous with respect to u, v, p, q in $\mathbb{R}^2 \times \mathbb{R}^{2N}$) which satisfies the following assumptions:

$$f(x, u, v, p, q) \geq 0 \text{ a.e. } x \in \Omega \text{ and for all } u, v \geq 0,\ p, q \in \mathbb{R}^N \tag{6}$$

$$|f(x, u, v, p, q)| \leq C(|u| + |v|) \left[|p|^2 + |q|^2 + K(x)\right] \tag{7}$$

where $C : [0, \infty) \to [0, \infty)$ is a non-decreasing function, $K \in L^1(\Omega)$.
And assume that

$$F, G \geq 0 \text{ and } F, G \in L^1(\Omega). \tag{8}$$

For the convenience of the reader we recall the definition of weak solutions we use in this paper.

Definition 2.1. We say that (u, v) is a weak solution of (1) if

$$\begin{cases} u, v \in W_0^{1,1}(\Omega),\ f(\cdot, u, v, \nabla u, \nabla v) \in L^1(\Omega), \\ -\Delta u = -f(x, u, v, \nabla u, \nabla v) + F & \text{in } D'(\Omega) \\ -\Delta v = f(x, u, v, \nabla u, \nabla v) + G & \text{in } D'(\Omega). \end{cases} \tag{9}$$

We will be interested in proving the existence of weak positive solutions of problem (1).

Theorem 2.2. *Under hypotheses* (6)–(8)*, system* (1) *has a positive weak solution.*

Remark 2.3. (a) It should be noted that there is no growth restriction on the "lower order nonlinearity" of f as a function of u and v and the growth of f with respect to the gradient can be quadratic. Hence the present theorem extends some results in [3] and in [16].

(b) The result of this work include the typical model (4) and the more general one

$$\begin{cases} -\Delta u = -(u + v)[|\nabla u|^2 + |\nabla v|^2] + F(x) & \text{in } \Omega \\ -\Delta v = (u + v)[|\nabla u|^2 + |\nabla v|^2] + G(x) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

3. Proof of Theorem 2.2

As explained before, Theorem 2.2 will be proved if we show the existence of solutions for (3). To do this, we will need the following functions. Let H be a function in $C^1(\mathbb{R})$, such that $0 \le H(s) \le 1$, and

$$H(s) = \begin{cases} 0 & \text{if } |s| \ge 1 \\ 1 & \text{if } |s| \le \dfrac{1}{2}. \end{cases}$$

And for a given real positive number $k,$ we define the function

$$T_k(s) = \max\{-k, \min(s,k)\}, \quad \text{for } s \in \mathbb{R}.$$

3.1. Approximating scheme.

In this paragraph, we define an approximated equation of (3) by truncating F and G as follows: for $n \ge 1,$ we choose F_n and G_n nonnegative functions such that

$$\begin{aligned} &F_n, G_n \in L^\infty(\Omega) \\ &\|F_n\|_{L^1(\Omega)} \le \|F\|_{L^1(\Omega)}, \ \|G_n\|_{L^1(\Omega)} \le \|G\|_{L^1(\Omega)}, \\ &F_n \longrightarrow F, \ G_n \longrightarrow G, \text{ strongly in } L^1(\Omega) \text{ as } n \to \infty. \end{aligned}$$

We then consider the following linear problem:

$$\begin{cases} -\Delta w_n = F_n + G_n & \text{in } D'(\Omega), \\ w_n \in W_0^{1,\infty}(\Omega). \end{cases} \tag{10}$$

It is well known that (10) has a positive solution w_n and there exists w, up to a subsequence still denoted by w_n, such that

$$\begin{aligned} &w_n \to w \text{ strongly in } W_0^{1,q}(\Omega), \ 1 \le q < \frac{N}{N-1} \\ &w_n \to w \text{ a.e. in } \Omega \\ &\nabla w_n \to \nabla w \text{ a.e. in } \Omega. \end{aligned}$$

We also define the nonlinear function f_n by:

$$f_n(x,u,\nabla u) = f(x,u,w_n - u, \nabla u, \nabla w_n - \nabla u)\rho_n(u)$$

where $\rho_n(x) = x^2/(x^2+n)$, then $\rho_n(0) = 0$, $\lim_{n\to\infty} \rho_n(x) = 0$, and $0 \le \rho_n \le 1$.

