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A FURTHER GENERALIZATION OF HARDY-HILBERT'S INTEGRAL INEQUALITY WITH PARAMETER AND APPLICATIONS

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Abstract. In this paper, by introducing some parameters and by employing a sharpening of Hölder's inequality, a new generalization of Hardy-Hilbert integral inequality involving the Beta function is established. At the same time, an extension of Widder's theorem is given.

1. Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f, g : (0, \infty) \to (0, \infty)$ are so that

$$0 < \int_0^\infty f^p(t)dt < \infty, \qquad 0 < \int_0^\infty g^q(t)dt < \infty.$$

Then we may state the following integral inequality

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$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(t)dt\right)^{1/p}\left(\int_0^\infty g^q(t)dt\right)^{1/q}, \tag{1.1}$$

in which the constant factor $\pi/\sin(\pi/p)$ is the best possible.

The inequality (1.1) is well known in the literature as Hardy-Hilbert's integral inequality.

Recently, some improvements and generalizations of Hardy-Hilbert's integral inequality have been given. For instance, we refer the reader to the papers [1]–[3], [5], [6], [8] and the bibliography therein.

The main purpose of this paper is to establish a new extended Hardy-Hilbert's type inequality, which includes improvements and generalisations of the corresponding results from [1]–[2].

2. Lemmas and their proofs

For convenience, we firstly introduce some notations:

$$(f^r, g^s) = \int_\alpha^\infty f^r(x)g^s(x)dx, \qquad \|f\|_p = \left(\int_\alpha^\infty f^p(x)dx\right)^{1/p},$$

$$\|f\|_2 = \|f\|, \qquad S_r(H,x) = \left(H^{r/2}, x\right)\|H\|_r^{-r/2},$$

where x is a parametric variable unit vector. Clearly, $S_r(H,x) = 0$ when the vector x selected is orthogonal to $H^{p/2}$.

Throughout this paper, m is taken to be

$$m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}.$$

In order to state our results, we need to point out the following lemmas.

Lemma 1. *Let* $f(x), g(x) > 0, x \in (0,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p > 1$. *If* $0 < ||f||_p < \infty$, $0 < ||g||_q < \infty$, *then*

$$(f,g) < ||f||_p ||g||_q (1-R)^m, \tag{2.1}$$

where $R = (S_p(f,h) - S_q(g,h))^2$, $||h|| = 1$, $f^{p/2}(x), g^{q/2}(x)$ *and* $h(x)$ *are linearly independent.*

The lemma is proved in [5], and we omit the details.

In the following, we define

$$k_\lambda = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right),$$

$$\theta_\lambda(r) = \int_0^1 \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du \qquad (r = p, q),$$

where

$$B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt \qquad (u, v > 0)$$

is the *Beta function.*

The following lemma also holds.

Lemma 2. *Let $b < 1$, $\lambda > 0$. Define the function*

$$\varphi(b,y) = y^{-1+b}\int_0^y \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^b du, \qquad y \in (0,1].$$

Then we have

$$\varphi(b,y) > \varphi(b,1), \qquad (0 < y < 1). \tag{2.2}$$

A proof of Lemma 2 is given in paper [8], and we omit it here.

Another technical result that will be required in the following is:

Lemma 3. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, $\alpha \geq -\beta$. Define the weight function ω_λ by*

$$\omega_\lambda(\alpha,\beta,r,x) = \int_\alpha^\infty \frac{1}{(x+y+2\beta)^\lambda}\left(\frac{x+\beta}{y+\beta}\right)^{(2-\lambda)/r} dy \qquad x \in (\alpha, \infty). \tag{2.3}$$

(i) *For $\alpha = -\beta$,*

$$\omega_\lambda(-\beta,\beta,r,x) = k_\lambda (x+\beta)^{1-\lambda} \qquad x \in (-\beta, \infty). \tag{2.4}$$

(ii) *For $\alpha > -\beta$,*

$$\omega_\lambda(\alpha,\beta,r,x) < \left[k_\lambda - \theta_\lambda(r)\left(\frac{\alpha+\beta}{x+\beta}\right)^{1+(\lambda-2)/r}\right](x+\beta)^{1-\lambda} \qquad x \in (\alpha, \infty). \tag{2.5}$$

Proof. Setting $u = (y+\beta)/(x+\beta)$, we have

$$\omega_\lambda(\alpha,\beta,r,x) = (x+\beta)^{1-\lambda}\int_{\frac{(\alpha+\beta)}{(x+\beta)}}^\infty \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du.$$

(i) For $\alpha = -\beta$, (2.4) is valid.

