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A NOTE ON P-TIMES AND TIME PROJECTIONS

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Abstract. We look at analogues of the σ-algebras of events occurring up to a time and the events which are strictly prior to a time of the classical (commutative) theory. In the second case, we define the *p-times* and investigate the order structure of time projections associated with these times in an abstract set up.

0. Introduction

In this paper we discuss essentially two topics; the analogues of the σ-algebras of events occurring up to a time and the events which are strictly prior to a time and various properties analogous to that in the classical (commutative) theory (cf. [8]) are investigated, the definition of *time* (or *p-time*) is given and the structure of time projections associated with these times in an arbitrary non-commutative filtration of von Neumann algebras is studied. Our aim in this part is to propose a general form (scheme) for the consideration of the order structure of time projections. The structure of time projections associated with random times was studied in [2, 3, 10] within an arbitrary non-commutative filtration of von Neumann algebras

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as well as those employed in quantum stochastic theory of the canonical anticommutation relations (CAR) and the canonical commutation relations (CCR) (cf. [4], [9]).

Section 2 contains a brief review of random times and the associated time projection. In Section 3 we introduce subspaces of a von Neumann algebra $\mathcal{A}$ analogous to the σ-algebras of events in the classical theory and compare the common properties of these subspaces with those σ-algebras in the classical case. Section 4 is devoted to the notion of *time* (or *p-time*) and the order structure of time projections associated with *p-times*. We give in this section the conditions under which the time projections associated with simple times form a lattice. Accordingly, we divide the family $\mathcal{T}$ of times into equivalent classes $\{[p] : p \text{ is a projection in } \mathcal{A}\}$ and we show that if p is projection in $\mathcal{A}_0$, the time projections associated with times in $[p]$ have all the corresponding properties from [2, 3, 10].

1. Notation and preliminiaries

Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ — the bounded linear operators on $\mathcal{H}$, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ — a von Neumann algebra, and let $(\mathcal{A}_t)$, $t \in \mathbb{R}^+$, be an increasing, right continuous family of von Neumann subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_{+\infty}$ is generated by the collection $\{\mathcal{A}_t\colon t \in [0, +\infty)\}$. We also suppose that there is a cyclic and separating unit vector Ω for $\mathcal{A}$ in $\mathcal{H}$, and that there is a family $(\mathcal{E}_t)$ of normal ω-invariant conditional expectations $\mathcal{E}_t\colon \mathcal{A} \to \mathcal{A}_t$, where ω is the vector state induced by Ω. If we denote the closure of $\mathcal{A}_t\Omega$ in $\mathcal{H}$ by $\mathcal{H}_t$, and the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_t$ by $\mathcal{P}_t$, we have

$$\mathcal{P}_t\,(a\Omega) = \mathcal{E}_t\,(a)\,\Omega$$

for any $a \in \mathcal{A}$. Furthermore, since $\mathcal{H}_t$ is invariant under $\mathcal{A}_t$, it follows that $\mathcal{P}_t \in \mathcal{A}_t'$ (see [1], [2] for a more detailed description). By an $\mathcal{A}$-valued process we mean a map from $[0, +\infty]$ into $\mathcal{A}$. An $\mathcal{A}$-valued process $f = (f_t)$ is called adapted if $f_t \in \mathcal{A}_t$ for all t. We have also the notion of $\mathcal{H}$-valued process.

2. Random times and time projections

We recall the definition and elementary properties of a random time and its associated time projection. For more details the reader is referred to [1, 2, 3].

Definition 2.1. A *random time*, τ, is an increasing family of projections $\tau = (q_t)$, $t \in [0, +\infty]$, where $q_t \in \mathcal{A}_t$, $q_0 = 0$ and $q_{+\infty} = \mathbf{1}$. A random time $\tau = (q_t)$ is called *simple*, if it assumes only finitely many distinct values.

Let Θ denote the set of all finite partitions of $[0, +\infty]$. Then, for $\theta \in \Theta$, say $\theta = \{0 = t_0 < t_1 < \ldots < t_n = +\infty\}$, the simple random time associated with τ and θ is given by $\tau(\theta) = \left(q_t^\theta\right)$, where

$$q_t^\theta = \sum_{i=0}^{n-1} q_{t_i} \chi_{[t_i, t_{i+1})}(t)$$

for $t \in [0, \infty)$, and $q_{+\infty}^\theta = \mathbf{1}$.

Definition 2.2. (i) Let $\tau = (q_t)$ and $\sigma = (q'_t)$ be random times. We say that $\tau \leq \sigma$, if $q'_t \leq q_t$ for each $t \in \mathbb{R}^+$. We define $\tau \wedge \sigma$ and $\tau \vee \sigma$ to be the random times $\tau \wedge \sigma = (q_t \vee q'_t)$ and $\tau \vee \sigma = (q_t \wedge q'_t)$. In a similar fashion, for any family Λ of random times, we define $\sup \Lambda$ and $\inf \Lambda$ as the random times consisting respectively of infima and suprema of the corresponding projections.

(ii) Let $\theta = \{0 = t_0 < t_1 < \ldots < t_n = +\infty\} \in \Theta$. We define

$$\mathcal{M}_{\tau(\theta)} = \sum_{i=1}^{n} \left(q_{t_i} - q_{t_{i-1}}\right) \mathcal{P}_{t_i} \equiv \sum_\theta \Delta q_{t_i} \mathcal{P}_{t_i}.$$

$\mathcal{M}_{\tau(\theta)}$ has the following properties (see [1, Theorem 2.3]):

1. $\mathcal{M}_{\tau(\theta)}$ is an orthogonal projection;
2. For $\theta, \eta \in \Theta$ with η finer than $\theta, \mathcal{M}_{\tau(\eta)} \leq \mathcal{M}_{\tau(\theta)}$;
3. If σ is another random time with $\tau \leq \sigma$, then $\mathcal{M}_{\tau(\theta)} \leq \mathcal{M}_{\sigma(\theta)}$ for each $\theta \in \Theta$.

These properties and the fact that Θ is a directed set ordered by inclusion, imply that $\{\mathcal{M}_{\tau(\theta)} \colon \theta \in \Theta\}$ is a decreasing net of orthogonal projections. Hence there exists a unique orthogonal projection

$$\mathcal{M}_\tau = \bigwedge_{\theta \in \Theta} \mathcal{M}_{\tau(\theta)};$$

moreover,

$$\mathcal{M}_{\tau(\theta)} \searrow \mathcal{M}_\tau$$

in the strong operator topology as θ refines. We shall call $\mathcal{M}_\tau$ the *time projection* for the random time τ ([1, Definition 2.4]). The next result summarises what is known about the order structure of random times.