One can see that f_n satisfies the same hypotheses as f, especially hypotheses (6)–(8). We remark here that (7) can be reformulated by

$$\begin{aligned} |f_n(x,u,\nabla u)| \le & C(|u| + |w_n - u|)(|\nabla u|^2 + |\nabla(w_n - u)|^2 + K(x)) \\ \le & C_1(u + w_n)(|\nabla u|^2 + |\nabla w_n|^2 + K(x)). \end{aligned} \tag{11}$$

We consider now the approximating problem

$$\begin{cases} -\Delta u_n + f_n(x, u_n, \nabla u_n) = F_n & \text{in } D'(\Omega), \\ u_n \in W_0^{1,\infty}(\Omega). \end{cases} \tag{12}$$

To show that equation (12) has a solution, one can see that $u_1 = 0$ is a subsolution and $u_2 = w_n$ is a supersolution. Then by virtue of the classical results in [14], [10], [5] there exists u_n solution of (12) such that

$$0 \leq u_n \leq w_n \quad \text{for all } n. \tag{13}$$

3.2. Estimates.

Lemma 3.2.1. *Let u_n and f_n be sequences defined as above. Then*

(i) $\displaystyle\int_\Omega f_n(x, u_n, \nabla u_n) dx \leq \|F\|_{L^1(\Omega)}$,

(ii) $\displaystyle\int_\Omega |\nabla T_k(u_n)|^2 \, dx \leq k \|F\|_{L^1(\Omega)}$.

Proof. (i) Integrating (12) over $\Omega,$ we obtain

$$-\int_\Omega \Delta u_n + \int_\Omega f_n = \int_\Omega F_n \ . \tag{14}$$

On the other hand, it is well known that for every function y in $W_0^{1.1}(\Omega)$ such that

$$\begin{cases} -\Delta y = H, \quad H \in L^1(\Omega) \\ y \geq 0, \end{cases}$$

there exists a sequence y_n in $C^2(\Omega) \cap C_0(\overline{\Omega})$ which satisfies

$$\begin{aligned} y_n &\longrightarrow y \text{ strongly in } W_0^{1.1}(\Omega) \\ \Delta y_n &\longrightarrow \Delta y \text{ strongly in } L^1(\Omega). \end{aligned}$$

The regularity of y_n allows us to write

$$\int_\Omega \Delta y_n = \int_{\partial\Omega} \frac{\partial y_n}{\partial \nu} d\sigma.$$

But $y_n \geq 0$ on Ω and $y_n = 0$ on $\partial\Omega$, then $\dfrac{\partial y_n}{\partial \nu} \leq 0$. We deduce by passing to the limit that $\int_\Omega \Delta y \leq 0$. Therefore

$$\int_\Omega \Delta u_n \leq 0.$$

Then relation (14) yields

$$\int_\Omega f_n \leq \int_\Omega F_n \leq \|F\|_{L^1(\Omega)}.$$

(ii) Multiplying the equation in (12) by $T_k(u_n)$ and integrating over Ω, we obtain

$$\int_\Omega |\nabla T_k(u_n)|^2 + \int_\Omega f_n T_k(u_n) = \int_\Omega F_n T_k(u_n).$$

The nonnegativity of f_n and $T_k(u_n)$ and the hypotheses on F_n allow us to write

$$\int_\Omega |\nabla T_k(u_n)|^2 \leq k \|F\|_{L^1(\Omega)}.$$

□

Remark 3.2.2. (a) Assertion (i) and the compactness of the operator defined from $L^1(\Omega)$ into $W_0^{1,q}(\Omega)$, $1 \leq q < N/(N-1)$, by: $g \mapsto v$ where v is a solution of the problem

$$\begin{cases} -\Delta v = g & \text{in } \mathfrak{D}'(\Omega) \\ v \in W_0^{1,q}(\Omega) & , \end{cases}$$

imply the existence of u, up to a subsequence still denoted by u_n for simplicity, such that u_n converges strongly to u, in $W_0^{1,q}(\Omega)$, $1 \leq q < N/(N-1)$, and $(u_n, \nabla u_n)$ converges almost everywhere in Ω (see [9]).

(b) Assertion (ii) and (a) imply that $T_k(u_n)$ converges weakly to $T_k(u)$ in $H_0^1(\Omega)$.