(ii) For $\alpha > -\beta$, we have

$$\begin{aligned} &\omega_\lambda(\alpha,\beta,r,x) \\ &= (x+\beta)^{1-\lambda}\left\{\int_0^\infty \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du \right. \qquad (2.6)\\ &\qquad \left. - \int_0^{\frac{\alpha+\beta}{x+\beta}} \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)/r} du\right\} \\ &= (x+\beta)^{1-\lambda}\left\{k_\lambda - \left(\frac{\alpha+\beta}{x+\beta}\right)^{1+(\lambda-2)/r}\varphi\left(\frac{2-\lambda}{r},\frac{\alpha+\beta}{x+\beta}\right)\right\}. \qquad (2.7)\end{aligned}$$

Putting $b = (2-\lambda)/r$, and since $\lambda > 2 - \min\{p,q\}, b < 1$ is valid, then by Lemma 2 we get

$$\varphi\left(\frac{2-\lambda}{r},\frac{\alpha+\beta}{x+\beta}\right) > \varphi\left(\frac{2-\lambda}{r},1\right) = \theta_\lambda(r) \qquad (x \in (0,\infty)). \qquad (2.8)$$

Substituting (2.8) into (2.7), we obtain (2.5). The proof is completed. □

Finally, the following result is needed as well.

Lemma 4. *Let a_n $(n = 0,1,2,3,\dots)$ be complex numbers. If*

$$A(z) := \sum_{n=0}^\infty a_n z^n$$

is analytic on unit disk $|z| \le 1$, and

$$A^*(z) := \sum_{n=0}^\infty \frac{a_n z^n}{n!}$$

is analytic on $|z| < \infty$, then

$$\int_0^1 |A(x)|^2\,dx = \int_0^1 \left|\int_0^\infty e^{-s/x} A^*(s)ds\right|^2 \frac{1}{x^2}dx, \qquad (2.9)$$

where $s \in (0,\infty)$, $\quad x \in (0,1]$.

Proof. Since $A^*(z)$ is analytic on the complex plane, the series

$$\sum_{n=0}^\infty \frac{e^{-t} a_n (xt)^n}{n!}$$

is uniformly convergent in $(0, \infty)$, and we obtain

$$\begin{aligned}\int_0^\infty e^{-t} A^*(tx)dt &= \int_0^\infty e^{-t} \sum_{n=0}^\infty \frac{a_n (xt)^n}{n!} dt \\ &= \sum_{n=0}^\infty \frac{a_n x^n}{n!} \int_0^\infty t^n e^{-t} dt \\ &= \sum_{n=0}^\infty a_n x^n = A(x).\end{aligned}$$

Setting $tx = s$, then

$$A(x) = \frac{1}{x} \int_0^\infty e^{-s/x} A^*(s) ds$$

whence

$$\int_0^1 |A(x)|^2 \, dx = \int_0^1 \left| \int_0^\infty e^{-s/x} A^*(s) ds \right|^2 \frac{1}{x^2} dx.$$

The lemma is thus proved. □

3. Main results

For the sake of convenience, we need the following notations:

$$\begin{aligned}F(x, y) =& \frac{f(x)}{(x + y + 2\beta)^{\lambda/p}} \left(\frac{x + \beta}{y + \beta} \right)^{(2-\lambda)/(pq)}, \\ G(x, y) =& \frac{g(y)}{(x + y + 2\beta)^{\lambda/q}} \left(\frac{y + \beta}{x + \beta} \right)^{(2-\lambda)/(pq)}, \\ \phi(r, x) =& \int_0^{\frac{\alpha+\beta}{x+\beta}} \frac{1}{(1 + u)^\lambda} \left(\frac{1}{u} \right)^{(2-\lambda)/r} du, \qquad (3.1) \\ S_p(F, h) =& \left\{ \int_\alpha^\infty \int_\alpha^\infty F^{p/2} h dx dy \right\} \\ & \times \left\{ \int_\alpha^\infty \left[k_\lambda - \phi(q, x) \right] (x + \beta)^{1-\lambda} f^p(x) dx \right\}^{-1/2},\end{aligned}$$

and

$$\begin{aligned}S_q(G, h) =& \left\{ \int_\alpha^\infty \int_\alpha^\infty G^{q/2} h dx dy \right\} \\ & \times \left\{ \int_\alpha^\infty \left[k_\lambda - \phi(p, x) \right] (x + \beta)^{1-\lambda} g^q(x) dx \right\}^{-1/2},\end{aligned}$$

where $h = h(x, y)$ is a unit vector satisfying the property

$$||h|| = \left\{ \int_\alpha^\infty \int_\alpha^\infty h^2(x, y) dx dy \right\}^{1/2} = 1$$

and $F^{p/2}$, $G^{q/2}$, h are linearly independent.