Let τ, σ be random times. For $\theta, \eta \in \Theta$ we have

$$\mathcal{M}_{\tau(\theta)} \vee \mathcal{M}_{\sigma(\eta)} = \mathcal{M}_{\tau(\theta) \vee \sigma(\eta)} \quad \text{and} \quad \mathcal{M}_{\tau(\theta)} \wedge \mathcal{M}_{\sigma(\eta)} = \mathcal{M}_{\tau(\theta) \wedge \sigma(\eta)}.$$

Also

$$\mathcal{M}_\tau \wedge \mathcal{M}_\sigma = \mathcal{M}_{\tau\wedge\sigma},$$

so that, in particular, if $\sigma \leq \tau$ then $M_\sigma \leq M_\tau$ (*Optional Stopping Theorem*). The complete proofs of these relations can be found in [2], [3]. One of our aims in this paper is to investigate them in the case of p-*times*.

3. Random times and subspaces

In this section we look at the analogues of the σ-algebras of events taking place up to a time and the events which are strictly before a time (see [8]), and prove some results analogous to those in the classical theory.

Definition 3.1. Let $\tau = (q_t)$ be a random time. By analogy with the commutative (classical) case, we define the subspace $\mathcal{A}_\tau \subseteq \mathcal{A}$ of all events taking place up to a time τ by

$$A_\tau = \{a \in \mathcal{A} : q_t a \in \mathcal{A}_t \text{ for all } t \in [0, +\infty]\}$$

and the subspace $\mathcal{A}_\tau^- \subseteq \mathcal{A}$ of all events taking place strictly before a time τ by

$$\mathcal{A}_\tau^- = \overline{\operatorname{span}}\{(1 - q_{t^+})\, a \colon a \in \mathcal{A}_t,\ t \in [0, +\infty)\},$$

where $q_{t^+} = \lim_{s>t} q_s$ (with $0^+ = 0$ and $\infty^+ = +\infty$) and the closure is taken in the strong operator topology.

Recall that an $\mathcal{A}$-valued adapted process is, by definition, a family (f_t) satisfying $f_t \in \mathcal{A}_t$ for each $t \in [0, +\infty]$. Let $\tau = (q_t)$ be a random time. We consider, for each partition, $\theta = \{0 = t_0 < t_1 < \ldots < t_n = +\infty\}$ of $[0, +\infty]$ the integral sum

$$\mathcal{S}_\theta^l (f; \tau) = \sum_{i=1}^{n} \left(q_{t_i} - q_{t_{i-1}}\right) f_{t_{i-1}} \equiv \sum_\theta \Delta q_{t_i} f_{t_{i-1}}.$$

Let us start with following lemma.

Lemma 3.2. *Let $\tau = (q_t)$ be a random time and (f_t) be an $\mathcal{A}$-valued adapted process. Then*

1. *$\mathcal{A}_\tau$ is closed in the strong operator topology;*
2. *$q_t \in \mathcal{A}_\tau$ for each $t \in [0, +\infty]$;*
3. *$\mathcal{S}_\theta^l (f; \tau) \in \mathcal{A}_\tau$ for each partition θ of $[0, +\infty]$;*
4. *$\overline{(\mathcal{A}_\tau \Omega)} \subseteq \mathcal{M}_\tau (\mathcal{H})$.*

Proof. The assertions 1 and 2 are obvious by the definition of $\mathcal{A}_\tau$. To prove 3 we note, for any $t \in [0,+\infty]$ and partition, θ, of $[0,+\infty]$, that

$$q_t \mathcal{S}_\theta^l (f;\tau) = \sum_\theta q_t \Delta q_{s_i} f_{s_{i-1}} \in \mathcal{A}_t$$

since $f_{s_i} \in \mathcal{A}_t$, for all $s_i \le t$, and $q_t \Delta q_{s_i} = q_t \wedge q_{s_i} - q_t \wedge q_{s_{i-1}} \in \mathcal{A}_t$ and equals zero for $s_{i-1} > t$. Hence $q_t \mathcal{S}_\theta^l (f;\tau) \in \mathcal{A}_t$ for every $t \in [0,+\infty]$. Thus $\mathcal{S}_\theta^l (f;\tau) \in \mathcal{A}_\tau$. To prove the assertion 4, we use Theorem 2.12 of [2], which states that $\zeta \in \mathcal{M}_\tau(\mathcal{H})$ if and only if $q_t\zeta \in \mathcal{H}_t$ for all $t \in [0,+\infty]$. Let $\zeta \in \overline{(\mathcal{A}_\tau \Omega)}$ then there exists $(a_n) \subset \mathcal{A}_\tau$ such that $a_n\Omega \to \zeta$ in $\mathcal{H}$. Hence $q_t a_n \Omega \to q_t\zeta$ in $\mathcal{H}$. Note that $q_t a_n \in \mathcal{A}_t$ for all $t \in [0,+\infty]$ and for all n. Also $q_t a_n \Omega \in \mathcal{A}_t\Omega \subseteq \mathcal{H}_t$ for all $t \in [0,+\infty]$. So $q_t\zeta \in \mathcal{H}_t$ for all $t \in [0,+\infty]$. Thus $\zeta \in \mathcal{M}_\tau(\mathcal{H})$, as required. □

The assertion 2 in Lemma 3.2 is an analogue of the classical result which states that τ is measurable with respect to the σ-algebra $\mathcal{F}_\tau$ ([8, Proposition 3.5 (a)]). Concerning assertion 3, note that $\mathcal{S}_\theta^l (f;\tau)$ is an operator in $\mathcal{A}$; if $\lim_\theta \mathcal{S}_\theta^l (f;\tau)\Omega = \zeta$ exists, then the left stochastic integral may be defined by

$$\left[\int d\tau(t) f(t)\right] \left(a'\Omega\right) = a'\zeta, \quad a' \in \mathcal{A}'.$$

Then $\int d\tau(t) f(t)$ is a densely defined closable operator whose closure is affiliated to $\mathcal{A}$, for more details see [11]. Moreover, if $\int d\tau(t) f(t)\Omega \in \mathcal{A}_\tau\Omega$, we obtain that $\overline{\int d\tau(t) f(t)} \in \mathcal{A}_\tau$. Indeed, if $\int d\tau(t) f(t)\Omega = a\Omega$ for some $a \in \mathcal{A}_\tau$, then for each $a' \in \mathcal{A}'$ we have

$$\left[\int d\tau(t) f(t)\right] \left(a'\Omega\right) = a' \int d\tau(t) f(t)\Omega = a'a\Omega = a\left(a'\Omega\right),$$

which means $\overline{\int d\tau(t) f(t)} = a$ on the dense subspace $\mathcal{A}'\Omega$, so $\overline{\int d\tau(t) f(t)} \in \mathcal{A}_\tau$. As for assertion 4, a natural question arises — is the converse true? Put another way, consider the von Neumann algebra generated by $\mathcal{A}_\tau$ which is denoted again by $\mathcal{A}_\tau$, is $\overline{(\mathcal{A}_\tau\Omega)} = \mathcal{M}_\tau(\mathcal{H})$? Below we give a partial answer to the question for deterministic times. Before that, we discuss this relation for bounded random times, for more details see [6].