Lemma 3.2.3. *Let w_n (respectively u_n) be solutions of* (10) *(respectively* (12)*), then*

(i) $\lim_{h\to\infty} \frac{1}{h} \int_\Omega |\nabla T_h(u_n)|^2 = 0$, $\lim_{h\to\infty} \frac{1}{h} \int_\Omega |\nabla T_h(w_n)|^2 = 0$ *uniformly on n.*

(ii) $\lim_{n\to\infty} T_h(w_n) = T_h(w)$ *strongly in $H_0^1(\Omega)$ for fixed positive h.*

Proof. (i) We first remark that u_n satisfies

$$-\Delta u_n \leq F \text{ in } \mathfrak{D}'(\Omega).$$

If we multiply this inequality by $T_h(u_n)$ and integrate on Ω, we obtain for every $0 < M < h$

$$\int_\Omega |\nabla T_h(u_n)|^2 \leq \int_{\Omega\cap[u_n\leq M]} T_h(u_n)F + \int_{\Omega\cap[u_n>M]} T_h(u_n)F$$

$$\leq M \int_\Omega F + h \int_\Omega F \chi_{[u_n > M]}.$$

Hence

$$\frac{1}{h} \int_\Omega |\nabla T_h(u_n)|^2 \leq \frac{M}{h} \|F\|_{L^1} + \int_\Omega F \chi_{[u_n > M]}.$$

Fix $\varepsilon > 0$. Since u_n is bounded in $L^1(\Omega)$, we have

$$|[u_n \geq k]| = \int_{[u_n \geq k]} dx \leq k^{-1} \|u_n\|_{L^1} \leq C k^{-1}.$$

Therefore, there exists k_ε independent of n such that

$$\int_\Omega F \chi_{[u_n > k_\varepsilon]} \leq \frac{\varepsilon}{2}.$$

Taking $M = k_\varepsilon$ an letting h tend to infinity, we obtain the desired conclusion.

For the second assertion, we remark that w_n satisfies the same hypotheses used in the proof of the first one. Then the same arguments are still valid for w_n.

(ii) We multiply the equation in (10) by $T_h(w_n) - T_h(w)$ and we integrate on Ω, we obtain

$$\int_\Omega |\nabla T_h(w_n) - \nabla T_h(w)|^2 + \int_\Omega \nabla T_h(w)(\nabla T_h(w_n) - \nabla T_h(w))$$
$$= \int_\Omega (F_n + G_n)(T_h(w_n) - T_h(w)).$$

We use then the fact that $|(F_n + G_n)(T_h(w_n) - T_h(w))| \leq 2h(|F| + |G|) \in L^1(\Omega)$ and $(F_n + G_n)(T_h(w_n) - T_h(w))$ converges almost everywhere to 0 in Ω, to conclude by virtue of Lebesgue's theorem that

$$\lim_{n\to\infty} \int_\Omega (F_n + G_n)(T_h(w_n) - T_h(w)) = 0. \tag{15}$$

On the other hand $\nabla T_h(w_n) - \nabla T_h(w)$ converges weakly to 0 in $L^2(\Omega)$ (since $\int_\Omega |\nabla T_h(w_n)|^2 \leq h(\|F\|_{L^1(\Omega)} + \|G\|_{L^1(\Omega)})$ and ∇w_n converges to ∇w almost everywhere in Ω) and $\nabla T_h(w) \in L^2(\Omega)$, consequently

$$\lim_{n\to\infty} \int_\Omega \nabla T_h(w)(\nabla T_h(w_n) - \nabla T_h(w)) = 0. \tag{16}$$

We conclude from (15) and (16) that

$$\lim_{n\to\infty}\int_\Omega |\nabla T_h(w_n) - \nabla T_h(w)|^2 = 0.$$

□

3.3. Convergence.

The aim of this paragraph is to prove that u (obtained in the previous section) is in fact a solution of problem (12). According to Definition 2.1, we have only to show that

$$-\Delta u + f(x, u, w-u, \nabla u, \nabla w - \nabla u) = F \quad \text{in } \mathfrak{D}'(\Omega).$$

We know by Lemma 3.2.1 that f_n is uniformly bounded in $L^1(\Omega)$. Moreover $f_n \geq 0$ and for almost every x in Ω

$$\lim_{n\to\infty} f_n(x, u_n, \nabla u_n) = f(x, u, w-u, \nabla u, \nabla w - \nabla u).$$

Then there exists μ a non-negative measure (see [22]) such that

$$\lim_{n\to\infty}(-\Delta u_n + f_n(x, u_n, \nabla u_n)) = -\Delta u + f(x, u, w-u, \nabla u, \nabla w - \nabla u) + \mu \text{ in } \mathfrak{D}'(\Omega).$$

