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ORTHOGONAL BASES FOR SPACES OF COMPLEX SPHERICAL HARMONICS

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Abstract. This paper proposes an inductive method to construct bases for spaces of spherical harmonics over the unit sphere Ω_{2q} of $\mathbb{C}^q$. The bases are shown to have many interesting properties, among them orthogonality with respect to the inner product of $L^2(\Omega_{2q})$. As a bypass, we study the inner product $[f,g] = f(\overline{D})(\overline{g(z)})(0)$ over the space $\mathbb{P}(\mathbb{C}^q)$ of polynomials in the variables $z, \overline{z} \in \mathbb{C}^q$, in which $f(\overline{D})$ is the differential operator with symbol $f(\overline{z})$. On the spaces of spherical harmonics, it is shown that the inner product $[\cdot,\cdot]$ reduces to a multiple of the $L^2(\Omega_{2q})$ inner product. Bi-orthogonality in $(\mathbb{P}(\mathbb{C}^q), [\cdot,\cdot])$ is fully investigated.

1. Introduction

This paper considers spaces of polynomials in the variables z and $\overline{z}$ of $\mathbb{C}^q$, $q \geq 1$. The unitary space $\mathbb{C}^q$ is assumed to be accompanied with its usual inner product

$$\langle z, w\rangle := z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_q\overline{w_q}, \quad z, w \in \mathbb{C}^q, \tag{1.1}$$

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where we are writing $z = (z_1, z_2, \ldots, z_q)$ and $w = (w_1, w_2, \ldots, w_q)$. The major polynomial space considered here is $\mathbb{P}(\mathbb{C}^q)$, the unitary space of polynomials in the independent variables z and $\overline{z}$ of $\mathbb{C}^q$. Elements of this space can be written in the form

$$p(z) := p(z, \overline{z}) = \sum_{|\alpha| \le m} \sum_{|\beta| \le n} p_{\alpha,\beta} z^{\alpha} \overline{z}^{\beta}, \quad p_{\alpha,\beta} \in \mathbb{C}, \quad \alpha, \beta \in \mathbb{Z}_+^q, \tag{1.2}$$

for nonnegative integers m and n, where standard multi-index notation is in force. The subspace of $\mathbb{P}(\mathbb{C}^q)$ composed of polynomials that are homogeneous of degree m in z and of degree n in $\overline{z}$ will be denoted by $\mathbb{P}_{m,n}(\mathbb{C}^q)$. The dimension of $\mathbb{P}_{m,n}(\mathbb{C}^q)$ is given by ([2, p.17])

$$\delta(q, m, n) := \binom{m+q-1}{q-1} \binom{n+q-1}{q-1}. \tag{1.3}$$

The subspace of $\mathbb{P}_{m,n}(\mathbb{C}^q)$ composed of harmonic elements, that is, elements that are in the kernel of the complex Laplacian

$$\Delta_{2q} := 4 \sum_{j=1}^{q} \frac{\partial^2}{\partial z_j \partial \overline{z_j}} \tag{1.4}$$

will be denoted by $\mathbb{H}_{m,n}(\mathbb{C}^q)$. Elements of this space play the role played by the solid harmonics in analysis on real spheres.

Next, we introduce spaces of polynomials restricted to the unit sphere

$$\Omega_{2q} := \{z \in \mathbb{C}^q : \langle z, z \rangle = 1\}. \tag{1.5}$$

The symbol $\mathcal{P}_{m,n}(\Omega_{2q})$ will stand for the space obtained from $\mathbb{P}_{m,n}(\mathbb{C}^q)$ by restricting its elements to Ω_{2q}. Finally, $\mathcal{H}_{m,n}(\Omega_{2q})$ will denote the space of *complex spherical harmonics* of degree m in z and degree n in $\overline{z}$, that is, the set of restrictions of elements of $\mathbb{H}_{m,n}(\mathbb{C}^q)$ to Ω_{2q}. The space $\mathcal{H}_{m,n}(\Omega_{2q})$ has dimension $d(q, m, n)$ given by ([2, p. 17])

$$d(q, m, n) = \delta(q, m, n) - \delta(q, m-1, n-1), \quad m, n \neq 0, \tag{1.6}$$

$$(q, m, 0) = \delta(q, m, 0), \quad \text{and} \quad \delta(q, 0, n) = \delta(q, 0, n). \tag{1.7}$$

This paper was motivated by the following three results: the orthogonal decomposition ([2])

$$\mathcal{P}_{m,n}(\Omega_{2q}) = \bigoplus_{j=0}^{m \wedge n} \mathcal{H}_{m-j,n-j}(\Omega_{2q}), \tag{1.8}$$

the dimension formula ([2], [8])

$$d(q, m, n) = \sum_{k=0}^{m} \sum_{l=0}^{n} d(q-1, k, l), \quad q \ge 2, \tag{1.9}$$

and the fact that some elements of $\mathcal{H}_{m,n}(\Omega_{2q})$ can be constructed from given elements in $\mathcal{H}_{m-k,n-l}(\Omega_{2q})$, $k < m$, $l < n$, by multiplying them by special elements of $\mathbb{H}_{k,l}(\mathbb{C}^q)$ (see proof of Theorem 5.1 in [3]).

Looking at the real version of (1.8) in either [1, p. 76] or [9, p. 139] one observes that the proof there requires a special inner product on spaces of homogeneous polynomials. In the first half of the paper, we endow our polynomial spaces with the following similar inner product

$$[f,g] := [f,g]_q := f(\overline{D})\left(\overline{g(z)}\right)(0), \quad f,g \in \mathbb{P}(\mathbb{C}^q), \tag{1.10}$$

in which

$$\overline{D} := \left(\frac{\partial}{\partial \overline{z_1}}, \frac{\partial}{\partial \overline{z_2}}, \cdots, \frac{\partial}{\partial \overline{z_q}}\right), \tag{1.11}$$

and extract a number of interesting properties. Among them, we show that there is a positive constant C, depending on m, n and q, such that

$$[f,g] = C\langle f,g\rangle_2, \quad f,g \in \mathcal{H}_{m,n}(\Omega_{2q}). \tag{1.12}$$

The inner product in the right-hand side of (1.12) is the usual one in $L^2(\Omega_{2q})$, that is,

$$\langle f,g\rangle_2 := \int_{\Omega_{2q}} f(z)\overline{g(z)}d\sigma_q(z), \quad f,g \in L^2(\Omega_{2q}), \tag{1.13}$$

where σ_q is a positive Borel measure invariant by isometries of C^q and uniquely determined by the condition

$$\sigma_q(\Omega_{2q}) = \frac{2\pi^q}{(q-1)!}. \tag{1.14}$$

The other properties we obtain are related to the Funk-Hecke formula ([5], [6]) and with properties of bi-orthogonal systems in the polynomial spaces endowed with the inner product in (1.10). All the results mentioned above form the contents of Sections 2 and 3.

Formula (1.9) suggests that one should be able to construct a basis for $\mathcal{H}_{m,n}(\Omega_{2q})$ from given bases for the spaces $\mathcal{H}_{k,l}(\Omega_{2q-2})$, $k = 0,1,\ldots,m$, $l = 0,1,\ldots,n$. We prove this is the case using the special polynomials introduced in [3, p. 3] as a generating function. In addition, we discuss orthogonality and representing properties that are implied by the result, completing the list of results forming Section 4.

2. The inner product $[\cdot,\cdot]$

To begin this section, we observe that the spaces $\mathcal{H}_{m,n}(\Omega_{2q})$ are pairwise orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_2$ ([3]). Throughout the paper, orthogonality will always refer to this inner product.