Definition 3.3. A random time $\tau = (q_\alpha)$ is bounded if there exists $s \in [0,+\infty)$ such that $q_\alpha = \mathbf{1}$, for each $\alpha \ge s$.

Proposition 3.4. *Let $\tau = (q_t)$ be a bounded random time. Then $\mathcal{A}_\tau \subseteq \mathcal{A}_s$ and*

$$\overline{(\mathcal{A}_\tau\Omega)} \subseteq \mathcal{M}_\tau(\mathcal{H}) \subseteq \overline{(\mathcal{A}_s\Omega)},$$

where $s = \inf\{t \colon q_t = \mathbf{1}\}$.

Proof. Let $a \in \mathcal{A}_\tau$ so that $q_t a \in \mathcal{A}_t$ for each $t \in [0, +\infty]$. Let $s = \inf\{t \colon q_t = \mathbf{1}\}$. So $a \in \mathcal{A}_t$ for each $t > s$ and hence $a \in (\bigcap_{t>s} \mathcal{A}_t) = \mathcal{A}_s$ (since the filtration $(\mathcal{A}_t)_{t\in[0,+\infty]}$ is right continous). Thus $\mathcal{A}_\tau \subseteq \mathcal{A}_s$. Moving on to the second part we have $\overline{(\mathcal{A}_\tau \Omega)} \subseteq \mathcal{M}_\tau (\mathcal{H})$, by Lemma 3.2 (4). Now let $\zeta \in \mathcal{M}_\tau (\mathcal{H})$ then $q_t \zeta \in \mathcal{H}_t$ for each $t \in [0, +\infty]$ ([2, Theorem 2.12]). This implies that $\zeta \in \mathcal{H}_t$ for each $t > s$ and hence $\zeta \in (\bigcap_{t>s} \mathcal{H}_t) = \mathcal{H}_s$ (since $s \mapsto \mathcal{P}_s$ is strongly continuous, see Proposition 3.2 of [4]). Note that $\mathcal{H}_s = \overline{(\mathcal{A}_s \Omega)}$. Thus $\overline{(\mathcal{A}_\tau \Omega)} \subseteq \mathcal{M}_\tau (\mathcal{H}) \subseteq \overline{(\mathcal{A}_s \Omega)}$, which shows the claim. □

Remark 3.5. If $\tau = (q_s)$ corresponds to the deterministic time $t \in (0, +\infty)$ defined by

$$q_s = \begin{cases} 0 & s \leq t \\ \mathbf{1} & s > t, \end{cases}$$

then we have

$$\begin{aligned} \mathcal{A}_\tau =& \{a \in \mathcal{A} \colon q_s a \in \mathcal{A}_s, \text{ for all } s\} = \mathcal{A}_t, \\ \mathcal{A}_\tau^- =& \overline{\text{span}}\{(\mathbf{1} - q_{s^+})\, a \colon a \in \mathcal{A}_s,\ s \in [0, +\infty)\} \\ =& \overline{\text{span}}\{a \in \mathcal{A}_s \colon s < t\} = \overline{(\bigcup_{s<t} \mathcal{A}_s)}. \end{aligned}$$

Its clear that $\overline{(\mathcal{A}_\tau \Omega)} = \overline{(\mathcal{A}_t \Omega)} = \mathcal{H}_t = \mathcal{P}_t (\mathcal{H}) = \mathcal{M}_\tau (\mathcal{H})$ and $\mathcal{A}_t \neq \overline{(\bigcup_{s<t} \mathcal{A}_s)}$ if we have a filtration in which $\bigcup_{s<t} \mathcal{A}_s$ is not dense in $\mathcal{A}_t$ (the filtration is not left continous).

The relation between $\mathcal{A}_\tau$ and $\mathcal{A}_\tau^-$ is as expected.

Proposition 3.6. *For any random time* $\tau = (q_t)$, $\mathcal{A}_\tau^- \subseteq \mathcal{A}_\tau$ *and the inclusion can be strict.*

Proof. Consider any element of the form $q_s (I - q_{t^+})\, a$ where $a \in \mathcal{A}_t$. If $s \leq t$ then $q_s \leq q_{t^+}$ and so $q_s (\mathbf{1} - q_{t^+})$ is zero. So $q_s (\mathbf{1} - q_{t^+})\, a \in \mathcal{A}_s$. If $s > t$ then $q_s (\mathbf{1} - q_{t^+}) = q_s - q_s q_{t^+} \in \mathcal{A}_s$, difference of elements in $\mathcal{A}_s$, and so $q_s (\mathbf{1} - q_{t^+})\, a \in \mathcal{A}_s$. Putting the two parts together and using the definition of $\mathcal{A}_\tau$ proves the first assertion. The above remark shows that the inclusion may be strict when τ is the deterministic time $t \in (0, +\infty)$ then $\mathcal{A}_\tau^- = \overline{\left(\bigcup_{s<t} \mathcal{A}_s\right)}$. If we have a filtration which is not left continuous then $\mathcal{A}_\tau^-$ will be strictly smaller than $A_\tau = \mathcal{A}_t$. □

