Journal of Applied Analysis
Vol. 12, No. 1 (2006), pp. 119–127

MINIMAX INEQUALITY AND EQUILIBRIA WITH A GENERALIZED COERCIVITY

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Received February 24, 2005 and, in revised form, June 7, 2005

Abstract. In the first part of this paper, we prove a minimax inequality for maps satisfying a generalized coercivity type condition. As a consequence, we prove a result on the solvability of complementarity problems. In the second part, a result on the existence of maximal element in non-compact domains is obtained and as application, we prove the existence of equilibrium for an abstract economy (or generalized game) with non-compact choice sets.

1. Introduction

This paper is a study of minimax inequality and equilibrium for maps satisfying a "coercivity" type condition. We firstly recall the notion of coercing family for set-valued maps (also called correspondences) defined by Ben-El-Mechaiekh, Chebbi and Florenzano in [2]. As an example, we give the very general coercivity condition obtained by Ding and Tan in [5].

2000 *Mathematics Subject Classification.* 49J35, 91A13.

Key words and phrases. Coercing family, minimax inequalities, complementarity problems, maximal elements, qualitative games, generalized games, abstract economies.

This research was supported by Ministère de l'Enseignement Supérieur, de la Recherche Scientifique et de la Technologie de la Tunisie (Service de la Recherche Scientifique et de la Technologie).

The Author would like to thank the anonymous referee for his useful comments and suggestions.

ISSN 1425-6908

In Section 2, we prove a minimax inequality for maps satisfying a generalized quasi-concavity condition and a coercivity type condition. Our result extends the minimax inequality obtained by Yen [11] to non-compact domains and generalizes also the minimax inequalities obtained in the non-compact case by Fan [6] and Ding and Tan [5]. As a consequence, we extend results on complementarity problems obtained by Karamardian [8] and Allen [1].

In Section 3, we prove the existence of maximal elements for preferences correspondences defined on non-compact subsets of a topological vector space and satisfying a coercivity type condition. As application, we prove an equilibrium existence result for generalized game (or abstract economy) with non-compact choice sets. The results of this section generalize corresponding results obtained in Borglin and Keiding [4], Toussaint [9], Tulcea [10] and Ding and Tan [5].

Throughout the paper, vector spaces are real and topological (vector) spaces are assumed to be Hausdorff. The convex hull of a subset A of a vector space is denoted by $\operatorname{co} A$, the closure of a subset A of a topological space is denoted by $\operatorname{cl} A$ and for any set X, $\langle X\rangle$ denotes the family of all non-empty finite subsets of X.

Let X be a subset of a topological vector space, Y a topological space and $F\colon X \to Y$ be a correspondence. In order to define the setting of this paper, we need the following definition given in [2]:

Definition 1. A family $\{(C_i, K_i)\}_{i\in I}$ of pair of sets is said to be *coercing* for F if and only if:

(i) For each $i \in I$, C_i is contained in a compact convex subset of X and K_i is a compact subset of Y.

(ii) For each $i, j \in I$, there exists $k \in I$ such that $C_i \cup C_j \subseteq C_k$.

(iii) For each $i \in I$, there exists $k \in I$ with $\displaystyle\bigcap_{x\in C_k} F(x) \subset K_i$.

For any correspondence $F\colon X \to Y$, let $F^*\colon Y \to X$ be the "dual" correspondence of F defined by $F^*(y) = X \setminus F^{-1}(y)$. Using the following equivalent formulation, we can easly see that (iii) is a coercivity type condition:

Remark 1. Let X be a subset of a topological vector space and $F\colon X \to Y$ be a correspondence. A family $\{(C_i, K_i)\}_{i\in I}$ of pair of sets is *coercing* for F if and only if it satisfies conditions (i), (ii) of Definition 1 and the following one:

$$\forall i \in I, \forall y \in X \setminus K_i, F^*(y) \cap C_k \neq \emptyset \quad \text{for some } k \in I. \qquad (\mathcal{C})$$

Definition 2. A family $\{(C_i, K_i)\}_{i\in I}$ of pair of sets is said to be $\mathcal{C}$-*coercing* for F if and only if it satisfies conditions (i), (ii) of Definition 1 and condition $(\mathcal{C})$ in Remark 1.