The first main result is incorporated in the following theorem.

Theorem 1. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, $\alpha \geq -\beta$, *and* $f, g > 0$. *Assume also that*

$$0 < \int_\alpha^\infty (t + \beta)^{1-\lambda} f^p(t) dt < \infty,$$

and

$$0 < \int_\alpha^\infty (t + \beta)^{1-\lambda} g^q(t) dt < \infty.$$

(i) *If* $\alpha > -\beta$, *then we have*

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x + y + 2\beta)^\lambda} dx dy \tag{3.2}$$
$$< \left\{ \int_\alpha^\infty \left(k_\lambda - \theta_\lambda(q) \left(\frac{\alpha + \beta}{t + \beta} \right)^{1 + (\lambda - 2)/q} \right) (t + \beta)^{1-\lambda} f^p(t) dt \right\}^{1/p}$$
$$\times \left\{ \int_\alpha^\infty \left(k_\lambda - \theta_\lambda(p) \left(\frac{\alpha + \beta}{t + \beta} \right)^{1 + (\lambda - 2)/p} \right) (t + \beta)^{1-\lambda} g^q(t) dt \right\}^{1/q} (1 - R_\lambda)^m.$$

(ii) *If* $\alpha = -\beta$, *then we have*

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x + y + 2\beta)^\lambda} dx dy \tag{3.3}$$
$$< k_\lambda \left(\int_{-\beta}^\infty (t + \beta)^{1-\lambda} f^p(t) dt \right)^{1/p} \left(\int_{-\beta}^\infty (t + \beta)^{1-\lambda} g^q(t) dt \right)^{1/q} (1 - R_\lambda)^m,$$

where

$$R_\lambda = \left(S_p(F, h) - S_q(G, h) \right)^2,$$

while the function h *is defined by*

$$h(x, y) = \left(\frac{2}{\pi} \right)^{1/2} \frac{e^{\alpha - x}}{(x + y - 2\alpha)^{1/2}} \left(\frac{x - \alpha}{y - \alpha} \right)^{1/4}. \tag{3.4}$$

Proof. By Lemma 1 and the equality (2.3), we have

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y+2\beta)^\lambda} dxdy \tag{3.5}$$
$$= \int_\alpha^\infty \int_\alpha^\infty FGdxdy$$
$$\leq \left\{\int_\alpha^\infty \int_\alpha^\infty F^p dxdy\right\}^{1/p} \left\{\int_\alpha^\infty \int_\alpha^\infty G^q dxdy\right\}^{1/q} (1-R_\lambda)^m$$
$$= \left(\int_\alpha^\infty \omega_\lambda(\alpha,\beta,q,t) f^p(t)dt\right)^{1/p} \left(\int_\alpha^\infty \omega_\lambda(\alpha,\beta,p,t) g^q(t)dt\right)^{1/q} (1-R_\lambda)^m.$$

Substituting (2.5) and (2.4) into the inequality (3.5) respectively, the inequalities (3.2) and (3.3) follow.

Next, let us discuss the expression R_λ.

We can choose the function h indicated by (3.4). Setting $s = x - \alpha$ and $t = y - \alpha$, we get

$$\|h\|^2 = \int_\alpha^\infty \int_\alpha^\infty h^2(x,y)dxdy = \frac{2}{\pi}\int_0^\infty e^{-2s} ds \int_0^\infty \frac{1}{s+t}\left(\frac{s}{t}\right)^{1/2} dt = 1.$$

Hence, $\|h\| = 1$.