Proposition 3.7. *Let $\tau = (q_t)$ and $\sigma = (p_t)$ be random times. Then*

$$\mathcal{A}_{\tau\wedge\sigma} = \mathcal{A}_\tau \cap \mathcal{A}_\sigma.$$

Proof. Let $a \in \mathcal{A}_{\tau\wedge\sigma}$ then $(q_t \vee p_t)\, a \in \mathcal{A}_t$ for each $t \in [0, +\infty]$. For each $t \in [0, +\infty]$, we have $q_t \vee p_t \in \mathcal{A}_t$ and

$$q_t a = q_t\,(q_t \vee p_t)\, a, \quad p_t a = p_t\,(q_t \vee p_t)\, a.$$

We conclude that $q_t a \in \mathcal{A}_t$ and $p_t a \in \mathcal{A}_t$ for each $t \in [0, +\infty]$. So $a \in \mathcal{A}_\tau$ and $a \in \mathcal{A}_\sigma$. Hence $a \in \mathcal{A}_\tau \cap \mathcal{A}_\sigma$. For the second part, let $a \in \mathcal{A}_\tau \cap \mathcal{A}_\sigma$. Then $q_t a \in \mathcal{A}_t$ and $p_t a \in \mathcal{A}_t$ for each $t \in [0, +\infty]$. Furthermore, we have $ua \in \mathcal{A}_t$ for each $t \in [0, +\infty]$, where $u \in \mathcal{A}_t$ is a finite linear combination of finite products of q_t and p_t. There is a net (u_j) of finite linear combinations of finite products of q_t and p_t which converges to $q_t \vee p_t$ in the strong-operator topology. This implies that the net $(u_j a)$ converges to $(q_t \vee p_t)\, a$ in the strong-operator topology and hence $(q_t \vee p_t)\, a \in \mathcal{A}_t$. This means that $a \in \mathcal{A}_{\tau\wedge\sigma}$. □

The above result is valid for any finite family of random times and the proof is obvious.

We recall from [6] that the set $\mathcal{T}$ of random times is partially ordered by the relation defined in 2.2 (i) and under this relation the set $\mathcal{T}$ forms a complete lattice. Now, let $\{\tau_\alpha \colon \alpha \in \Lambda\}$ be a set of times. By adjoining to this subset the infima of each finite subset of this family, we generate a decreasing net of random times whose infimum, τ, is identical with that of the original family. Similarly, we can construct an increasing net of random times whose supremum is identical with the supremum of the original family.

Let $\{\tau_\alpha\} = \{(q_t^{(\alpha)})\}$ be a net of random times, and $\tau = (q_t)$ be a random time. Then τ_α is said to converge strongly to τ, if $q_t^{(\alpha)} \to q_t$ strongly for each $t \in [0, +\infty]$.

The next result is an analogue of the classical results for a descending family of random times (see Proposition 3.5 (b) and Theorem 6.3 (a) of [8]). Also note that the classical result is considered for a countable family only.

Theorem 3.8. *Let σ and τ be random times with $\sigma \leq \tau$. Then $\mathcal{A}_\sigma \subseteq \mathcal{A}_\tau$. If $\{\tau_\alpha \colon \alpha \in \Lambda\}$ is a family of random times, with $\tau = \inf_\alpha \tau_\alpha$, then*

$$\mathcal{A}_\tau = \bigcap_{\alpha\in\Lambda} \mathcal{A}_{\tau_\alpha}.$$

Proof. The relation $\sigma \leq \tau$ entails $\tau\,(t) \leq \sigma\,(t)$ for each $t \in [0, +\infty]$. Let $a \in \mathcal{A}_\sigma$ then $\sigma\,(t)\, a \in \mathcal{A}_t$ for each $t \in [0, +\infty]$. Since $\tau\,(t) \in \mathcal{A}_t$ for each $t \in [0, \infty]$ and $\tau\,(t)\, a = \tau\,(t)\, \sigma\,(t)\, a$ for each $t \in [0, +\infty]$, we conclude that

$\tau(t)\, a \in \mathcal{A}_t$ for each $t \in [0, +\infty]$ and so for $a \in \mathcal{A}_\tau$. So $\mathcal{A}_\sigma \subseteq \mathcal{A}_\tau$. Moving on to the second part we see immediately that

$$\mathcal{A}_\tau \subseteq \bigcap_{\alpha \in \Lambda} \mathcal{A}_{\tau_\alpha},$$

by the first part. Now let $a \in \bigcap_{\alpha\in\Lambda} \mathcal{A}_{\tau_\alpha}$ then $a \in \mathcal{A}_{\tau_\alpha}$ for each $\alpha \in \Lambda$ and $\tau_\alpha(t)\, a \in \mathcal{A}_t$ for each $t \in [0, +\infty]$. As we noted above, we may include the infima of finite subsets of the set of times without altering the infima of the collection. Thus we may assume that $\{\tau_\alpha\}$ is a decreasing directed family of random times. Then $\tau_\alpha(t)$ increase to $\tau(t)$ and hence converges to it strongly and it follows that $\tau_\alpha(t)\, a$ converges strongly to $\tau(t)\, a$. Thus $\tau(t)\, a \in \mathcal{A}_t$, that is, $a \in \mathcal{A}_\tau$. This shows that $\bigcap_{\alpha\in\Lambda} \mathcal{A}_{\tau_\alpha} \subseteq \mathcal{A}_\tau$ and so they agree. □

Theorem 3.8 and Remark 3.5 show that $\{\mathcal{A}_\tau : \tau \text{ is a random time}\}$ is an increasingly directed family of subspaces of $\mathcal{A}$ that contains the filtration $(\mathcal{A}_t)_{t\in[0,+\infty]}$. Analogous to Proposition 6.1 (d) and Theorem 6.3 (c) of [8] of the classical case which is given in countable case, we have

Theorem 3.9. *Let σ and τ be random times with $\sigma \le \tau$. Then $\mathcal{A}_\sigma^- \subseteq \mathcal{A}_\tau^-$. If $\{\tau_\alpha : \alpha \in \Lambda\}$ is an increasing family of random times, with $\tau = \sup_\alpha \tau_\alpha$, then*

$$\mathcal{A}_\tau^- = \overline{\left(\bigcup_{\alpha\in\Lambda} \mathcal{A}_{\tau_\alpha}^-\right)}.$$

Proof. The relation $\sigma \le \tau$ implies $\tau(t) \le \sigma(t)$ for each $t \in [0, +\infty]$ and hence $\tau(t^+) \le \sigma(t^+)$ for each $t \in [0, +\infty)$. Thus $(\mathbf{1} - \sigma(t^+)) \le (\mathbf{1} - \tau(t^+))$ for each $t \in [0, +\infty)$ and so for $a \in \mathcal{A}_t$,

$$\begin{aligned}\left(\mathbf{1} - \sigma\left(t^+\right)\right) a &= \left(\mathbf{1} - \sigma\left(t^+\right)\right)\left(\mathbf{1} - \tau\left(t^+\right)\right) a \\ &= \left(\mathbf{1} - \tau\left(t^+\right)\right)\left[\left(\mathbf{1} - \sigma\left(\mathbf{t}^+\right)\right) a\right] \in \mathcal{A}_\tau^-,\end{aligned}$$

since we have $(\mathbf{1} - \sigma(t^+))\, a \in \mathcal{A}_{t^+} = \mathcal{A}_t$. Thus $\mathcal{A}_\sigma^- \subseteq \mathcal{A}_\tau^-$. Moving on to the second part we see immediately that

