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HADAMARD PRODUCT OF CERTAIN CLASSES OF FUNCTIONS

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Abstract. In this paper we consider the Hadamard product $\star$ of regular functions using the concept of subordination. Let $P(A,B)$ denote the class of regular functions subordinated to the linear fractional transformation $(1+Az)/(1-Bz)$, where $A+B \neq 0$ and $|B| \leq 1$. By $P(A,B) \star P(C,D)$ we denote the set $\{f \star g : \ f \in P(A,B), \ g \in P(C,D)\}$. It is known ([3], [7]), that for some complex numbers A, B, C, D there exist X and Y such that $P(A,B) \star P(C,D) \subset P(X,Y)$. The purpose of this note is to find the necessary and sufficient conditions for the equality of the classes $P(A,B) \star P(C,D)$ and $P(X,Y)$.

1. Introduction

By $\mathcal{H}$ we denote the family of all functions which are regular in the unit disc $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$ and let $\mathcal{N}$ denote the set of functions from $\mathcal{H}$ normalized by the condition $f(0) = 1$. Let Ω be the family of all functions ω of the class $\mathcal{H}$ such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \Delta$. We say, that a function $f \in \mathcal{H}$ is subordinated to a function $g \in \mathcal{H}$ in Δ (and write $f \prec g$

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or $f(z) \prec g(z)$) if there exists a function ω in Ω such that $f(z) = g(\omega(z))$ for $z \in \Delta$. For two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

of the class $\mathcal{H}$ we define the Hadamard product $f \star g$ as follows

$$(f \star g)(z) = f(z) \star g(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \ z \in \Delta.$$

Let Q_1 and Q_2 be given subclasses of the class $\mathcal{H}$. The Hadamard product $Q_1 \star Q_2$ of the classes Q_1 and Q_2 we define as follows

$$Q_1 \star Q_2 = \{f \star g \colon f \in Q_1, \ g \in Q_2\}.$$

For the given complex A, B such that $A + B \neq 0$ and $|B| \leq 1$ we define

$$P(A,B) = \left\{ f \in \mathcal{N} \colon f(z) \prec \frac{1+Az}{1-Bz}, \ z \in \Delta \right\}.$$

Note that for $A = B = 1$ the class $P(1,1) = P$ is the class of functions in $\mathcal{N}$ with positive real part in Δ (the Carathéodory functions), which plays an important role in the representation of certain univalent functions (see [1, Vol. 1, p. 88]). So, the classes $P(A,B)$ are the natural generalizations of the class P.

J. Stankiewicz and Z. Stankiewicz [7] proved the following

Theorem A. *If $A + B \neq 0$, $C + D \neq 0$, $|B| \leq 1$ and $|D| \leq 1$, then*

$$P(A,B) \star P(C,D) \subset P(AD + AC + BC, BD).$$

Moreover, if $|B| = 1$ or $|D| = 1$, then

$$P(A,B) \star P(C,D) = P(AD + AC + BC, BD).$$

K. Piejko, J. Sokół and J. Stankiewicz [5] showed that for $|B| < 1$ and $|D| < 1$ we have $P(A,B) \star P(C,D) \neq P(AD + AC + BC, BD)$. R. R. London [3] obtained the result for some generalization of Hadamard product.

Theorem B. *Let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ and let $f \in P(A,B)$, $g \in P(C,D)$, where $A + B \neq 0$, $C + D \neq 0$, $|B| \leq 1$ and $|D| \leq 1$. If moreover $X + Y \neq 0$, $|Y| \leq 1$ and $w \in \mathbb{C}$ then*

$$1 + \frac{w(X+Y)}{(A+B)(C+D)} \sum_{n=1}^{\infty} a_n b_n z^n \in P(X,Y)$$

if and only if $|w| + |BD - wY| \leq 1$.

A new proof and a generalization of this very interesting result one can find in [4]. Note that putting

$$w = \frac{(A+B)(C+D)}{X+Y}$$

in the Theorem B we obtain

Corollary C. *If* $A+B \neq 0$, $C+D \neq 0$, $X+Y \neq 0$, $|B| \leq 1$, $|D| \leq 1$ *and* $|Y| \leq 1$, *then*

$$P(A,B) \star P(C,D) \subset P(X,Y) \tag{1}$$

if and only if

$$\left|\frac{(A+B)(C+D)}{X+Y}\right| + \left|BD - \frac{Y(A+B)(C+D)}{X+Y}\right| \leq 1. \tag{2}$$

It is easy to observe, that for the equality in (1) it does not suffice the equality in (2). For instance, if we put $B = D = X = Y = 1$ and $A = C = 0$, then we obtain:

$$\left|\frac{(A+B)(C+D)}{X+Y}\right| + \left|BD - \frac{Y(A+B)(C+D)}{X+Y}\right| = \frac{1}{2} + \frac{1}{2} = 1$$

and $P(0,1) \star P(0,1) \subset P(1,1)$, but $P(0,1) \star P(0,1) \neq P(1,1)$.