Note that in case where the family is reduced to one element, condition $(\mathcal{C})$ appeared first in this generality (with two sets K and C) in [3] and generalizes condition of Karamardian [8] and Allen [1]. Condition $(\mathcal{C})$ is also an extension of the coercivity condition given by Fan [6]. Fore more examples about correspondences admitting a coercing family (when I is a singleton), see [2]. By the following example, we can see that the notion of coercing family is very general:

Example 1. If $F\colon X \to X$ is a correspondence satisfying the following condition given in [5]: There exists X_0 contained in a compact convex subset of X and K a compact subset of X such that:

$$\forall y \in X \setminus K, F(y) \cap \mathrm{co}(X_0 \cup y) \neq \emptyset.$$

Then F admits a $\mathcal{C}$-coercing family.

Proof. Take the family:

$$\{(C_{A_y}), K)\}_{\{y\in\langle X\setminus K\rangle, A_y\in\langle X\rangle\}},$$

where for each $y \in \langle X\setminus K\rangle$ and for each $A_y \in \langle X\rangle$, $C_{A_y} = \mathrm{co}(X_0 \cup A_y)$. This family verifies conditions (i) and (ii) of Definition 1, by putting $A_y = \{y\}$ for every $y \in X \setminus K$, condition $\mathcal{C}$ is satisfied. □

2. Minimax inequalities

Let us recall that if X is a subset of a vector space Y, a correspondence $F\colon X \to Y$ is called *KKM* if for any $A \in \langle X\rangle$:

$$\mathrm{co}(A) \subset \bigcup_{x\in A} F(x).$$

A subset X of a topological space Y is compactly closed (open, respectively) if for every compact set C of Y, $X \cap C$ is closed (open, respectively) in C.

The following minimax inequality is an equivalent analytic formulation of Theorem 3.1 in [2]:

Theorem 1. *Let X be a non-empty convex subset of a topological vector space E and $f\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function such that:*

(*a*) *For each fixed $x \in X$, the function: $y \mapsto f(x, y)$ is lower semi-continuous on each non-empty compact subset of X.*

(*b*) *For each* $A \in \langle X\rangle$, $\sup_{y\in \operatorname{co} A}\min_{x\in A} f(x,y) \le 0$.

(*c*) *There exists a family* $\{(C_i, K_i)\}_{i\in I}$ *satisfying conditions* (*i*) *and* (*ii*) *of Definition* 1 *and the following one: For each* $i \in I$, *there exists* $k \in I$ *such that:*

$$\{y \in X,\ f(x,y) \le 0,\ \forall x \in C_k\} \subset K_i.$$

Then there exists $y_0 \in X$ *such that* $f(x, y_0) \le 0$ *for all* $x \in X$.

Proof. For each $x \in X$, let $F(x) = \{y \in X \colon f(x,y) \le 0\}$. We have to show that F satisfies all conditions of Theorem 3.1 in [2]. By (a), $F(x)$ is compactly closed in X for each $x \in X$. If F is not KKM, there exist $A \in \langle X\rangle$ and $y \in \operatorname{co} A$ such that $f(x,y) > 0$ for all $x \in A$, which contradicts (b). Condition (c) implies that F admits a coercing family, it follows that $\bigcap_{x\in X} F(x) \ne \emptyset$. Let $y_0 \in \bigcap_{x\in X} F(x)$, then $f(x,y_0) \le 0$, for all $x \in X$. □

Theorem 1 extends Theorem 6 of Fan [6]. Using Example 1, Theorem 1 is also a generalization of Theorem 1 in [5].