$$\overline{(\bigcup_{\alpha\in\Lambda} \mathcal{A}_{\tau_\alpha}^-)} \subseteq \mathcal{A}_\tau^-,$$

by the first part. Now for $t \in [0, +\infty]$ we have $\tau_\alpha(t)$ decreases to $\tau(t)$ and hence converges to it strongly. Let $t \in [0, +\infty)$ be fixed and $\zeta \in \mathcal{H}$. Choose $s > t$ so that $\|(\tau(t^+) - \tau(s))\zeta\|$ is small (since $\tau(t^+) = \lim_{s>t} \tau(s)$). Now choose $\alpha \in \Lambda$ so that $\|(\tau_\alpha(s) - \tau(s))\zeta\|$ is small. Note that $\tau_\alpha(s) \ge \tau_\alpha(t^+) \ge \tau(t^+)$ because $\tau_\alpha \le \tau$, so

$$0 \le \tau_\alpha\left(t^+\right) - \tau\left(t^+\right) \le \tau_\alpha(s) - \tau\left(t^+\right) = \tau_\alpha(s) - \tau(s) + \tau(s) - \tau\left(t^+\right).$$

By using the triangle inequality (which works in $\|\cdot\|$ at ζ) we get that $\tau_\alpha(t^+)$ decreases strongly to $\tau(t^+)$. So $(\mathbf{1}-\tau_\alpha(t^+))$ increases strongly to $(\mathbf{1}-\tau(t^+))$. It follows that, for $t \in [0,+\infty)$ and $a \in \mathcal{A}_t$, $(\mathbf{1}-\tau_\alpha(t^+))\,a$ converges strongly to $(\mathbf{1}-\tau(t^+))\,a$. We know that $(\mathbf{1}-\tau_\alpha(t^+))\,a \in \mathcal{A}^-_{\tau_\alpha}$ for each α. Thus $(\mathbf{1}-\tau(t^+))\,a \in \overline{(\bigcup_{\alpha\in\Lambda}\mathcal{A}^-_{\tau_\alpha})}$. And we have $\mathcal{A}^-_\tau \subseteq \overline{(\bigcup_{\alpha\in\Lambda}\mathcal{A}^-_{\tau_\alpha})}$. □

4. Order structure of p-times

In this section we define a general notion of random time which we shall call *p-time* or *time* and as in the theory of random times (see [2, 3, 10]), we shall discuss the structure of time projections associated with *p-times*, the Optional Stopping Theorem and the range of time projection associated with *p-time*. Subsequently, we divide the family $\mathbb{T}$ of all times into equivalent classes $[p]$, where p is a projection in $\mathcal{A}$ and we see that if $p \in \mathcal{A}_0$, then the results of [2, 3, 10] concerning the structure of random times are still valid for time projections associated with times in $[p]$.

Definition 4.1. By a *time* (or *p-time)* we mean an increasing adapted family of projections (q_t), $t \in [0,+\infty]$, where $q_0 = 0$ and $q_\infty = p$. Note that p is not necessarily $\mathbf{1}$. Accordingly, the random times are $\mathbf{1}$-*times* which forms a subfamily of the family $\mathbb{T}$ of all times.

Lemma 4.2. *Let $\tau = (q_t)$ be a random time and p a projection in $\mathcal{A}_\tau$. Then $(q_t \wedge p)$ is a p-time. Moreover, each p-time arises in this way for some projection p in $\mathcal{A}_\tau$.*

Proof. The relation $p \in \mathcal{A}_\tau$ entails $q_t p \in \mathcal{A}_t$ for all t and so $(q_t p)^k \in \mathcal{A}_t$ for all t, for any $k = 1, 2, \ldots$. Letting $k \to +\infty$, we obtain $q_t \wedge p \in \mathcal{A}_t$ for all t. This shows that the increasing family $(q_t \wedge p)$ of projections is adapted, taking the value 0 at $t = 0$ and the value p at $t = \infty$. This means that $(q_t \wedge p)$ is a *p-time*. Now let $\sigma = (e_t)$ be a *p-time*. Define a random time $\tau = (q_t)$ as follows: $q_t = e_t$ for all $t \in [0,+\infty)$ and $q_{+\infty} = \mathbf{1}$. Then $q_t \wedge p = e_t$ for all t. Also note that $q_t p = e_t \in \mathcal{A}_t$ for all $t \in [0,+\infty)$ and $q_{+\infty} p = p \in \mathcal{A}_{+\infty}$. By Definition 3.1, $p \in \mathcal{A}_\tau$, which gives the claim. □

Recall that a random time $\sigma = (q_t)$ is less than $\tau = (p_t)$ if and only if $p_t \le q_t$ for every $t \in [0,+\infty]$. We extend this definition to all *times* in this context. The requirement is exactly as before: the projections of the "larger" family should be smaller than the projections of the smaller family at each point $t \in [0,+\infty]$. Then the family $\mathbb{T}$ is partially ordered by the above relation, $\leq$, and under this relation the family $\mathbb{T}$ form a complete lattice. The proof is essentially the same as that for random times (see [6, Lemma 2.3]).

In preparation for a discussion of the order structure of the projections associated with times, for any *p-time* $\tau = (q_t)$, we set as for the random times ([1, Definition 2.2])

$$\mathcal{M}_{\tau(\theta)} = \sum_{i=0}^{n-1} \left(q_{t_{i+1}} - q_{t_i}\right) \mathcal{P}_{t_{i+1}} = \sum_{i=0}^{n-1} \Delta q_{t_{i+1}} \mathcal{P}_{t_{i+1}} \equiv \sum_{\theta} \Delta q_{t_{i+1}} \mathcal{P}_{t_{i+1}}$$

where $\theta = \{0 = t_0 < t_1 < \ldots < t_n = \infty\} \in \Theta$. $\mathcal{M}_{\tau(\theta)}$ is an operator on $\mathcal{H}$.