If $\mathcal{O}(2q)$ is the group of isometries of $\mathbb{C}^q$ that fix the origin then σ_q is $\mathcal{O}(2q)$-invariant in the following sense: $\sigma_q(\rho B) = \sigma_q(B)$ if $\rho \in \mathcal{O}(2q)$ and B is a Borel subset of Ω_{2q}. As a consequence, the following invariance property holds:

$$\langle f \circ \rho, g \circ \rho \rangle_2 = \langle f, g \rangle_2, \quad f, g \in L^2(\Omega_{2q}), \quad \rho \in \mathcal{O}(2q). \tag{2.1}$$

The following well-known result establishes the $\mathcal{O}(2q)$-invariance of complex spherical harmonics.

Lemma 2.1. *The space $\mathcal{H}_{m,n}(\Omega_{2q})$ is $\mathcal{O}(2q)$-invariant, that is, if $f \in \mathcal{H}_{m,n}(\Omega_{2q})$ and $\rho \in \mathcal{O}(2q)$ then $f \circ \rho \in \mathcal{H}_{m,n}(\Omega_{2q})$.*

Proof. It will be left to the reader. □

Next, we return to formula (1.10).

Lemma 2.2. *Formula* (1.10) *defines an inner product in $\mathbb{P}(\mathbb{C}^q)$.*

Proof. It is very easy to see from the definitions that if $(i, j) \neq (k, l)$ then the spaces $\mathbb{P}_{i,j}(\mathbb{C}^q)$ and $\mathbb{P}_{k,l}(\mathbb{C}^q)$ are orthogonal with respect to $[\cdot, \cdot]$. In particular, we have

$$[z^\alpha \overline{z}^\beta, z^\gamma \overline{z}^\delta] = \begin{cases} \alpha!\beta!, & (\alpha, \beta) = (\gamma, \delta) \\ 0, & (\alpha, \beta) \neq (\gamma, \delta). \end{cases} \tag{2.2}$$

Now, let $f, g \in \mathbb{P}(\mathbb{C}^q)$. There are pairs of indices (k, l) and (m, n) in $\mathbb{Z}_+^2$ such that

$$f(z) = \sum_{i=0}^{k} \sum_{j=0}^{l} f_{i,j}(z), \quad g(z) = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} g_{\mu,\nu}(z),$$
$$f_{i,j} \in \mathbb{P}_{i,j}(\mathbb{C}^q), \quad g_{\mu,\nu} \in \mathbb{P}_{\mu,\nu}(\mathbb{C}^q). \tag{2.3}$$

Hence,

$$[f, g] = \sum_{i=0}^{k} \sum_{j=0}^{l} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} [f_{i,j}, g_{\mu,\nu}] = \sum_{\mu=0}^{k \wedge m} \sum_{\nu=0}^{l \wedge n} [f_{\mu,\nu}, g_{\mu,\nu}]. \tag{2.4}$$

Expanding $f_{\mu,\nu}$ and $g_{\mu,\nu}$ in the form

$$f_{\mu,\nu}(z) = \sum_{|\alpha|=\mu} \sum_{|\beta|=\nu} a_{\alpha,\beta} z^\alpha \overline{z}^\beta, \quad g_{\mu,\nu}(z) = \sum_{|\gamma|=\mu} \sum_{|\delta|=\nu} b_{\gamma,\delta} z^\gamma \overline{z}^\delta,$$
$$a_{\alpha,\beta}, b_{\gamma,\delta} \in \mathbb{C}, \tag{2.5}$$

we finally deduce that

$$[f, g] = \sum_{\mu=0}^{k \wedge m} \sum_{\nu=0}^{l \wedge n} \sum_{|\alpha|=\mu} \sum_{|\beta|=\nu} \alpha!\beta! a_{\alpha,\beta} \overline{b_{\alpha,\beta}}. \tag{2.6}$$

Using this representation, it is now easy to verify that $[\cdot,\cdot]$ defines an inner product in the space $\mathbb{P}(\mathbb{C}^q)$. $\square$

As an example, we observe that the set

$$\bigcup_{m,n\in\mathbb{Z}_+}\left\{\frac{z^\alpha}{\sqrt{\alpha!}}\frac{\overline{z}^\beta}{\sqrt{\beta!}} : |\alpha|=m,\ |\beta|=n\right\} \tag{2.7}$$

is an orthonormal basis for $(\mathbb{P}(\mathbb{C}^q),[\cdot,\cdot])$. Another remark at this time is that formula (1.10) reduces to

$$[f,g]=f(\overline{D})\left(\overline{g(z)}\right), \tag{2.8}$$

when the space $\mathbb{P}(\mathbb{C}^q)$ is replaced with its subspace $\mathbb{P}_{m,n}(\mathbb{C}^q)$. At last, we observe that formula (2.6) is a complex extension of that appearing in Theorem 5.14 in [1].

Lemma 2.3. *The inner product* $[\cdot,\cdot]$ *possesses the following invariance property*

$$[f\circ\rho, f\circ\rho]=[f,f], \quad f\in\mathbb{P}_{m,n}(\mathbb{C}), \quad \rho\in\mathcal{O}(2q). \tag{2.9}$$

Proof. Since every element of $\mathbb{P}_{m,n}(\mathbb{C}^q)$ is a linear combination of elements of the form $z^\alpha\overline{z}^\beta$, it suffices to verify the formula in the statement of the lemma for elements of this type. However, since

$$[z^\alpha\overline{z}^\beta\circ\rho, z^\gamma\overline{z}^\delta\circ\rho]=[z^\alpha\circ\rho, z^\gamma\circ\rho][\overline{z}^\beta\circ\rho,\overline{z}^\delta\circ\rho], \quad \rho\in\mathcal{O}(2q), \tag{2.10}$$

it suffices to prove the formula in the case in which $f(z)=z^\alpha$ and $g(z)=z^\gamma$, $|\alpha|=|\gamma|$, and in the conjugate case of this one. Let $\rho\in\mathcal{O}(2q)$ be described as

$$\rho(z)=\left(\sum_{j=1}^q a_{1j}z_j, \sum_{j=1}^q a_{2j}z_j,\ldots,\sum_{j=1}^q a_{qj}z_j\right), \quad a_{lj}\in\mathbb{C}, \quad z\in\mathbb{C}^q. \tag{2.11}$$

If f and g are as above then the formula to be proven is

$$[f\circ\rho, g\circ\rho]=\prod_{l=1}^q D_l^{\alpha_l}\overline{\rho(z)^\gamma}, \tag{2.12}$$

where

$$D_l := a_{l1}\frac{\partial}{\partial\overline{z}_1}+a_{l2}\frac{\partial}{\partial\overline{z}_2}+\cdots+a_{lq}\frac{\partial}{\partial\overline{z}_q}, \quad l=1,2,\ldots,q. \tag{2.13}$$

First consider the case $\alpha=\gamma$. Using the relation

$$\sum_{k=1}^q\overline{a_{jk}}a_{lk}=\begin{cases}0 & \text{if } l\neq j\\ 1 & \text{if } l=j,\end{cases} \tag{2.14}$$

it is easily seen that

$$D_l^{\alpha_l}\,\overline{\rho(z)}^{\alpha} = \alpha_l!\,\overline{\rho(z)}^{\alpha-\alpha_l\varepsilon_l}, \quad l = 1, 2, \dots, q. \tag{2.15}$$

It follows that $[f\circ\rho, g\circ\rho] = \alpha! = [z^\alpha, z^\alpha]$. If $\alpha \neq \gamma$, we can assume without loss of generality, that $\alpha_j > \gamma_j$ for some j. In this case, $D_j^{\alpha_j}\,\overline{\rho(z)}^{\gamma} = 0$, that is, $[f\circ\rho, g\circ\rho] = 0 = [z^\alpha, z^\gamma]$. The conjugate case is dealt with in a similar manner. □

Next, we employ the vector space isomorphism

$$f \in \mathbb{H}_{m,n}(\mathbb{C}^q) \longmapsto f|_{\Omega_{2q}} \in \mathcal{H}_{m,n}(\Omega_{2q}) \tag{2.16}$$

to bring the inner product (1.10) into the space $\mathcal{H}_{m,n}(\Omega_{2q})$. If $f \in \mathcal{H}_{m,n}(\Omega_{2q})$ write $\widehat{f}$ to denote the unique element of $\mathbb{H}_{m,n}(\mathbb{C}^q)$ such that $\widehat{f}|_{\Omega_{2q}} = f$. Then the formula

$$[f, g] := [\widehat{f}, \widehat{g}], \quad f, g \in \mathcal{H}_{m,n}(\Omega_{2q}) \tag{2.17}$$

defines an inner product in $\mathcal{H}_{m,n}(\Omega_{2q})$.