By Lemma 1 and the given h, we have

$$R_\lambda = \left\{\left(\int_\alpha^\infty \int_\alpha^\infty F^{p/2} h\, dxdy\right)\left(\int_\alpha^\infty \int_\alpha^\infty F^p\, dxdy\right)^{-1/2} - \left(\int_\alpha^\infty \int_\alpha^\infty G^{q/2} h\, dxdy\right)\left(\int_\alpha^\infty \int_\alpha^\infty G^q\, dxdy\right)^{-1/2}\right\}^2. \tag{3.6}$$

Substituting (2.3), (2.6) and (3.1) into (3.6), we get

$$R_\lambda = (S_p(F,h) - S_q(G,h))^2.$$

It is obvious that $F^{p/2}$, $G^{q/2}$ and h are linearly independent, so it is impossible for equality to hold in (3.5).

The proof is thus completed. □

Owing to $p, q > 1$, when $\lambda = 1, 2$; the condition $\lambda > 2 - \min\{p, q\}$ is satisfied. We have

$$\theta_1(r) = \int_0^1 \frac{1}{1+u}\left(\frac{1}{u}\right)^{1/r} du > \int_0^1 \frac{1}{1+u} du = \ln 2,$$
$$k_1 = B\left(\frac{1}{p}, \frac{1}{q}\right) = \frac{\pi}{\sin(\pi/p)},$$

$$\theta_2(r) = \int_0^1 \frac{1}{(1+u)^2} du = \frac{1}{2},$$
$$k_2 = B\left(\frac{p+2-2}{p}, \frac{q+2-2}{q}\right) = B(1,1) = 1.$$

The following results are natural consequences of Theorem 1.

Corollary 1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq \beta$, $f, g > 0$,

$$0 < \int_\alpha^\infty f^p(t)dt < \infty,$$

and

$$0 < \int_\alpha^\infty g^q(t)dt < \infty,$$

then

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{x+y+2\beta} dxdy < \left\{\int_\alpha^\infty \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\alpha+\beta}{t+\beta}\right)^{1/p} \ln 2\right) f^p(t)dt\right\}^{1/p} \times \left\{\int_\alpha^\infty \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\alpha+\beta}{t+\beta}\right)^{1/q} \ln 2\right) g^q(t)dt\right\}^{1/q} (1-R_1)^m, \tag{3.7}$$

$$\int_{-\beta}^\infty \int_{-\beta}^\infty \frac{f(x)g(y)}{x+y+2\beta} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_{-\beta}^\infty f^p(t)dt\right)^{1/p} \left(\int_{-\beta}^\infty g^q(t)dt\right)^{1/q} (1-R_1)^m, \tag{3.8}$$

and

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)dt\right)^{1/p} \left(\int_0^\infty g^q(t)dt\right)^{1/q} (1-R_1)^m. \tag{3.9}$$

Remark 1. When $p = q = 2$, the inequality (3.9) reduces, after some simple computation, to an inequality obtained in [2].

Corollary 2. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq \beta$, $f, g > 0$,

$$0 < \int_{\alpha}^{\infty} (t+\beta)^{-1} f^p(t)dt < \infty$$

and

$$0 < \int_{\alpha}^{\infty} (t+\beta)^{-1} g^q(t)dt < \infty,$$

then

$$\begin{aligned}
&\int_{\alpha}^{\infty}\int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y+2\beta)^2} dxdy \\
&< \left\{ \int_{\alpha}^{\infty} \left(1 - \frac{\alpha+\beta}{2(t+\beta)}\right) \frac{1}{t+\beta} f^p(t)dt \right\}^{1/p} \\
&\times \left\{ \int_{\alpha}^{\infty} \left(1 - \frac{\alpha+\beta}{2(t+\beta)}\right) \frac{1}{t+\beta} g^q(t)dt \right\}^{1/q} (1-R_2)^m
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&\int_{-\beta}^{\infty}\int_{-\beta}^{\infty} \frac{f(x)g(y)}{(x+y+2\beta)^2} dxdy \\
&< \left(\int_{-\beta}^{\infty} \frac{1}{t+\beta} f^p(t)dt \right)^{1/p} \left(\int_{-\beta}^{\infty} \frac{1}{t+\beta} g^q(t)dt \right)^{1/q} (1-R_2)^m.
\end{aligned} \tag{3.11}$$

Remark 2. The inequalities (3.2), (3.3) and (3.7)–(3.9) are generalizations of (1.1).

Remark 3. We can also define $h(x, y)$ as

$$h(x,y) = \begin{cases} 1 & (x,y) \in [0,1] \times [0,1] \\ 0 & (x,y) \in (0,\infty) \times (0,\infty) \setminus [0,1] \times [0,1]. \end{cases}$$

In this case, the expression of R_λ will be much simpler. The details are omitted.