Theorem 4.3. *Let $\tau = (p_t)$ be a time (p-time) and $\theta \in \Theta$ a finite partition; then the operator $\mathcal{M}_{\tau(\theta)}$ has the following properties:*

1. *$\mathcal{M}_{\tau(\theta)}$ is an orthogonal projection;*
2. *For $\theta, \eta \in \Theta$ with η finer than θ, $\mathcal{M}_{\tau(\eta)} \leq \mathcal{M}_{\tau(\theta)}$;*
3. *Let $\sigma = (q_t)$ be another time (q-time) with $\sigma \leq \tau$. Then $\mathcal{M}_{\sigma(\theta)} \leq \mathcal{M}_{\tau(\theta)}$ if and only if $p = q$.*

Proof. The assertions 1 and 2 follow immediately from Proposition 1.3 of [7], by setting $e_t = p_t$ and $f_t = \mathcal{P}_t$. Moving on to the third assertion we see immediately that $p \leq q$ (since $\sigma \leq \tau$). Then

$$\mathcal{M}_{\tau(\theta)}\mathcal{M}_{\sigma(\theta)} = \sum_{i,j=1}^{n} \Delta p_{t_i} \mathcal{P}_{t_i} \Delta q_{t_j} \mathcal{P}_{t_j} = \sum_{i,j=1}^{n} \mathcal{P}_{t_i} \Delta p_{t_i} \Delta q_{t_j} \mathcal{P}_{t_j}.$$

If $j \geq i+1$, then $p_{t_i} q_{t_{j-1}} = p_{t_i}$, since $\sigma \leq \tau$, and so $\Delta p_{t_i} \Delta q_{t_j} = 0$ whenever $j \geq i+1$. Hence

$$\begin{aligned} \mathcal{M}_{\tau(\theta)}\mathcal{M}_{\sigma(\theta)} &= \sum_{j \leq i}^{n} \mathcal{P}_{t_i} \Delta p_{t_i} \Delta q_{t_j} \mathcal{P}_{t_j} = \sum_{j \leq i}^{n} \Delta p_{t_i} \mathcal{P}_{t_j} \Delta q_{t_j} \\ &= \sum_{j} \left(\sum_{i \geq j} \Delta p_{t_i} \right) \mathcal{P}_{t_j} \Delta q_{t_j} = \sum_{j} \left(p - p_{t_{j-1}}\right) \mathcal{P}_{t_j} \Delta q_{t_j} \\ &= p \sum_{j} \mathcal{P}_{t_j} \Delta q_{t_j} - \sum_{j} \mathcal{P}_{t_j} p_{t_{j-1}} \Delta q_{t_j} = p \mathcal{M}_{\sigma(\theta)}. \end{aligned} \qquad (\star)$$

Now assume that $\mathcal{M}_{\sigma(\theta)} \leq \mathcal{M}_{\tau(\theta)}$, then $\mathcal{M}_{\sigma(\theta)} = p\mathcal{M}_{\sigma(\theta)}$. Taking into account that $\mathcal{M}_{\sigma(\theta)}\Omega = q\Omega$, we get that $q\Omega = pq\Omega$ and hence $q = qp$ (since Ω is a separating vector for $\mathcal{A}$). This means that $q \leq p$ and so $p = q$. Now consider $p = q$, from equality $(\star)$ we get that $\mathcal{M}_{\sigma(\theta)}\mathcal{M}_{\tau(\theta)} = q\mathcal{M}_{\sigma(\theta)} = \mathcal{M}_{\sigma(\theta)}$. This means that $\mathcal{M}_{\sigma(\theta)} \leq \mathcal{M}_{\tau(\theta)}$, which shows the claim. □

The properties of Theorem 4.3 and the fact that Θ is a directed set ordered by inclusion, imply that $\{\mathcal{M}_{\tau(\theta)} \colon \theta \in \Theta\}$ is a decreasing net of orthogonal projections. Hence there exists a unique orthogonal projection

$$\mathcal{M}_\tau = \bigwedge_{\theta\in\Theta} \mathcal{M}_{\tau(\theta)};$$

moreover,

$$\mathcal{M}_{\tau(\theta)} \searrow \mathcal{M}_\tau$$

in the strong operator topology as θ refines. We shall call $\mathcal{M}_\tau$ again the *time projection* for the *p-time* τ. Note that the equality $(\star)$ imply that the time projection associated with a *p-time* is less or equal to p.

As an immediate corollary to Theorem 4.3, we have the following

Theorem 4.4 (**Optional Stopping).** *Let* $\tau = (p_t)$ *be a* p*-time and* $\sigma = (q_t)$ *be a* q*-time with* $\sigma \leq \tau$. *We have* $\mathcal{M}_\sigma \leq \mathcal{M}_\tau$ *if and only if* $p = q$.

Proof. Since $\mathcal{M}_{\tau(\theta)}\mathcal{M}_{\sigma(\theta)} = p\mathcal{M}_{\sigma(\theta)}$ for each θ and multiplication is continuous in the strong operator topology on $\mathcal{B}(\mathcal{H})$ and jointly continuous on bounded parts of $\mathcal{B}(\mathcal{H})$, we have

$$\mathcal{M}_{\tau(\theta)}\mathcal{M}_{\sigma(\theta)} \to \mathcal{M}_\tau\mathcal{M}_\sigma, \quad \text{and} \quad p\mathcal{M}_{\sigma(\theta)} \to p\mathcal{M}_\sigma.$$

Thus we obtain that $\mathcal{M}_\tau\mathcal{M}_\sigma = p\mathcal{M}_\sigma$. The remaining of the proof is essentially the same as that of Theorem 4.3 (3). □

We would like to explain here why we consider the notion of *p-times*. We do so because for this kind of times we can assume that the filtration is indexed by a compact interval $[0, T]$ and so the time is an operator monotone projection valued adapted process taking value 0 at $t = 0$ (we will not pursue this here). The second reason is that we obtain that the von Neumann algebra $\mathcal{A}$ is a subalgebra of $\mathcal{V}$, where $\mathcal{V}$ is the von Neumann algebra generated by the family $\{\mathcal{M}_\tau \colon \tau \text{ is a time}\}$. To clarify this relation, we observe that each projection $p \in \mathcal{A}$ can be considered as a time projection associated with the *p-time* $\tau = (p_t)$ which is given by

$$p_t = \begin{cases} 0 & t \in [0, +\infty) \\ p & t = +\infty. \end{cases}$$

Then $\mathcal{M}_{\tau(\theta)} = p$ for each $\theta \in \Theta$ and hence $\mathcal{M}_\tau = p$. Also note that, it is a simple matter to verify that the time projection associated with the p-deterministic time $t \in [0, +\infty]$, which is defined by

$$p_s = \begin{cases} 0 & s \leq t \\ p & s > t, \end{cases}$$

agrees with $p\mathcal{P}_t$.