On the other hand

$$\lim_{n\to\infty}(-\Delta u_n + f_n(x, u_n, \nabla u_n)) = \lim_{n\to\infty} F_n = F \quad \text{in } L^1(\Omega).$$

Consequently

$$-\Delta u + f(x, u, w-u, \nabla u, \nabla w - \nabla u) \leq F \quad \text{in } \mathfrak{D}'(\Omega).$$

Therefore to conclude the proof of Theorem 2.2, we must establish the opposite inequality. To this end we introduce the following test function

$$\psi \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right),$$

where H denotes the function defined above, $C_2(s) = \int_0^s C_1(t)dt$ (C_1 is given by relation (11))and $\psi \leq 0$, $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We multiply the equation satisfied by u_n in (12) by this test function and we integrate on Ω, we obtain

$$\int_\Omega F_n \psi \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right) = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \int_\Omega \nabla u_n \nabla \psi \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right)$$

$$I_2 = -\int_\Omega \psi \nabla u_n \nabla w_n C_1(u_n + w_n) \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right)$$

$$I_3 = \frac{1}{k}\int_\Omega |\nabla u_n|^2 \psi \exp(-C_2(u_n + w_n)) H'\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right)$$

$$I_4 = \frac{1}{k}\int_\Omega \nabla u_n \nabla w_n \psi \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H'\left(\frac{w_n}{k}\right)$$

$$I_5 = \int_\Omega (f_n - |\nabla u_n|^2 C_1(u_n + w_n))\psi \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right).$$

By investigating separately each term, we get for the first one

$$\begin{aligned}\lim_{n\to\infty} I_1 &= \lim_{n\to\infty} \int_\Omega \nabla T_k(u_n) \nabla\psi \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right) \\ &= \int_\Omega \nabla u \nabla \psi \exp(-C_2(u + w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right),\end{aligned}$$

since

$$\nabla\psi \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right)$$

converges strongly to

$$\nabla\psi \exp(-C_2(u + w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right) \text{ in } L^2(\Omega)$$

and $\nabla T_k(u_n)$ converges weakly to $\nabla T_k(u)$ in $L^2(\Omega)$, see [15, Lemma 1.3, p. 12]. Concerning the second term, we first remark that

$$I_2 = -\int_\Omega \psi \nabla T_k(u_n) \nabla T_k(w_n) C_1(u_n + w_n) \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right).$$

Then by using the same argument, we have

$$\lim_{n\to\infty} I_2 = -\int_\Omega \psi \nabla u \nabla w C_1(u + w) \exp(-C_2(u + w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right),$$

since

$$\psi \nabla T_k(w_n) C_1(u_n + w_n) \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right)$$

converges to

$$\psi \nabla T_k(w) C_1(u + w) \exp(-C_2(u + w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right)$$

strongly in $L^2(\Omega)$ as n tends to infinity and $\nabla T_k(u_n)$ converges weakly to $\nabla T_k(u)$ in $L^2(\Omega)$, see [15, Lemma 1.3, p. 12].

In order to deal with I_3 and I_4 we use Lemma 3.2.3 (i). We have

$$I_3 \leq \|\psi\|_{L^\infty(\Omega)} \frac{1}{k} \int_\Omega |\nabla T_k(u_n)|^2 ,$$

since $\exp(-C_2(u_n + w_n)) \leq 1$. Then

$$\lim_{k\to\infty} I_3 = 0 \text{ uniformly on } n.$$

For I_4 we first use Holder's inequality to write

$$\begin{aligned} I_4 \leq & \frac{1}{k} \left(\int_\Omega |\nabla T_k(u_n)|^2 \left| \psi \exp(-C_2(u_n + w_n) H\left(\frac{u_n}{k}\right) H'\left(\frac{w_n}{k}\right) \right| \right)^{1/2} \\ & \times \frac{1}{k} \left(\int_\Omega |\nabla T_k(w_n)|^2 \left| \psi \exp(-C_2(u_n + w_n) H\left(\frac{u_n}{k}\right) H'\left(\frac{w_n}{k}\right) \right| \right)^{1/2} \\ & \leq \left(\|\psi\|_{L^\infty(\Omega)} \frac{1}{k} \int_\Omega |\nabla T_k(u_n)|^2 \right)^{1/2} \left(\|\psi\|_{L^\infty(\Omega)} \frac{1}{k} \int_\Omega |\nabla T_k(w_n)|^2 \right)^{1/2} . \end{aligned}$$