In this note we want to find necessary and sufficient conditions for equality of the classes $P(A,B) \star P(C,D)$ and $P(X,Y)$.

2. Main result

We are now in position to show the following

Theorem 1. *Let* $A+B \neq 0$, $C+D \neq 0$, $X+Y \neq 0$, $|B| \leq 1$, $|D| \leq 1$ *and* $|Y| \leq 1$. *Then* $P(A,B) \star P(C,D) = P(X,Y)$ *if and only if* $|B| = 1$ *or* $|D| = 1$, $|BD| = |Y|$ *and* $(AC + AD + BC)Y = BDX$.

Proof. It is well known result due to E. Landau [2] (see also [1, Vol. 2, p. 37] or [5]), that there exists a sequence of functions $\omega_\nu \in \Omega$ such that for all positive integers ν

$$\omega_\nu(z) = \sum_{n=1}^{\infty} \gamma_{\nu,n} z^n, \tag{3}$$

and that the coefficients of power series have the following properties:

$$\gamma_{\nu,n} > 0 \quad \text{for } n \in \{1, 2, \ldots, \nu+1\} \tag{4}$$

and

$$\lim_{\nu\to\infty} s_\nu = +\infty, \tag{5}$$

where $s_\nu = \gamma_{\nu,1} + \gamma_{\nu,2} + \cdots + \gamma_{\nu,\nu} + \gamma_{\nu,\nu+1}$.

Let us assume that

$$P(A,B) \star P(C,D) = P(X,Y) \tag{6}$$

for some complex numbers A, B, C, D, X, Y such that $A+B \neq 0$, $C+D \neq 0$, $X+Y \neq 0$, $|B| \leq 1$, $|D| \leq 1$ and $|Y| \leq 1$.

First we prove that $|B| = 1$ or $|D| = 1$. Suppose, contrary to our claim that $|B| < 1$ and $|D| < 1$. Since

$$P(A,B) = P(C,D) \quad \text{if and only if } C = Ae^{i\varphi} \text{ and } D = Be^{i\varphi}, \tag{7}$$

where φ is a real number, then there is no loss of generality in the assuming that $0 \leq B < 1$, $0 \leq D < 1$ and $0 \leq Y \leq 1$.

For a fixed positive integer ν, let h_ν be given by

$$h_\nu(z) = \frac{1 + X\omega_\nu(z)}{1 - Y\omega_\nu(z)}, \tag{8}$$

where ω_ν is a Schwarz function given by (3). It is clear that $h_\nu \in P(X,Y)$ for all integers ν. From the assumption (6) we can see that there exist $f \in P(A,B)$ and $g \in P(C,D)$ such that

$$f(z) \star g(z) = h_\nu(z). \tag{9}$$

Let the functions f and g have the following forms:

$$f(z) = 1 + (A+B)\tilde{f}(z) \quad \text{and} \quad g(z) = 1 + (C+D)\tilde{g}(z),$$

where

$$\tilde{f}(z) = \frac{\tilde{\omega}_1(z)}{1 - B\tilde{\omega}_1(z)}, \quad \tilde{g}(z) = \frac{\tilde{\omega}_2(z)}{1 - D\tilde{\omega}_2(z)} \tag{10}$$

and $\tilde{\omega}_1, \tilde{\omega}_2 \in \Omega$.