Corollary 1. *Let* X *be a non-empty convex subset of a topological vector space* E *and* $f, g \colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ *be such that:*

(*i*) *For each* $(x,y) \in X \times X$, $f(x,y) \le g(x,y)$.

(*ii*) *For each* $x \in X$, $g(x,x) \le 0$.

(*iii*) *For each fixed* $x \in X$, *the function:* $y \mapsto f(x,y)$ *is lower semi-continuous on each non-empty compact subset of* X.

(*iv*) *For each fixed* $y \in X$, *the set* $\{x \in X \colon g(x,y) > 0\}$ *is convex.*

(*v*) *There exists a family* $\{(C_i, K_i)\}_{i\in I}$ *satisfying conditions* (*i*) *and* (*ii*) *of Definition* 1 *and the following one: For each* $i \in I$, *there exists* $k \in I$ *such that:*

$$\{y \in X \colon f(x,y) \le 0,\ \forall x \in C_k\} \subset K_i.$$

Then there exists $y_0 \in X$ *such that* $f(x, y_0) \le 0$ *for all* $x \in X$.

Proof. It is sufficient to show that (i), (ii) and (iv) imply condition (b) of Theorem 1. If not, there exist $A \in \langle X\rangle$ and $y \in \operatorname{co} A$ such that $\min_{x\in A} f(x,y) > 0$. Then by (i), $\min_{x\in A} g(x,y) > 0$. It follows by (iv) that $g(y,y) > 0$, which contradicts (ii). □

The following minimax inequality, which includes a generalization of Theorem 1 of Yen [11], can be deduced from Corollary 1:

Corollary 2. *Let X be a non-empty convex subset of a topological vector space E and $f, g\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that:*

(*a*) *For each $(x, y) \in X \times X$, $f(x, y) \leq g(x, y)$.*

(*b*) *For each fixed $x \in X$, the function: $y \mapsto f(x, y)$ is lower semi-continuous on each non-empty compact subset of X.*

(*c*) *For each fixed $y \in X$, the function: $x \mapsto g(x, y)$ is quasi-concave.*

(*d*) *For any $\alpha \in \mathbb{R}$, there exists a family $\{(C_i, K_i)\}_{i \in I}$ satisfying conditions (i), (ii) of Definition 1 and the following one: For each $i \in I$, there exists $k \in I$ such that:*

$$\{y \in X\colon f(x, y) \leq \alpha, \forall x \in C_k\} \subset K_i.$$

Then the following minimax inequality holds:

$$\inf_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} g(x, x).$$

Proof. We can assume that $\lambda = \sup_{x \in X} g(x, x)$ is finite, otherwise there is nothing to prove. The functions $f - \lambda$ and $g - \lambda$ satisfy conditions of Corollary 1, then there exists $y_0 \in X$ such that $f(x, y_0) \leq \lambda$, $\forall x \in X$. Hence the minimax inequality follows. □

The following extension of Theorem 3.1 of Karamardian [8] on the solvability of complementarity problems follows immediately from Theorem 1:

Corollary 3. *Let X be a non-empty convex subset of a topological vector space E and $f\colon X \to E^*$, where E^* denotes the topological dual of E, be such that:*

(*i*) *For each fixed $x \in X$, the function: $y \mapsto \langle f(y), y - x\rangle$ is lower semi-continuous on each non-empty compact subset of X.*

(*ii*) *There exists a family $\{(C_i, K_i)\}_{i \in I}$ satisfying conditions (i), (ii) of Definition 1 and the following one: For each $i \in I$, there exists $k \in I$ such that:*

$$\{y \in X\colon \langle f(y), y - x\rangle \leq 0, \forall x \in C_k\} \subset K_i.$$

Then there exists $y^* \in X$ such that $\langle f(y^*), y^* - x\rangle \leq 0$ for all $x \in X$.