Theorem 2.7 below will reveal that the spaces $(\mathcal{H}_{m,n}(\Omega_{2q}), [\cdot,\cdot])$ and $(\mathcal{H}_{m,n}(\Omega_{2q}), \langle\cdot,\cdot\rangle_2)$ are isomorphic. The following results will be helpful in proving that theorem. Details about them can be found in [2]. The proof of the first one can also be adapted from results proved in [6, p. 17]. From now on, the symbol ε_j will stand for the vector of $\mathbb{C}^q$ having 1 in its j^{th} component and zeros elsewhere.

Lemma 2.4. *If W is a nonzero finite-dimensional $\mathcal{O}(2q)$-invariant space of continuous functions on Ω_{2q} then there exists a unique f in $W \setminus \{0\}$ such that $f\circ\rho = f$, when $\rho \in \mathcal{O}(2q)$ and $\rho(\varepsilon_q) = \varepsilon_q$.*

Lemma 2.5. *Let f be in $\mathcal{H}_{m,n}(\Omega_{2q})$. The following assertions are equivalent:*

i) $f\circ\rho = f$ *if* $\rho \in \mathcal{O}(2q)$ *and* $\rho(\varepsilon_q) = \varepsilon_q$;

ii) *There exists a complex number C such that*

$$f(z) = Ce^{i(m-n)\theta}|\langle z, \varepsilon_q\rangle|^{|m-n|}P_{m\wedge n}^{(q-2,|m-n|)}(2|\langle z, \varepsilon_q\rangle|^2 - 1), \quad z \in \Omega_{2q}, \tag{2.18}$$

in which θ is an argument of $\langle z, \varepsilon_q\rangle$ in $[0, 2\pi)$.

Proposition 2.6. *Let $\mathcal{N}$ be a subspace of $\mathcal{H}_{m,n}(\Omega_{2q})$. If $\mathcal{N}$ is $\mathcal{O}(2q)$-invariant then either $\mathcal{N} = \{0\}$ or $\mathcal{N} = \mathcal{H}_{m,n}(\Omega_{2q})$.*

Proof. If $\mathcal{N} \neq \{0\}$ then $\mathcal{H}_{m,n}(\Omega_{2q}) = \mathcal{N} \oplus \mathcal{N}^{\perp}$, in which $\mathcal{N}^{\perp}$ is the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}_{m,n}(\Omega_{2q})$. Obviously, $\mathcal{N}^{\perp}$ is $\mathcal{O}(2q)$-invariant. The rest of the proof will show that $\mathcal{N}^{\perp} = \{0\}$. Indeed, if not, we may use Lemma 2.4 to choose $f \in \mathcal{N} \setminus \{0\}$ and $g \in \mathcal{N}^{\perp} \setminus \{0\}$ such that $f \circ \rho = f$ and $g \circ \rho = g$, when $\rho \in \mathcal{O}(2q)$ and $\rho(\varepsilon_q) = \varepsilon_q$. Lemma 2.5 furnishes a complex number C such that $f = C\,g$. It follows that $f = g = 0$, a clear contradiction. □

Theorem 2.7. *There exists a positive constant C, depending on m, n and q, such that*

$$[f, g] = C\langle f, g\rangle_2, \quad f, g \in \mathcal{H}_{m,n}(\Omega_{2q}). \tag{2.19}$$

Proof. Since $F := \{f \in \mathcal{H}_{m,n}(\Omega_{2q}) : \langle f, f\rangle_2 = 1\}$ is a compact subset of $\mathcal{H}_{m,n}(\Omega_{2q})$, the continuous function

$$f \in F \longmapsto [f, f] \in \mathbb{R} \tag{2.20}$$

attains its maximum in a point f_0 of F. It follows that,

$$[f, f] \leq [f_0, f_0]\langle f, f\rangle_2, \quad f \in \mathcal{H}_{m,n}(\Omega_{2q}). \tag{2.21}$$

We will use this information to show that the bilinear form

$$\varphi : \mathcal{H}_{m,n}(\Omega_{2q}) \times \mathcal{H}_{m,n}(\Omega_{2q}) \longrightarrow \mathbb{C} \tag{2.22}$$

given by

$$\varphi(f, g) = [f_0, f_0]\langle f, g\rangle_2 - [f, g], \quad f, g \in \mathcal{H}_{m,n}(\Omega_{2q}) \tag{2.23}$$

is identically zero. Equivalently, we will show that

$$\mathcal{N} := \{f \in \mathcal{H}_{m,n}(\Omega_{2q}) : \varphi(f, g) = 0, \quad g \in \mathcal{H}_{m,n}(\Omega_{2q})\} \tag{2.24}$$

is the whole space $\mathcal{H}_{m,n}(\Omega_{2q})$. Since $\mathcal{N}$ is a subspace of $\mathcal{H}_{m,n}(\Omega_{2q})$, Proposition 2.6 tells us that it suffices to show that $\mathcal{N}$ is nonzero and $\mathcal{O}(2q)$-invariant. Let $\rho \in \mathcal{O}(2q)$ and $f \in \mathcal{N}$. Due to (2.21), φ is positive definite. Hence, we may apply Schwarz's inequality [4, p. 375] to obtain

$$|\varphi(f \circ \rho, g)|^2 \leq \varphi(f \circ \rho, f \circ \rho)\varphi(g, g), \quad g \in \mathcal{H}_{m,n}(\Omega_{2q}). \tag{2.25}$$

However, Lemma 2.3 and property (2.1) imply that $\varphi(f \circ \rho, f \circ \rho) = \varphi(f, f) = 0$. It follows that $f \circ \rho \in \mathcal{N}$. Since a similar argument shows that $\varphi(f_0, g) = 0$, $g \in \mathcal{H}_{m,n}(\Omega_{2q})$, it is clear that $\mathcal{N}$ is nonzero. □

Corollary 2.8. *There exists a positive constant C such that*

$$[f, g] = C\langle f|_{\Omega_{2q}}, g|_{\Omega_{2q}}\rangle_2, \quad f, g \in \mathbb{H}_{m,n}(\mathbb{C}^q). \tag{2.26}$$

Next, we compute the constant C in Theorem 2.7. The following lemma is taken from Rudin's book [8, p. 16].

Lemma 2.9. *For multi-indices α and β we have*

$$\int_{\Omega_{2q}} z^\alpha \overline{z}^\beta d\sigma_q(z) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \dfrac{2\pi^q \alpha!}{(|\alpha| + q - 1)!} & \text{if } \alpha = \beta. \end{cases} \tag{2.27}$$

Take $f(z) = g(z) = z_1^m \overline{z_2}^n$ in the space $\mathbb{H}_{m,n}(\mathbb{C}^q)$. Formula (2.2) implies that $[f, g] = m!n!$ while Lemma 2.9 produces

$$\langle f, g\rangle_2 = \frac{2\pi^q m!n!}{(m + n + q - 1)!}. \tag{2.28}$$

This proves the following theorem.

Theorem 2.10. *The constant C in Theorem 2.7 equals to $(m + n + q - 1)!(2\pi^q)^{-1}$.*

We close the section by showing that Theorem 2.7 cannot hold in the bigger space $\mathcal{P}_{m,n}(\Omega_{2q})$. In fact, if $h(z) = z_1^m \overline{z_1}^n$ then $[h, h] = m!n!$ while Lemma 2.9 yields $\langle h, h\rangle_2 = 2\pi^q (m + n)!/(m + n + q - 1)!$. Now, it is easily seen that the equality $[h, h] = C\langle h, h\rangle_2$ holds if and only if $C = m!n!(m + n + q - 1)!(2\pi^q)^{-1}/(m + n)!$. This is not the value of C we have encountered in Theorem 2.10.