Let $\tau = (q_t)$ be a *q-time*, $\theta \in \Theta$. Then we have

$$\begin{aligned}\mathcal{M}_{\tau(\theta)} &= \sum_{\theta} \Delta q_{t_{i+1}} \mathcal{P}_{t_{i+1}} = \sum_{i=0}^{n-1} \left(q_{t_{i+1}} - q_{t_i}\right) \mathcal{P}_{t_{i+1}} \\ &= q_{t_n} \mathcal{P}_{t_n} - q_{t_0} \mathcal{P}_{t_0} - \sum_{i=0}^{n-1} q_{t_i} \left(\mathcal{P}_{t_{i+1}} - \mathcal{P}_{t_i}\right) \\ &= q - \sum_{i=0}^{n-1} q_{t_i} \left(\mathcal{P}_{t_{i+1}} - \mathcal{P}_{t_i}\right) = q - \sum_{i=0}^{n-1} q_{t_i} \Delta \mathcal{P}_{t_{i+1}},\end{aligned}$$

where we have used $q_0 = 0$, $q_{+\infty} = q$ and $\mathcal{P}_{+\infty} = \mathbf{1}$ in $\mathcal{B}(\mathcal{H})$. Note that

$$q \sum_{i=1}^{n} \Delta \mathcal{P}_{t_i} = q - q\mathcal{P}_0,$$

so we can write $\mathcal{M}_{\tau(\theta)}$ as

$$\mathcal{M}_{\tau(\theta)} = q\mathcal{P}_0 + \sum_{i=1}^{n} \left(q - q_{t_{i-1}}\right) \Delta \mathcal{P}_{t_i}.$$

Our next results show under which conditions the time projections associated with simple times form a lattice.

Theorem 4.5. *Let $\tau = (q_t)$ be a q-time and $\sigma = (p_t)$ a p-time, and let $\theta \in \Theta$. Then $\mathcal{M}_{(\sigma\vee\tau)(\theta)} = \mathcal{M}_{\sigma(\theta)} \bigvee \mathcal{M}_{\tau(\theta)}$ if and only if $q = p$.*

Proof. Let $\theta = \{0 = t_0 < t_1 < \ldots < t_n = +\infty\} \in \Theta$. Then we have

$$\begin{aligned}q - \mathcal{M}_{\tau(\theta)} &= \sum_{i=0}^{n-1} q_{t_i} \left(\mathcal{P}_{t_{i+1}} - \mathcal{P}_{t_i}\right) \\ p - \mathcal{M}_{\sigma(\theta)} &= \sum_{i=0}^{n-1} p_{t_i} \left(\mathcal{P}_{t_{i+1}} - \mathcal{P}_{t_i}\right),\end{aligned}$$

and hence, for $\zeta \in \mathcal{H}$,

$$\begin{aligned}\left(q - \mathcal{M}_{\tau(\theta)}\right)\left(p - \mathcal{M}_{\sigma(\theta)}\right)\zeta &= \sum_{j=0}^{n-1} q_{t_j} \Delta \mathcal{P}_{t_{j+1}} \left(\sum_{i=0}^{n-1} p_{t_i} \Delta \mathcal{P}_{t_{i+1}} \zeta\right) \\ &= \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} q_{t_j} \Delta \mathcal{P}_{t_{j+1}} p_{t_i} \Delta \mathcal{P}_{t_{i+1}} \zeta\end{aligned}$$

$$= \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} q_{t_j} \Delta\mathcal{P}_{t_{j+1}} \Delta\mathcal{P}_{t_{i+1}} p_{t_i} \zeta$$

$$= \sum_{j=0}^{n-1} q_{t_j} p_{t_j} \Delta\mathcal{P}_{t_{j+1}} \zeta,$$

since $\Delta\mathcal{P}_{t_j} \Delta\mathcal{P}_{t_i} = 0$ if $i \neq j$. It follows that

$$\left[\left(q - \mathcal{M}_{\tau(\theta)}\right)\left(p - \mathcal{M}_{\sigma(\theta)}\right)\right]^k \zeta = \sum_{j=0}^{n-1} \left(q_{t_j} p_{t_j}\right)^k \Delta\mathcal{P}_{t_{j+1}} \zeta,$$

for any $k = 1, 2, \ldots$. Letting $k \to +\infty$, we obtain

$$\begin{aligned} \left(q - \mathcal{M}_{\tau(\theta)}\right) \wedge \left(p - \mathcal{M}_{\sigma(\theta)}\right) \zeta &= \sum_{j=0}^{n-1} \left(q_{t_j} \wedge p_{t_j}\right) \Delta\mathcal{P}_{t_{j+1}} \zeta \\ &= (q \wedge p)\, \zeta - \left[q \wedge p - \sum_{j=0}^{n-1} \left(q_{t_j} \wedge p_{t_j}\right) \Delta\mathcal{P}_{t_{j+1}} \right] \zeta \\ &= (q \wedge p)\, \zeta - \mathcal{M}_{(\sigma\vee\tau)(\theta)} \zeta. \end{aligned}$$

So

$$\left(q - \mathcal{M}_{\tau(\theta)}\right) \wedge \left(p - \mathcal{M}_{\sigma(\theta)}\right) = q \wedge p - \mathcal{M}_{(\sigma\vee\tau)(\theta)}.$$

Now assume that $p = q$. Then

$$\begin{aligned} \left(q - \mathcal{M}_{\tau(\theta)}\right) \wedge \left(p - \mathcal{M}_{\sigma(\theta)}\right) =& p - \mathcal{M}_{\tau(\theta)} \vee \mathcal{M}_{\sigma(\theta)} \\ =& p - \mathcal{M}_{(\sigma\vee\tau)(\theta)}, \end{aligned}$$

which yields that

$$\mathcal{M}_{\tau(\theta)} \vee \mathcal{M}_{\sigma(\theta)} = \mathcal{M}_{(\sigma\vee\tau)(\theta)}.$$

Conversely, we see that $\sigma \leq \sigma \vee \tau$ and $\mathcal{M}_{\sigma(\theta)} \leq \mathcal{M}_{\tau(\theta)} \vee \mathcal{M}_{\sigma(\theta)} = \mathcal{M}_{(\sigma\vee\tau)(\theta)}$. Then by Theorem 4.3 (3) we obtain that $p = p \wedge q$ and similarly $q = p \wedge q$. This means that $p = q$, as required. □

Theorem 4.6. *Let $\tau = (q_t)$ be a q-time and $\sigma = (p_t)$ a p-time, and let $\theta \in \Theta$. If $p, q \in \mathcal{A}_0$, then $\mathcal{M}_{(\sigma\wedge\tau)(\theta)} = \mathcal{M}_{\sigma(\theta)} \wedge \mathcal{M}_{\tau(\theta)}$ if and only if $q = p$.*

Proof. Let $\theta = \{0 = t_0 < t_1 < \ldots < t_n = \infty\} \in \Theta$. Then we have

$$\mathcal{M}_{\tau(\theta)} = q\mathcal{P}_0 + \sum_{i=0}^{n-1} \left(q - q_{t_i}\right) \Delta\mathcal{P}_{t_{i+1}},$$

$$\mathcal{M}_{\sigma(\theta)} = p\mathcal{P}_0 + \sum_{i=0}^{n-1} (p - p_{t_i}) \Delta\mathcal{P}_{t_{i+1}}.$$