If moreover X is a cone and if X^0 denotes the polar cone of X, then $-f(y^*) \in X^0$ and $\langle f(y^*), y^*\rangle = 0$.

3. Equilibria in an abstract economy

Correspondences play a central role in the theory of economic equilibria. They usually represent preference relations (the value $P(x)$ of a correspondence P consists of all those commodities preferred to x). The issue there is to determine the existence of a so-called *maximal element* for a given preference P, i.e. an element $\bar{x}$ with $P(\bar{x}) = \emptyset$.

Definition 3. Using the terminology of Borglin and Keiding [4], given a map $P\colon X \to X$ of a non-empty subset X of a topological vector space, we say that:

(i) P is a KF *correspondence* if:
   (a) for all $y \in X$, $P^{-1}(y)$ is compactly open in X;
   (b) for all $x \in X$, $x \notin \operatorname{co} P(x)$.

(ii) A correspondence $\Psi_x\colon X \to X$ is a *KF-majorant of P* at $x \in X$ if Ψ_x is KF and $P(x') \subseteq \Psi_x(x')$, for all x' in some open neighborhood U_x of x in X.

(iii) P is *KF-majorized* if it admits a KF-majorant at each $x \in X$ with $P(x) \neq \emptyset$.

Remark 2. The concept of KF majoration is hereditary in the sense that it becomes global in the presence of paracompactness. More precisely, if a correspondence $P\colon X \to X$ is KF-majorized and if X is paracompact, then P is majorized by a KF correspondence Ψ, i.e., $P(x) \subseteq \Psi(x), \forall x \in X$ (see [4]).

Theorem 3.2 in [2] can be rephrased in terms of the existence of maximal elements as follows:

Proposition 1. *Let X be a non-empty convex and paracompact subset of a topological vector space. A correspondence $P\colon X \to X$ admits a maximal element provided that it is KF-majorized and has a $\mathcal{C}$-coercing family.*

Proof. Suppose that, for all $x \in X$, $P(x) \neq \emptyset$. Since P is KF-majorized and X is paracompact, it follows from Remark 2 that there exists a KF correspondence Ψ such that $P(x) \subseteq \Psi(x)$, $\forall x \in X$. By Theorem 3.2 in [2], the correspondence $\operatorname{co} \Psi$ admits a maximal element, which is also a maximal element for P. □

Theorem 1 in [5] follows from Example 1 and Proposition 1:

Corollary 4. *Let X be a non-empty convex and paracompact subset of a topological vector space and $P\colon X \to X$ a KF-majorized correspondence. If P satisfies the following coercivity condition: There exist X_0 contained in a compact convex subset of X and K a compact subset of X such that:*

$$\forall y \in X \setminus K, P(y) \cap \operatorname{co}(X_0 \cup y) \neq \emptyset.$$

Then P admits a maximal element.

Let now J be a (possibly infinite) set of agents. We consider the situation where each agent $j \in J$ has a non-empty choice set (or strategy set) X^j and a preference correspondence $P^j\colon X = \prod_{j\in J} X^j \to X^j$ such that $x^j \notin P^j(x)$, $x \in X$. Following Gale and Mas-Colell [7], we say that the collection $(X^j, P^j)_{j\in J}$ is a *qualitative game*.

Using Proposition 1, we obtain the following existence result for qualitative games:

Proposition 2. *Let $(X^j, P^j)_{j\in J}$ be a qualitative game such that the set $X = \prod_{j\in J} X^j$ is paracompact and satisfying the following conditions for each $j \in J$:*

(i) X^j is a non-empty convex subset of a topological vector space E^j.

(ii) P^j is KF-majorized.

(iii) $\{x \in X\colon P^j(x) \neq \emptyset\}$ is open in X.

(iv) P^j admits a $\mathcal{C}$-coercing family.

Then the game $(X^j, P^j)_{j\in J}$ has an equilibrium.