3. Bi-orthogonality in $(\mathbb{P}(\mathbb{C}^q), [\cdot, \cdot])$

In this section we investigate orthogonality in the space $(\mathbb{P}(\mathbb{C}^q), [\cdot, \cdot])$. We begin with a result related to basic elements of $(\mathbb{P}_{m,n}(\mathbb{C}^q), [\cdot, \cdot])$.

Theorem 3.1. *Let $\{f_\mu : \mu = 1, 2, \ldots, \delta(q, m, n)\}$ and $\{g_\nu : \nu = 1, 2, \ldots, \delta(q, m, n)\}$ be bases for $(\mathbb{P}_{m,n}(\mathbb{C}^q), [\cdot, \cdot])$. If $[f_\mu, g_\nu] = 0$, $\mu \neq \nu$ then*

$$\langle z, w\rangle^m \langle w, z\rangle^n = m!\, n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(w)}}{[f_\mu, g_\mu]}, \quad z, w \in \mathbb{C}^q. \tag{3.1}$$

Proof. Since $\{f_\mu : \mu = 1, 2, \ldots, \delta(q, m, n)\}$ is a basis for $\mathbb{P}_{m,n}(\mathbb{C}^q)$, there are polynomials p_μ, $\mu = 1, 2, \ldots, \delta(q, m, n)$ such that

$$\langle z, w\rangle^m \langle w, z\rangle^n = \sum_{\mu=1}^{\delta(q,m,n)} p_\mu(w) f_\mu(z), \quad z, w \in \mathbb{C}^q. \tag{3.2}$$

Due to the hypothesis,

$$[\langle \cdot, w\rangle^m \langle w, \cdot\rangle^n, g_\nu] = \sum_{\mu=1}^{\delta(q,m,n)} p_\mu(w)[f_\mu, g_\nu]$$

$$=p_\nu(w)[f_\nu, g_\nu], \quad \nu = 1,2,\dots,\delta(q,m,n), \quad w \in \mathbb{C}^q.$$

On the other hand, writing g_ν in the form

$$g_\nu(z) = \sum_{|\alpha|=m} \sum_{|\beta|=n} c_{\alpha,\beta} z^\alpha \overline{z}^\beta \tag{3.3}$$

and computing, we obtain

$$\begin{aligned}[\langle\cdot, w\rangle^m \langle w, \cdot\rangle^n, g_\nu] =& m!\,n! \sum_{|\gamma|=m} \sum_{|\delta|=n} \sum_{|\alpha|=m} \sum_{|\beta|=n} \frac{\overline{w}^\gamma}{\gamma!} \frac{w^\delta}{\delta!} \overline{c_{\alpha,\beta}} [z^\gamma \overline{z}^\delta, z^\alpha \overline{z}^\beta] \\ =& m!\,n! \sum_{|\alpha|=m} \sum_{|\beta|=n} \overline{c_{\alpha,\beta}}\, \overline{w}^\alpha w^\beta \\ =& m!\,n!\, \overline{g_\nu(w)}, \quad \nu = 1,2,\dots,\delta(m,n).\end{aligned}$$

Thus,

$$m!\,n!\,\overline{g_\nu(w)} = p_\nu(w)[f_\nu, g_\nu], \quad \nu = 1,2,\dots,\delta(q,m,n), \quad w \in \mathbb{C}^q, \tag{3.4}$$

and, in particular, since each g_μ is not identically zero, $[f_\mu, g_\mu] \neq 0$, $\mu = 1,2,\dots,\delta(m,n)$. Concluding,

$$p_\mu = m!n! \frac{\overline{g_\mu}}{[f_\mu, g_\mu]}, \quad \mu = 1,2,\dots,\delta(m,n) \tag{3.5}$$

and the result follows. □

If we let $z = w$ in the previous theorem we get the Pythagorian identity

$$\frac{\langle z, z\rangle^{m+n}}{m!\,n!} = \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(z)}}{[f_\mu, g_\mu]}, \quad z \in \mathbb{C}^q. \tag{3.6}$$

When $z \in \Omega_{2q}$, it reduces to

$$\frac{1}{m!\,n!} = \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(z)}}{[f_\mu, g_\mu]}. \tag{3.7}$$

If both bases in the previous theorem are equal and orthonormal with respect to $[\cdot,\cdot]$ then we deduce the addition formula

$$\langle z, w\rangle^m \langle w, z\rangle^n = m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} f_\mu(z)\overline{f_\mu(w)}, \quad z, w \in \mathbb{C}^q. \tag{3.8}$$

This formula has a structure very similar to that of the addition formula for complex spherical harmonics ([2]). Finally, the following extension of (3.1) can be proved in a similar manner:

$$\langle z,u\rangle^m\langle v,z\rangle^n = m!\,n!\sum_{\mu=1}^{\delta(q,m,n)}\frac{f_\mu(z)\overline{g_\mu(u,\overline{v})}}{[f_\mu,g_\mu]},\quad z,u,v\in\mathbb{C}^q. \tag{3.9}$$

Here, $g_\mu(u,\overline{v})$ is obtained from $g_\mu(u)=g_\mu(u,\overline{u})$, substituting $\overline{u}$ by $\overline{v}$.

In our next result, we establish a Funk-Hecke type theorem for elements in the space $(\mathbb{P}(\mathbb{C}^q),[\cdot,\cdot])$.

Theorem 3.2. *Let f be an element of $\mathbb{P}_{m,n}(\mathbb{C}^q)$ and g an element of $\mathbb{P}(\mathbb{C})$. Then, for each $w\in\mathbb{C}^q$, the map $z\in\mathbb{C}^q\mapsto g(\langle z,w\rangle)$ belong to $\mathbb{P}(\mathbb{C}^q)$. In addition, there exists a nonnegative constant λ, depending on m and n, such that*

$$[g(\langle\cdot,w\rangle),f]=\lambda\,\overline{f(w)},\quad w\in\mathbb{C}^q. \tag{3.10}$$

Proof. For each pair (k,l), we will denote by $\{g^\mu_{k,l}:\mu=1,2,\ldots,\delta(q,k,l)\}$ an orthonormal basis for $(\mathbb{P}_{k,l}(\mathbb{C}^q),[\cdot,\cdot])$. Assume g has degree α in z and degree β in $\overline{z}$. Recalling Theorem 3.1, we can write

$$g(\langle z,w\rangle)=\sum_{k=0}^{\alpha}\sum_{l=0}^{\beta}\sum_{\mu=1}^{\delta(q,k,l)}k!\,l!\,g^\mu_{k,l}(z)\overline{g^\mu_{k,l}(w)},\quad z,w\in\mathbb{C}^q. \tag{3.11}$$

We can find complex numbers a_j such that

$$f=\sum_{j=1}^{\delta(q,m,n)}a_j g^j_{m,n}. \tag{3.12}$$

It follows that

$$\begin{aligned}[g(\langle\cdot,w\rangle),f]&=\sum_{j=1}^{\delta(q,m,n)}\sum_{k=0}^{\alpha}\sum_{l=0}^{\beta}\sum_{\mu=1}^{\delta(q,k,l)}k!\,l!\,\overline{a_j}\,\overline{g^\mu_{k,l}(w)}[g^\mu_{k,l},g^j_{m,n}]\\&=\sum_{j=1}^{\delta(q,m,n)}\sum_{k=0}^{\alpha}\sum_{l=0}^{\beta}\sum_{\mu=1}^{\delta(q,k,l)}k!\,l!\,\overline{a_j}\,\overline{g^\mu_{k,l}(w)}\delta_{km}\delta_{ln}\delta_{\mu j},\quad w\in\mathbb{C}^q.\end{aligned}$$

Thus,

$$[g(\langle\cdot,w\rangle),f]=\begin{cases}m!\,n!\,\overline{f(w)}, & \alpha\geq m \text{ and } \beta\geq n\\ 0, & \text{otherwise,}\end{cases} \tag{3.13}$$

completing the proof of the theorem. □

Corollary 3.3. *The following formula holds*

$$[\langle\cdot,w\rangle^m\langle w,\cdot\rangle^n,\langle\cdot,\zeta\rangle^m\langle\zeta,\cdot\rangle^n]=m!\,n!\langle\zeta,w\rangle^m\langle w,\zeta\rangle^n,\quad w,\zeta\in\mathbb{C}^q. \tag{3.14}$$

The following theorem is a converse of Theorem 3.1.