Thus

$$\lim_{k\to\infty} I_4 = 0 \text{ uniformly on } n.$$

Now we investigate the remaining term I_5. Since f_n satisfies the inequality (11) and $\psi \leq 0$, we have

$$\begin{aligned} & (f_n - |\nabla u_n|^2 C_1(u_n + w_n))\psi \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right) \\ & \geq \psi \exp(-C_2(u_n+w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right) C_1(u_n+w_n)(|\nabla T_k(w_n)|^2 + K(x)). \end{aligned}$$

Therefore by applying Fatou's lemma, we obtain

$$\lim_{n\to\infty} I_5 \geq \int_\Omega (f - |\nabla u|^2 C_1(u + w))\psi \exp(-C_2(u + w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right).$$

On the other hand we have by Lebesgue's theorem

$$\begin{aligned} & \lim_{n\to\infty} \int_\Omega F_n \psi \exp(-C_2(u_n + w_n)) H\left(\frac{u_n}{k}\right) H\left(\frac{w_n}{k}\right) \\ & = \int_\Omega F\psi \exp(-C_2(u + w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right). \end{aligned}$$

Finally we have shown

$$\int_\Omega \nabla u \nabla \psi \exp(-C_2(u+w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right)$$
$$+ \int_\Omega \psi(f - C_1(u+w)\,|\nabla u|^2\, u) \exp(-C_2(u+w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right) + \omega\left(\frac{1}{k}\right)$$
$$- \int_\Omega \psi \nabla u \nabla w C_1(u+w) \exp(-C_2(u+w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right)$$
$$\leq \int_\Omega F\psi \exp(-C_2(u+w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right),$$

where $\omega(\varepsilon)$ denotes a quantity that tends to 0 when ε tends to 0. Now we choose

$$\psi = -\varphi \exp(C_2(u+w)) H\left(\frac{u}{k}\right) H\left(\frac{w}{k}\right),$$

where $\varphi \geq 0$, $\varphi \in \mathfrak{D}(\Omega)$ and we replace ψ by this value in the previous inequality to get

$$- \int_\Omega \nabla u \nabla \varphi H^2\left(\frac{u}{k}\right) H^2\left(\frac{w}{k}\right) - \int_\Omega \nabla u \nabla(u+w) C_1(u+w) \varphi H^2\left(\frac{u}{k}\right) H^2\left(\frac{w}{k}\right)$$
$$- \frac{1}{k}\int_\Omega |\nabla u|^2 \varphi H\left(\frac{u}{k}\right) H'\left(\frac{u}{k}\right) H^2\left(\frac{w}{k}\right)$$
$$- \frac{1}{k}\int_\Omega \nabla u \nabla w \varphi H^2\left(\frac{u}{k}\right) H'\left(\frac{w}{k}\right) H\left(\frac{w}{k}\right) + \omega\left(\frac{1}{k}\right)$$
$$- \int_\Omega \varphi(f - C_1(u+w)\,|\nabla u|^2) H\left(\frac{u}{k}\right)^2 H^2\left(\frac{w}{k}\right)$$
$$+ \int_\Omega \varphi \nabla u \nabla w C_1(u+w) H^2\left(\frac{u}{k}\right) H^2\left(\frac{w}{k}\right)$$
$$\leq - \int_\Omega F\varphi H^2\left(\frac{u}{k}\right) H^2\left(\frac{w}{k}\right).$$

By developing calculations and remarking that the second and the third terms are equivalent to $\omega\left(\frac{1}{k}\right)$, we can write

$$- \int_\Omega \nabla u \nabla \varphi H^2\left(\frac{u}{k}\right) H^2\left(\frac{w}{k}\right) - \int_\Omega \varphi f H\left(\frac{u}{k}\right)^2 H^2\left(\frac{w}{k}\right) + \omega\left(\frac{1}{k}\right)$$

$$\leq -\int_\Omega F\varphi H^2\left(\frac{u}{k}\right)H^2\left(\frac{w}{k}\right).$$

We finally pass to the limit as k tends to infinity and we use the fact that

$$\lim_{k\to\infty} H\left(\frac{u}{k}\right) = 1 \quad \text{and} \quad \lim_{k\to\infty} H\left(\frac{w}{k}\right) = 1,$$

to conclude that for every $\varphi \geq 0$, $\varphi \in \mathfrak{D}(\Omega)$ that

$$\int_\Omega \nabla u\nabla\varphi + \int_\Omega \varphi f \geq \int_\Omega F\varphi.$$

This finishes the proof of Theorem 2.2. □

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