Proof. For each $x \in X$, let $J(x) = \{j \in J\colon P^j(x) \neq \emptyset\}$. Define $\Phi\colon X \to X$ by:

$$\Phi(x) = \begin{cases} \displaystyle\bigcap_{j\in J(x)} \operatorname{conv}(P'^j(x)) & \text{if } J(x) \neq \emptyset \\ \emptyset & \text{if } J(x) = \emptyset \end{cases}$$

where $P'^j\colon X \to X$ is defined by: $y \in P'^j(x) \iff y^j \in P^j(x)$. Using (ii), (iii), a standard argument (see [9]) shows that Φ is KF-majorized. Hypothesis (iv) implies that Φ admits a $\mathcal{C}$-coercing family. Hence, there exists an $\overline{x} \in X$ such that $\Phi(\overline{x}) = \emptyset$ i.e. $P^j(\overline{x}) = \emptyset$ for all $j \in J$. □

More generally, if each agent j is restricted in his choices to some non-empty subset of his strategy set due to the actions of the other players; this is formalized in terms of a *constraint correspondence* $B^j\colon X \to X^j$. The family $(X^j, B^j, P^j)_{j\in J}$ is called a *generalized qualitative game* or an *abstract economy*. We say that $\overline{x} \in X$ is an *equilibrium* of the game if for each $j \in J$:

$$\overline{x}^j \in \operatorname{cl}_{X^j} B^j(\overline{x}) \quad \text{and} \quad B^j(\overline{x}) \cap P^j(\overline{x}) = \emptyset.$$

Proposition 3. *Let* $(X^j, B^j, P^j)_{j\in J}$ *be a generalized qualitative game such that the set* $X = \prod_{j\in J} X^j$ *is paracompact and satisfying the following conditions for each* $j \in J$*:*

(*i*) X^j *is a non-empty convex subset of a topological vector space.*

(*ii*) *For each* $x \in X$, $B^j(x)$ *is non-empty and convex.*

(*iii*) *For each* $y^j \in X^j$, $(B^j)^{-1}(y^j)$ *is open in* X.

(*iv*) $\mathrm{cl}_{X^j}(B^j)\colon X \to X^j$ *is upper semi-continuous.*

(*v*) $B^j \cap P^j$ *is* KF*-majorized.*

(*vi*) $\{x \in X\colon (B^j \cap P^j)(x) \neq \emptyset\}$ *is open in* X.

(*vii*) $P^j \cap B^j$ *admits a* $\mathcal{C}$*-coercing family.*

Then the abstract economy $(X^j, B^j, P^j)_{j\in J}$ *has an equilibrium.*

Proof. For each $j \in J$, let $F^j = \{x \in X\colon x^j \notin \mathrm{cl}_{X^j} B^j(x)\}$. The set F^j is open in X by (iv). Define $Q^j\colon X \to X$ by:

$$Q^j(x) = \begin{cases} (B^j \cap P^j)(x) & \text{if } x \notin F^j \\ B^j(x) & \text{if } x \in F^j. \end{cases}$$

We can also show by a standard argument (see [9]) that the qualitative game $(X^j, Q^j)_{j\in J}$ satisfies the hypotheses (i)–(iii) of Proposition 2. By (vii) Q^j admits a $\mathcal{C}$-coercing family. We conclude that the qualitative game $(X^j, Q^j)_{j\in J}$ admits an equilibrium $\overline{x}$. Since $B^j(x)$ is non-empty for all $x \in X$, this implies that for each $j \in J$, $\overline{x}^j \in \mathrm{cl}_{X^j}(B^j(\overline{x}))$ and $B^j(\overline{x}) \cap P^j(\overline{x}) = \emptyset$. □

Proposition 3 generalizes Theorem 4 in [5]. If X_j is compact for each $j \in J$, then Proposition 2 reduces to Corollary 3 in [4], Theorem 2.5 in [9] and Proposition 3 in [10].

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