Theorem 3.4. *Let* $\{f_\mu : \mu = 1, 2, \ldots, \delta(q,m,n)\}$ *be a linearly independent subset of* $\mathbb{P}_{m,n}(\mathbb{C}^q)$. *Assume there is a subset* $\{g_\mu : \mu = 1, 2, \ldots, \delta(q,m,n)\}$ *of* $\mathbb{P}(\mathbb{C}^q)$ *such that* $[f_\mu, g_\mu] \neq 0$, $\mu = 1, 2, \ldots, \delta(q,m,n)$ *and*

$$\langle z, w\rangle^m \langle w, z\rangle^n = m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(w)}}{[f_\mu, g_\mu]} \quad z, w \in \mathbb{C}^q. \qquad (3.15)$$

Then $\{f_\mu : \mu = 1, 2, \ldots, \delta(q,m,n)\}$ *and* $\{g_\mu : \mu = 1, 2, \ldots, \delta(q,m,n)\}$ *are bases for* $\mathbb{P}_{m,n}(\mathbb{C}^q)$ *satisfying* $[f_\mu, g_\nu] = 0$, $\mu \neq \nu$.

Proof. The use of (3.15) yields

$$\begin{aligned} m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(\lambda w)}}{[f_\mu, g_\mu]} &= \langle z, \lambda w\rangle^m \langle \lambda w, z\rangle^n \\ &= \langle \overline{\lambda} z, w\rangle^m \langle w, \overline{\lambda} z\rangle^n \\ &= m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(\overline{\lambda} z)\overline{g_\mu(w)}}{[f_\mu, g_\mu]} \\ &= m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \overline{\lambda}^m \lambda^n \frac{f_\mu(z)\overline{g_\mu(w)}}{[f_\mu, g_\mu]}, \ z, w \in \mathbb{C}^q, \ \lambda \in \mathbb{C}. \end{aligned}$$

Hence

$$\sum_{\mu=1}^{\delta(q,m,n)} \left(\overline{g_\mu(\lambda w)} - \overline{\lambda}^m \lambda^n \overline{g_\mu(w)}\right) \frac{f_\mu(z)}{[f_\mu, g_\mu]} = 0, \quad z, w \in \mathbb{C}^q, \quad \lambda \in \mathbb{C}. \quad (3.16)$$

Since the set $\{f_\mu : \mu = 1, 2, \ldots, \delta(q,m,n)\}$ is linearly independent, it follows that

$$g_\mu(\lambda w) - \lambda^m \overline{\lambda}^n g_\mu(w) = 0, \quad w \in \mathbb{C}^q, \quad \lambda \in \mathbb{C}, \qquad (3.17)$$

that is, $g_\mu \in \mathbb{P}_{m,n}(\mathbb{C}^q)$, $\mu = 1, 2, \ldots, \delta(q,m,n)$. To conclude the proof we apply Theorem 3.2 and formula (3.15) appropriately to obtain

$$m!\,n!\, f_\nu(z) = [\langle \cdot, z\rangle^m \langle z, \cdot\rangle^n, \overline{f_\nu}] = m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)[f_\nu, g_\mu]}{[f_\mu, g_\mu]},$$
$$\nu = 1, 2, \ldots, \delta(q,m,n).$$

The linear independence hypothesis allows us to conclude that $[f_\nu, g_\mu] = 0$, $\mu \neq \nu$. □

Corollary 3.5. *If a linearly independent subset* $\{f_\mu : \mu = 1, 2, \dots, \delta(q, m, n)\}$ *of* $\mathbb{P}_{m,n}(\mathbb{C}^q)$ *satisfies*

$$\langle z, w\rangle^m \langle w, z\rangle^n = m!\, n! \sum_{\mu=1}^{\delta(q,m,n)} f_\mu(z)\overline{f_\mu(w)}, \quad z, w \in \mathbb{C}^q, \tag{3.18}$$

then it is orthonormal with respect to $[\cdot,\cdot]$.

Proof. It suffices to observe that, under the given hypotheses, the denominator in the sum on the right-hand side of the last equation in the proof of Theorem 3.4 disappears. □

4. Generating bases

This section presents a method to construct bases for the space $\mathcal{H}_{m,n}(\Omega_{2q})$. The method is inductive over the dimension of the sphere, that is, it presupposes the knowledge of a basis for $\mathcal{H}_{m,n}(\Omega_{2q-2})$. We begin with a technical lemma that exhibits a very special kernel in $\mathbb{H}_{m,n}(\mathbb{C}^q)$. As we said before, the idea behind the use of this kernel comes from the proof of Theorem 5.1 in [3].

For a fixed $q_1 \in \{1, 2, \dots, q\}$ we will employ the decomposition $\mathbb{C}^q = W^{q_1} \oplus V^{q-q_1}$, where $W^{q_1} = \{z \in \mathbb{C}^q : z_j = 0, j = q_1 + 1, q_1 + 2, \dots, q\}$ and $V^{q-q_1} = \{z \in \mathbb{C}^q : z_j = 0,\, j = 1, 2, \dots, q_1\}$.

Lemma 4.1. *Let* $w \in W^{q_1} \cap \Omega_{2q}$ *and* $v \in V^{q-q_1} \cap \Omega_{2q}$. *Then*

$$G^{w,v}_{m,n}(z) := \langle z, v + w\rangle^m \langle v - w, z\rangle^n, \quad z \in \mathbb{C}^q \tag{4.1}$$

is an element of $\mathbb{H}_{m,n}(\mathbb{C}^q)$.

Proof. First observe that

$$\frac{\partial}{\partial \overline{z_j}}\, G^{w,v}_{m,n} = \begin{cases} -n\langle z, v + w\rangle^m \langle v - w, z\rangle^{n-1} w_j & j = 1, 2, \dots, q_1 \\ n\langle z, w + v\rangle^m \langle v - w, z\rangle^{n-1} v_j & j = q_1 + 1, q_1 + 2, \dots, q. \end{cases}$$

Next, notice that

$$\sum_{j=q_1+1}^{q} \frac{\partial^2}{\partial z_j \partial \overline{z_j}} G^{w,v}_{m,n} = -\sum_{j=1}^{q_1} \frac{\partial^2}{\partial z_j \partial \overline{z_j}} G^{w,v}_{m,n}. \tag{4.2}$$

It follows that $\Delta_{2q}(G^{w,v}_{m,n}) = 0$. The homogeneity of $G^{w,v}_{m,n}$ with respect to z and $\overline{z}$ is clear. □

If $q_1 = q-1$ in the previous lemma then $W^{q-1} \cap \Omega_{2q}$ is a copy of Ω_{2q-2}. In other words, elements of $W^{q-1} \cap \Omega_{2q}$ are of the form $\widehat{w} = (w, 0)$ with $w \in \Omega_{2q-2}$. Denoting the elements of $\mathbb{C}^q$ by $\widehat{z} = (z, z_q)$, $z \in \mathbb{C}^{q-1}$, and taking $v = \varepsilon_q = (0, 0, \ldots, 0, 1)$, the function in the previous lemma takes the form

$$G^{\widehat{w},v}_{m,n}(\widehat{z}) = (\langle z, w\rangle + z_q)^m(-\langle w, z\rangle + \overline{z_q})^n. \tag{4.3}$$

From now on, we will adopt the following simplified notation: $G^w_{m,n} := G^{\widehat{w},v}_{m,n}$. The main result of this section is as follows.