4. Applications

We start with the following result:

Theorem 2. *Suppose that a_n $(n = 0,1,2,3,\dots)$ are complex numbers. Also, define*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad A^*(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!},$$

and the function f as:

$$f(x) = e^{-x} A^*(x), \qquad x \in (0,\infty). \tag{4.1}$$

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^1 |A(x)|^2\,dx < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty |f(x)|^p\,dx\right)^{1/p} \left(\int_0^\infty |f(x)|^q\,dx\right)^{1/q} (1-\overline{R})^m, \tag{4.2}$$

where

$$\overline{R} := \left(S_p\left(\overline{F},h\right) - S_q\left(\overline{G},h\right)\right)^2 > 0,$$

with $\|h\| = 1$, and

$$\overline{F} := \frac{|f(s)|}{(s+t)^{1/p}}\left(\frac{s}{t}\right)^{1/(pq)}; \qquad \overline{G} := \frac{|f(t)|}{(s+t)^{1/q}}\left(\frac{t}{s}\right)^{1/(pq)},$$

$$\psi(t) := \int_0^\infty \frac{e^{-s}}{(s+t)}\left(\frac{t}{s}\right)^{(q-p)/(2pq)} ds,$$

$$S_p(\overline{F},h) := \sqrt{2}\left\{\int_0^\infty e^{-s}\,|f(s)|^{p/2}\,ds\right\}\left\{\int_0^\infty |f(s)|^p\,ds\right\}^{-1/2},$$

$$S_q(\overline{G},h) := \frac{\sqrt{2}\sin(\pi/p)}{\pi}\left\{\int_0^\infty \psi(t)\,|f(t)|^{q/2}\,dt\right\}\left\{\int_0^\infty |f(s)|^q\,ds\right\}^{-1/2}$$

and $(\overline{F})^{p/2}$, $(\overline{G})^{q/2}$, h are linearly independent.

Proof. Setting $y = 1/x$ on the right-hand side of the equality (2.9), we have

$$\int_0^1 |A(x)|^2\,dx = \int_1^\infty \left|\int_0^\infty e^{-sy} A^*(s)ds\right|^2 dy. \tag{4.3}$$

Next, put $u = y - 1$. According to the equalities (4.1) and (4.3), we get

$$\int_0^1 |A(x)|^2\,dx = \int_0^\infty du \left|\int_0^\infty e^{-su} f(s)ds\right|^2.$$

Using Hardy's technique, we may state that

$$\begin{aligned}\int_0^1 |A(x)|^2\,dx &= \int_0^\infty du \left|\int_0^\infty e^{-su} f(s)ds\right|^2 \\ &= \int_0^\infty du \int_0^\infty e^{-su} f(s)ds \overline{\int_0^\infty e^{-tu} f(t)dt} \\ &\le \int_0^\infty \int_0^\infty \left(\int_0^\infty e^{-(s+t)u} du\right) |f(s)|\,|f(t)|\,dsdt \\ &= \int_0^\infty \int_0^\infty \frac{|f(s)|\,|f(t)|}{s+t} dsdt \\ &\le \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty |f(x)|^p\right)^{1/p} \left(\int_0^\infty |f(x)|^q\right)^{1/q} (1-\overline{R})^m.\end{aligned}$$

Let us choose the function $h(s,t)$ to be defined by

$$h(s,t) := \left(\frac{2}{\pi}\right)^{1/2} \frac{e^{-s}}{(s+t)^{1/2}} \left(\frac{s}{t}\right)^{1/4},$$

then

$$\|h\| = \left\{\int_0^\infty \int_0^\infty h^2(s,t)dsdt\right\}^{1/2} = 1.$$

Notice that

$$k_1(p) = B\left(\frac{1}{q}, \frac{1}{p}\right) = \frac{\pi}{\sin(\pi/p)},$$

and, in a similar way to the one in Theorem 1, the expression of $\overline{R}$ is easily given. We omit the details. □

Remark 4. In particular, when $p = q = 2$, it follows from (4.2) that

$$\int_0^1 A^2(x)dx = \pi(1-r)^{1/2} \int_0^\infty f^2(x)dx. \tag{4.4}$$

If r in (4.4) is replaced by zero, then Widder's theorem (see [7]) can be recaptured.

Remark 5. After simple computation, the inequality (4.4) is equivalent to the inequality (3.4) in [2]. Consequently, inequality (4.2) is an extension of (3.4) in [2].

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