Using these notation we can rewrite (9) as

$$\tilde{f}(z) \star \tilde{g}(z) = \frac{X+Y}{(A+B)(C+D)}\tilde{h}_\nu(z), \tag{11}$$

where

$$\tilde{h}_\nu(z) = \frac{\omega_\nu(z)}{1 - Y\omega_\nu(z)}.$$

Let the functions $\tilde{f}, \tilde{g}$ and $\tilde{h}_\nu$ have the following expansions in Δ:

$$\tilde{f}(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \tilde{g}(z) = \sum_{n=1}^{\infty} b_n z^n, \quad \tilde{h}_\nu(z) = \sum_{n=1}^{\infty} c_{\nu,n} z^n. \tag{12}$$

From (10) it follows, that

$$\tilde{f}(z) \prec \frac{z}{1-Bz} \quad \text{and} \quad \tilde{g}(z) \prec \frac{z}{1-Dz}.$$

It is well known result due to W. Rogosinski [6], that if the function $p_1(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ is subordinated to the function $p_2(z) = \sum_{n=0}^{\infty} \beta_n z^n$ in the unit disc, then $\sum_{n=0}^{\infty} |\alpha_n|^2 \leq \sum_{n=0}^{\infty} |\beta_n|^2$. Hence, since $0 \leq B < 1$ and $0 \leq D < 1$, we obtain

$$\sum_{n=1}^{\infty} |a_n|^2 \leq \frac{1}{1-B^2} \quad \text{and} \quad \sum_{n=1}^{\infty} |b_n|^2 \leq \frac{1}{1-D^2}. \tag{13}$$

From (11) and (12) we obtain

$$a_n b_n = \frac{X+Y}{(A+B)(C+D)} c_{\nu,n},$$

for all positive integers n, therefore (13) yields

$$\begin{aligned}\sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 \leq & \frac{1}{1-|B|^2} + \frac{1}{1-|D|^2} \\ & - \left| \frac{2(X+Y)}{(A+B)(C+D)} \right| \sum_{n=1}^{\infty} |c_{\nu,n}|.\end{aligned} \tag{14}$$

Let us note, that for all positive integers ν

$$[1 - Y\omega_\nu(z)]\, \tilde{h}_\nu(z) = \omega_\nu(z), \quad z \in \Delta$$

and so in the view of (3) and (12) we have

$$c_{\nu,1} = \gamma_{\nu,1} \quad \text{and} \quad c_{\nu,n} = \gamma_{\nu,n} + Y \sum_{k=1}^{n-1} c_{\nu,n-k}\gamma_{\nu,k} \quad \text{for } n > 1.$$

Since $Y \geq 0$ and in view of (4) the above condition gives

$$c_{\nu,n} \geq \gamma_{\nu,n} \text{ for } n \in \{1, 2, \ldots, \nu+1\}.$$

Hence

$$\sum_{n=1}^{\infty} |c_{\nu,n}| \geq \sum_{n=1}^{\nu+1} |c_{\nu,n}| \geq \gamma_{\nu,1} + \gamma_{\nu,2} + \gamma_{\nu,3} + \ldots + \gamma_{\nu,\nu} + \gamma_{\nu,\nu+1} = s_\nu. \tag{15}$$

Combining (14) and (15) we obtain

$$0 \leq \sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 \leq \frac{1}{1-B^2} + \frac{1}{1-D^2} - \left| \frac{2(X+Y)}{(A+B)(C+D)} \right| s_\nu. \tag{16}$$

It follows from (5) that we are able to choose a suitable ν such that the right side of (16) is negative. In this way (16) follows the contradiction and so we proved that

$$|B| = 1 \quad \text{or} \quad |D| = 1. \tag{17}$$

In view of (6), (17) and Theorem A we have

$$P(X,Y) = P(AC + AD + BC, BD),$$

and so (7) yields

$$|BD| = |Y| \quad \text{and} \quad (AC + AD + BC)Y = BDX. \tag{18}$$

This ends the first part of the proof.

Note that if we assume (17) and (18) then by Theorem A and by (7) we immediately obtain $P(A,B) \star P(C,D) = P(X,Y)$. □

Remark. Since for $|B| < 1$ the class $P(A,B)$ is a class of bounded functions:

$$P(A,B) = \left\{ f \in \mathcal{N} \colon \left| f(z) - \frac{1 + A\overline{B}}{1 - |B|^2} \right| < \frac{|A+B|}{1-|B|^2},\ z \in \Delta \right\},$$

we can deduce from Theorem 1 that there exist some bounded functions which can not be represented as the Hadamard product of two bounded functions. Namely, there does not exist complex A, B, C, D, X, Y, such that $A + B \neq 0$, $C + D \neq 0$, $X + Y \neq 0$, $|B| < 1$, $|D| < 1$, $Y \leq 1$ and $P(X,Y) \subset P(A,B) \star P(C,D)$.

This fact seems to be very surprising.

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