Theorem 4.2. *Let $\{g_j : j = 1, 2, \ldots, d(q,m,n)\}$ be a linearly independent subset of $\bigcup_{k=0}^m \bigcup_{l=0}^n \mathcal{H}_{k,l}(\Omega_{2q-2})$. Then there exists a subset $\{f_j : j = 1, 2, \ldots, d(q,m,n)\}$ of $\mathbb{H}_{m,n}(\mathbb{C}^q)$ such that*

$$G^w_{m,n}(\widehat{z}) = \sum_{j=1}^{d(q,m,n)} f_j(\widehat{z}) g_j(w), \quad \widehat{z} = (z, z_q) \in \mathbb{C}^q, \quad w \in \Omega_{2q-2}. \tag{4.4}$$

Proof. Initially, we expand the right-hand side of (4.3) to write

$$G^w_{m,n}(\widehat{z}) = \sum_{|\alpha|+\mu=m} \frac{m!}{\mu!\,\alpha!} z^\alpha z_q^\mu \overline{w}^\alpha \sum_{|\beta|+\nu=n} \frac{n!}{\nu!\,\beta!}(-w)^\beta \overline{z}^\beta \overline{z}_q^\nu, \quad \widehat{z} \in \mathbb{C}^q. \tag{4.5}$$

Since $|\alpha| \le m$ and $|\beta| \le n$, a help of (1.8) allows us to find constants $a_j(\alpha, \beta)$ such that

$$\overline{w}^\alpha(-w)^\beta = \sum_{j=1}^{d(q,m,n)} a_j(\alpha,\beta) g_j(w), \quad w \in \Omega_{2q-2}. \tag{4.6}$$

Hence,

$$G^w_{m,n}(\widehat{z}) = \sum_{j=1}^{d(q,m,n)} \left(\sum_{|\alpha|+\mu=m} \sum_{|\beta|+\nu=n} a_j(\alpha,\beta) \frac{m!}{\mu!\,\alpha!} z^\alpha z_q^\mu \frac{n!}{\nu!\,\beta!} \overline{z}^\beta \overline{z}_q^\nu \right) g_j(w). \tag{4.7}$$

We now show that the expression

$$f_j(\widehat{z}) := \sum_{|\alpha|+\mu=m} \sum_{|\beta|+\nu=n} a_j(\alpha,\beta) \frac{m!}{\mu!\,\alpha!} z^\alpha z_q^\mu \frac{n!}{\nu!\,\beta!} \overline{z}^\beta \overline{z}_q^\nu, \tag{4.8}$$

defines an element of $\mathbb{H}_{m,n}(\mathbb{C}^q)$, for $j = 1, 2, \ldots, d(q,m,n)$. The homogeneity of f_j of degree m with respect to $\widehat{z}$ and of degree n with respect to $\overline{\widehat{z}}$ is obvious. Applying the Laplacian in (4.7) we deduce

$$0 = \Delta_{2q}(G^w_{m,n})(\widehat{z}) = \sum_{j=1}^{d(q,m,n)} \Delta_{2q}(f_j)(\widehat{z}) g_j(w), \ \widehat{z} \in \mathbb{C}^q, \ w \in \Omega_{2q-2}. \tag{4.9}$$

The linear independence of the g_j implies that $\Delta_{2q}(f_j) = 0$. □

The following lemma describes an integral operator that reproduces complex spherical harmonics. It is a complex version of the famous Funk-Hecke formula. A proof for this version can be found in [5] and [7]. In the statement of the lemma, $B[0,1]$ is the closed unit disk in $\mathbb{C}$, $d\nu_q(z)$ is the normalized Lebesgue measure given by

$$d\nu_q(z) := \frac{q-1}{\pi}\left(1 - x^2 - y^2\right)^{q-2} dx\, dy, \quad z = x + iy \in B[0,1], \tag{4.10}$$

$L^{p,q}(B[0,1])$ is the class of complex functions that are p-integrable in $B[0,1]$ with respect to ν_q and $P^{q-2}_{m,n}$ is the disk polynomial of degree $m+n$ associated with the integer $q-2$.

Lemma 4.3. *Let Y be an element of $\mathcal{H}_{m,n}(\Omega_{2q})$, and K an element of $L^{1,q}(B[0,1])$. Then for every w in Ω_{2q}, the mapping $z \in \Omega_{2q} \longmapsto K(\langle z, w\rangle)Y(z)$ is in $L^1(\Omega_{2q})$ and*

$$\int_{\Omega_{2q}} K(\langle z, w\rangle)Y(z)d\sigma_q(z) = \lambda^{q-2}_{n,m}(K)Y(w), \quad w \in \Omega_{2q}, \tag{4.11}$$

in which

$$\lambda^{q-2}_{n,m}(K) := \frac{2\pi^q}{(q-1)!}\int_{B[0,1]} K(z)\overline{P^{q-2}_{n,m}(z)}d\nu_q(z). \tag{4.12}$$

Theorem 4.4. *Let $\{g_j : j = 1, 2, \ldots, d(q, m, n)\}$ be a linearly independent subset of $\bigcup_{k=0}^m \bigcup_{l=0}^n \mathcal{H}_{k,l}(\Omega_{2q-2})$ and let $\{f_j : j = 1, 2, \ldots, d(q, m, n)\}$ be as in Theorem 4.2. If the set $\{g_j : j = 1, 2, \ldots, d(q, m, n)\}$ is orthonormal then $\{f_j : j = 1, 2, \ldots, d(q, m, n)\}$ is an orthogonal basis for $(\mathbb{H}_{m,n}(\mathbb{C}^q), [\cdot, \cdot])$.*

Proof. In the first step of the proof we show that $[G^w_{m,n}, G^\zeta_{m,n}] = K(\langle \zeta, w\rangle)$, for some function K. Indeed, recalling the hat notation introduced in the beginning of the section, we see that

$$[G^w_{m,n}, G^\zeta_{m,n}] = D_1^m \left[(\langle \zeta, z\rangle + \overline{z_q})^m\right] D_2^n \left[(-\langle z, \zeta\rangle + z_q)^n\right], \tag{4.13}$$

in which

$$D_1 := \overline{w_1}\frac{\partial}{\partial \overline{z_1}} + \overline{w_2}\frac{\partial}{\partial \overline{z_2}} + \cdots + \overline{w_{q-1}}\frac{\partial}{\partial \overline{z}_{q-1}} + \frac{\partial}{\partial \overline{z}_q} \tag{4.14}$$

and

$$D_2 := -w_1\frac{\partial}{\partial z_1} - w_2\frac{\partial}{\partial z_2} - \cdots - w_{q-1}\frac{\partial}{\partial z_{q-1}} + \frac{\partial}{\partial z_q}. \tag{4.15}$$

However,

$$D_1^m(\langle \zeta, z\rangle + \overline{z_q})^m = m!(\langle \zeta, w\rangle + 1)^m \tag{4.16}$$

and

$$D_2^n(-\langle z, \zeta\rangle + z_q)^n = n!(\langle w, \zeta\rangle + 1)^n \tag{4.17}$$

so that

$$[G^w_{m,n}, G^\zeta_{m,n}] = m!\,n!(\langle \zeta, w\rangle + 1)^m(\langle w, \zeta\rangle + 1)^n := K(\langle \zeta, w\rangle), \quad w, \zeta \in \Omega_{2q-2}. \quad (4.18)$$

Next, we use the previous theorem to deduce that

$$[G^w_{m,n}, G^\zeta_{m,n}] = \sum_{l=1}^{d(q,m,n)} p_l(w)\overline{g_l(\zeta)}, \quad w, \zeta \in \Omega_{2q-2}, \quad (4.19)$$

in which

$$p_l(w) = \sum_{j=1}^{d(q,m,n)} [f_j, f_l] g_j(w), \quad w \in \Omega_{2q-2}. \quad (4.20)$$

If the g_l form an orthonormal set we can apply the previous lemma to obtain

$$\lambda(j) g_j(w) = \int_{\Omega_{2q-2}} K(\langle \zeta, w\rangle) g_j(\zeta) d\sigma_{q-1}(\zeta) = p_j(w), \quad j = 1, 2, \ldots, d(q, m, n), \quad (4.21)$$

in which $\lambda(j)$ is a positive constant depending on g_j and K. Thus,

$$[G^w_{m,n}, G^\zeta_{m,n}] = \sum_{l=1}^{d(q,m,n)} \lambda(l) g_l(w)\overline{g_l(\zeta)}, \quad w, \zeta \in \Omega_{2q-2}. \quad (4.22)$$

A comparison with (4.19) yields the relation

$$\lambda(l) g_l(w) = \sum_{j=1}^{d(q,m,n)} [f_j, f_l] g_j(w), \quad w \in \Omega_{2q-2}, \quad l = 1, 2, \ldots, d(q, m, n). \quad (4.23)$$

It is now evident that $[f_j, f_l] = 0$, $j \neq l$ and that $[f_l, f_l] = \lambda(l)$, $l = 1, 2, \ldots, d(q, m, n)$. □

Example 4.5. Let $m = n = 1$ and $q = 3$. Due to Lemma 2.9, the polynomials

$$g_1(w) = \frac{1}{\sqrt{2}\,\pi}, \quad g_2(w) = \frac{1}{\pi} w_1, \quad g_3(w) = \frac{1}{\pi} w_2, \quad g_4(w) = \frac{1}{\pi}\overline{w_1}, \quad g_5(w) = \frac{1}{\pi}\overline{w_2}, \quad (4.24)$$

$$g_6(w) = \frac{\sqrt{3}}{\pi} w_1\overline{w_2}, \quad g_7(w) = \frac{\sqrt{3}}{\pi}\overline{w_1} w_2 \quad \text{and} \quad g_8(w) = \frac{\sqrt{6}}{2\pi}(w_1\overline{w_1} - w_2\overline{w_2}) \quad (4.25)$$

define an orthonormal subset of $\mathcal{H}_{0,0}(\Omega_4)\cup\mathcal{H}_{0,1}(\Omega_4)\cup\mathcal{H}_{1,0}(\Omega_4)\cup\mathcal{H}_{1,1}(\Omega_4)$. The kernel $G^w_{1,1}(\widehat{z})$ takes the form

$$z_3\overline{z_3}-\overline{z_1}z_3w_1-\overline{z_2}z_3w_2+z_1\overline{z_3}\,\overline{w_1}+z_2\overline{z_3}\,\overline{w_2}-\overline{z_1}z_2w_1\overline{w_2}-z_1\overline{z_2}w_2\overline{w_1}\\ -z_1\overline{z_1}w_1\overline{w_1}-z_2\overline{z_2}w_2\overline{w_2}.$$

Computing the coefficients $a_j(\alpha,\beta)$ in (4.6), here written as $a_j(\alpha;\beta)$, we obtain

$$a_1(0,0;0,0)=\sqrt{2}\pi,\ a_1(1,0;1,0)=-\frac{\sqrt{2}\pi}{2},\ a_8(1,0;1,0)=-\frac{\pi}{\sqrt{6}} \tag{4.26}$$

$$a_1(0,1;0,1)=-\frac{\sqrt{2}\pi}{2},\ a_8(0,1;0,1)=\frac{\pi}{\sqrt{6}},\ a_2(0,0;1,0)=-\pi, \tag{4.27}$$

$$a_4(1,0;0,0)=\pi,\quad a_5(0,1;0,0)=\pi,\quad a_6(0,1;1,0)=-\frac{\pi}{\sqrt{3}} \tag{4.28}$$

$$a_7(1,0;0,1)=-\frac{\pi}{\sqrt{3}},\quad a_3(0,0;0,1)=-\pi, \tag{4.29}$$

while all the others equal zero. Looking at (4.8), we encounter

$$f_1(\widehat{z})=\frac{\sqrt{2}\,\pi}{2}\left(-z_1\overline{z_1}-z_2\overline{z_2}+2z_3\overline{z_3}\right),\quad f_2(\widehat{z})=-\pi z_3\overline{z_1},\\ f_3(\widehat{z})=-\pi z_3\overline{z_2}, \tag{4.30}$$

$$f_4(\widehat{z})=\pi z_1\overline{z_3},\quad f_5(\widehat{z})=\pi z_2\overline{z_3},\quad f_6(\widehat{z})=-\frac{\pi}{\sqrt{3}}z_2\overline{z_1}, \tag{4.31}$$

and

$$f_7(\widehat{z})=-\frac{\pi}{\sqrt{3}}z_1\overline{z_2},\quad f_8(\widehat{z})=\frac{\pi}{\sqrt{6}}\left(-z_1\overline{z_1}+z_2\overline{z_2}\right). \tag{4.32}$$

Theorem 4.4 implies that $\{f_j : j=1,2,\ldots,8\}$ is an orthogonal basis for $(\mathbb{H}_{1,1}(\mathbb{C}^3),[\cdot,\cdot])$. The isomorphism (2.11) provides us with an orthogonal basis for $\mathcal{H}_{1,1}(\Omega_6)$.

Corollary 4.6. *Assume the hypotheses in Theorem* 4.4. *If* $\{g_j : j=1,2,\ldots,d(q,m,n)\}$ *is orthonormal then*

$$f_j(\widehat{z})=p_j(z_q)\overline{g_j(z)},\quad z\in\Omega_{2q-2},\quad j=1,2,\ldots,d(q,m,n), \tag{4.33}$$

in which $\{p_j : j=1,2,\ldots,d(q,m,n)\}$ *is a subset of* $\mathbb{P}(\mathbb{C})$.

Proof. If $\{g_j : j = 1, 2, \ldots, d(q,m,n)\}$ is orthonormal, we can use (4.4) to deduce

$$f_j(\widehat{z}) = \int_{\Omega_{2q-2}} G^w_{m,n}(\widehat{z})\, \overline{g_j(w)}\, d\sigma_{q-1}(w), \quad z \in \Omega_{2q-2}. \tag{4.34}$$

Expanding $G^w_{m,n}$ in the form

$$G^w_{m,n}(\widehat{z}) = \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} \frac{(-1)^\nu m!\, n!}{\mu!\nu!(m-\mu)!(n-\nu)!} z_q^{m-\mu}\overline{z_q}^{n-\nu} K_{\mu,\nu}(\langle w, z\rangle), \tag{4.35}$$

where $K_{\mu,\nu}(\langle z, w\rangle) = \langle z, w\rangle^\mu \langle w, z\rangle^\nu$, using Lemma 4.3 and arranging we obtain

$$f_j(\widehat{z}) = \left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} \frac{(-1)^\nu m!\, n!}{\mu!\nu!(m-\mu)!(n-\nu)!} b_j(\mu,\nu) z_q^{m-\mu}\overline{z_q}^{n-\nu}\right) \overline{g_j(z)}, \quad z \in \Omega_{2q-2}, \tag{4.36}$$

where the $b_j(\mu,\nu)$ are constants produced by the Funk-Hecke formula. Defining

$$p_j(z) = \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} \frac{(-1)^\nu m!\, n!}{\mu!\nu!(m-\mu)!(n-\nu)!} b_j(\mu,\nu) z^{m-\mu}\overline{z}^{n-\nu}, \quad z \in \mathbb{C} \tag{4.37}$$

concludes the proof. □

Corollary 4.7. *Assume the hypotheses in Theorem* 4.4. *If* $\{g_j : j = 1, 2, \ldots, d(q,m,n)\}$ *is orthonormal then*

$$(\langle \zeta, w\rangle + 1)^m (\langle w, \zeta\rangle + 1)^n = D \sum_{j=1}^{d(q,m,n)} \langle f_j, f_j\rangle_2 g_j(w)\overline{g_j(\zeta)}, \quad w, \zeta \in \Omega_{2q-2}, \tag{4.38}$$

in which $D = (m+n+q-1)!(2\pi^q\, m!n!)^{-1}$.

Proof. First we manipulate the sum in the right-hand side of (4.38) to obtain

$$\begin{aligned} &\sum_{\mu=1}^{d(q,m,n)} \langle f_\mu, f_\mu\rangle_2 g_\mu(w)\overline{g_\mu(\zeta)} \\ &= \sum_{\mu=1}^{d(q,m,n)} \sum_{\nu=1}^{d(q,m,n)} \left(\int_{\Omega_{2q}} f_\mu(\widehat{z})\overline{f_\nu(\widehat{z})} d\sigma_q(\widehat{z})\right) g_\mu(w)\overline{g_\nu(\zeta)} \end{aligned}$$

$$= \int_{\Omega_{2q}} \sum_{\mu=1}^{d(q,m,n)} f_\mu(\widehat{z}) g_\mu(w) \overline{\sum_{\nu=1}^{d(q,m,n)} f_\nu(\widehat{z}) g_\nu(\zeta)}\, d\sigma_q(\widehat{z})$$
$$= \int_{\Omega_{2q}} G^w_{m,n}(\widehat{z}) \overline{G^\zeta_{m,n}(\widehat{z})}\, d\sigma_q(\widehat{z}), \quad w, \zeta \in \Omega_{2q-2}.$$

Recalling Lemma 4.1, Theorem 2.7 and Theorem 2.10, we conclude that

$$\sum_{\mu=1}^{d(q,m,n)} \langle f_\mu, f_\mu \rangle_2 g_\mu(w) \overline{g}_\mu(\zeta) = \frac{2\pi^q}{(m+n+q-1)!} [G^w_{m,n}, G^\zeta_{m,n}], \qquad w, \zeta \in \Omega_{2q-2}. \quad (4.39)$$

Finally, (4.18) reduces (4.39) to

$$\sum_{\mu=1}^{d(q,m,n)} \langle f_\mu, f_\mu \rangle_2 g_\mu(w) \overline{g_\mu(\zeta)} = D^{-1}(\langle \zeta, w\rangle + 1)^m (\langle w, \zeta \rangle + 1)^n, \qquad w, \zeta \in \Omega_{2q-2}, \quad (4.40)$$

with D as described in the statement of the corollary. □

By letting $w = \zeta$ in Corollary 4.7 we deduce the following identity

$$\sum_{\mu=1}^{d(q,m,n)} \langle f_\mu, f_\mu \rangle_2 |g_\mu(w)|^2 = \frac{2^{m+n+1}\pi^q\, m!n!}{(m+n+q-1)!}, \quad w \in \Omega_{2q-2}. \quad (4.41)$$

We close this section presenting two independent results, one giving an estimate for the sum $\sum_{\mu=1}^{d(q,m,n)} \langle f_\mu, f_\mu \rangle_2$ and the other explaining why the construction in Theorem 4.2 preserves bi-orthogonality.

Corollary 4.8. *Assume the hypotheses in Theorem* 4.4. *If* $\{g_j : j = 1, 2, \ldots, d(q,m,n)\}$ *is orthonormal then*

$$\sum_{j=1}^{d(q,m,n)} \langle f_j, f_j \rangle_2 \le \frac{2^{m+n+2}\pi^{2q-1}}{(q-1)!(q-2)!}. \quad (4.42)$$

Proof. First apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} G^w_{m,n}(\widehat{z}) \overline{G^w_{m,n}(\widehat{z})} &\le \langle \widehat{z}, \widehat{z} \rangle^m \langle (w,1), (w,1) \rangle^m \langle \widehat{z}, \widehat{z} \rangle^n \langle (-w,1), (-w,1) \rangle^n \\ &\le 2^{m+n} \langle \widehat{z}, \widehat{z} \rangle^{m+n}, \quad \widehat{z} \in \mathbb{C}^q, \quad w \in \Omega_{2q-2}. \end{aligned}$$

Integration yields

$$\int_{\Omega_{2q}} G^w_{m,n}(\widehat{z}) \overline{G^w_{m,n}(\widehat{z})} d\sigma_q(\widehat{z}) \le 2^{m+n} \int_{\Omega_{2q}} d\sigma_q(\widehat{z}) = \frac{2^{m+n+1}\pi^q}{(q-1)!},$$

$$w \in \Omega_{2q-2}. \quad (4.43)$$

On the other hand, if $\{g_j : j = 1, 2, \dots, d(q, m, n)\}$ is orthonormal, the arguments at beginning of the proof of Corollary 4.7 imply that

$$\sum_{j=1}^{d(q,m,n)} |g_j(w)|^2 \langle f_j, f_j \rangle_2 \leq \frac{2^{m+n+1}\pi^q}{(q-1)!}, \quad w \in \Omega_{2q-2}. \tag{4.44}$$

Finally,

$$\begin{aligned}\sum_{j=1}^{d(q,m,n)} \langle f_j, f_j \rangle_2 &= \sum_{j=1}^{d(q,m,n)} \langle f_j, f_j \rangle_2 \int_{\Omega_{2q-2}} |g_j(w)|^2 d\sigma_{q-1}(w) \\ &\leq \frac{2^{m+n+1}\pi^q}{(q-1)!} \int_{\Omega_{2q-2}} d\sigma_{q-1}(w) = \frac{2^{m+n+2}\pi^{2q-1}}{(q-1)!(q-2)!},\end{aligned}$$

completing the proof. □

Corollary 4.9. *Let* $\{g_j : j = 1, 2, \dots, d(q, m, n)\}$ *and* $\{g_j' : j = 1, 2, \dots, d(q, m, n)\}$ *be orthonormal subsets of* $\bigcup_{k=0}^{m} \bigcup_{l=0}^{n} \mathcal{H}_{k,l}(\Omega_{2q-2})$ *and let* $\{f_j : j = 1, 2, \dots, d(q, m, n)\}$ *and* $\{f_j' : j = 1, 2, \dots, d(q, m, n)\}$ *be the corresponding sets resulting from the use of Theorem 4.2. If* $\langle g_j, g_k' \rangle_2 = 0$, $j \neq k$, *then* $[f_j, f_k'] = 0$, $j \neq k$.

Proof. We use Corollary 4.6 to write

$$f_j(\widehat{z}) = p_j(z_q)\overline{g_j(z)}, \quad z \in \Omega_{2q-2}, \quad j = 1, 2, \dots, d(q, m, n), \tag{4.45}$$

and

$$f_j'(\widehat{z}) = p_j'(z_q)\overline{g_j'(z)}, \quad z \in \Omega_{2q-2}, \quad j = 1, 2, \dots, d(q, m, n), \tag{4.46}$$

in which $\{p_j : j = 1, 2, \dots, d(q, m, n)\}$ and $\{p_j' : j = 1, 2, \dots, d(q, m, n)\}$ are subsets of $\mathbb{P}(\mathbb{C})$. It follows, with a help of Theorem 2.7, that

$$\begin{aligned}[f_j, f_k']_q =& p_j\left(\frac{\partial}{\partial \overline{z_q}}\right)\left(\overline{p_k'(z_q)}\right)[g_j, g_k']_{q-1} \\ =& \frac{m+n+q-1)!}{2\pi^q}\, p_j\left(\frac{\partial}{\partial \overline{z_q}}\right)\left(\overline{p_k'(z_q)}\right)\langle g_j, g_k' \rangle_2.\end{aligned}$$

The conclusion in the statement of the Corollary follows. □

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