Then, for any $\zeta \in \mathcal{H}$,

$$\begin{aligned}
\mathcal{M}_{\tau(\theta)}\mathcal{M}_{\sigma(\theta)}\zeta =& q\mathcal{P}_0 p\mathcal{P}_0\zeta + q\mathcal{P}_0 \sum_{i=0}^{n-1} (p - p_{t_i}) \Delta\mathcal{P}_{t_{i+1}}\zeta \\
&+ \sum_{i=0}^{n-1} (q - q_{t_i}) \Delta\mathcal{P}_{t_{i+1}} p\mathcal{P}_0\zeta \\
&+ \sum_{i=0}^{n-1} (q - q_{t_i}) \Delta\mathcal{P}_{t_{i+1}} \sum_{j=0}^{n-1} \left(p - p_{t_j}\right) \Delta\mathcal{P}_{t_{j+1}}\zeta \\
=& qp\mathcal{P}_0\zeta + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} (q - q_{t_i}) \Delta\mathcal{P}_{t_{i+1}} \left(p - p_{t_j}\right) \Delta\mathcal{P}_{t_{j+1}}\zeta \\
=& qp\mathcal{P}_0\zeta + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} (q - q_{t_i}) \left(p - p_{t_j}\right) \Delta\mathcal{P}_{t_{i+1}} \Delta\mathcal{P}_{t_{j+1}}\zeta \\
=& qp\mathcal{P}_0\zeta + \sum_{i=0}^{n-1} (q - q_{t_i}) (p - p_{t_i}) \Delta\mathcal{P}_{t_{i+1}}\zeta,
\end{aligned}$$

since $\Delta\mathcal{P}_{t_j}\Delta\mathcal{P}_{t_i} = 0$ if $i \neq j$ and $\mathcal{P}_0\Delta\mathcal{P}_{t_j} = 0$. Now if $p = q$, we obtain

$$\mathcal{M}_{\tau(\theta)}\mathcal{M}_{\sigma(\theta)}\zeta = q\mathcal{P}_0\zeta + \sum_{\theta} (q - q_{t_i}) (q - p_{t_i}) \Delta\mathcal{P}_{t_{i+1}}\zeta.$$

As in the proof of Theorem 4.5, we see that

$$\left(\mathcal{M}_{\tau(\theta)}\mathcal{M}_{\sigma(\theta)}\right)^k \zeta = (q\mathcal{P}_0)^k \zeta + \sum_{\theta} \left[(q - q_{t_i}) (q - p_{t_i})\right]^k \Delta\mathcal{P}_{t_{i+1}}\zeta.$$

Letting $k \to \infty$, we get

$$\begin{aligned}
\mathcal{M}_{\tau(\theta)} \wedge \mathcal{M}_{\sigma(\theta)}\zeta =& q \wedge \mathcal{P}_0\zeta + \sum_{\theta} \left[(q - q_{t_i}) \wedge (q - p_{t_i})\right] \Delta\mathcal{P}_{t_{i+1}}\zeta \\
=& q\mathcal{P}_0\zeta + \sum_{\theta} (q - q_{t_i} \vee p_{t_i}) \Delta\mathcal{P}_{t_{i+1}}\zeta \\
=& \mathcal{M}_{(\sigma\wedge\tau)(\theta)}\zeta.
\end{aligned}$$

For the converse, we see that $\sigma \wedge \tau \leq \sigma$ and $\mathcal{M}_{\sigma\wedge\tau(\theta)} = \mathcal{M}_{\tau(\theta)} \wedge \mathcal{M}_{\sigma(\theta)} \leq \mathcal{M}_{\sigma(\theta)}$. Then by using Theorem 4.3 (3) we get $p = p \vee q$ and similarly we get that $q = p \vee q$. This means that $p = q$, as required. □

Corollary 4.7. *Let* $\tau = (q_t)$ *be a* q*-time and* $\sigma = (p_t)$ *a* p*-time. If* $p, q \in \mathcal{A}_0$*, then* $\mathcal{M}_{\sigma\wedge\tau} = \mathcal{M}_\sigma \wedge \mathcal{M}_\tau$ *if and only if* $q = p$.

Proof. Suppose that $\mathcal{M}_{\sigma\wedge\tau} = \mathcal{M}_\sigma \wedge \mathcal{M}_\tau$. We have $\sigma\wedge\tau \leq \sigma$, $\sigma\wedge\tau \leq \tau$ and $\mathcal{M}_{\sigma\wedge\tau} \leq \mathcal{M}_\sigma$, $\mathcal{M}_{\sigma\wedge\tau} \leq \mathcal{M}_\tau$. By virtue of Optional Stopping Theorem (Theorem 4.4), we obtain that $p = p \vee q$ and $q = p \vee q$, so that $q = p$. Conversely, the assumption $q = p$ implies that $\mathcal{M}_{(\sigma\wedge\tau)(\theta)} = \mathcal{M}_{\sigma(\theta)} \wedge \mathcal{M}_{\tau(\theta)}$ for each $\theta \in \Theta$ (Theorem 4.6). Again by Optional Stopping Theorem we obtain that $\mathcal{M}_{\sigma\wedge\tau} \leq \mathcal{M}_\sigma \wedge \mathcal{M}_\tau$. Since $\mathcal{M}_\sigma \wedge \mathcal{M}_\tau \leq \mathcal{M}_{\sigma(\theta)} \wedge \mathcal{M}_{\tau(\theta)} = \mathcal{M}_{(\sigma\wedge\tau)(\theta)}$, for all $\theta \in \Theta$, we obtain $\mathcal{M}_\sigma \wedge \mathcal{M}_\tau \leq \mathcal{M}_{\sigma\wedge\tau}$, from which the result follows. □

Next we prove the structure of the suprema and infima (without the assumption that $p, q \in \mathcal{A}_0$) of time projections under different conditions.

Proposition 4.8. *Let* $\tau = (q_t)$ *be a* q*-time and* $\sigma = (p_t)$ *a* p*-time with* $p_t q_t = q_t p_t$, $\forall t$*. Then* $\mathcal{M}_\sigma$ *and* $\mathcal{M}_\tau$ *commute and* $\mathcal{M}_{\sigma\wedge\tau} = \mathcal{M}_\sigma \wedge \mathcal{M}_\tau$ *and* $\mathcal{M}_\tau \vee \mathcal{M}_\sigma = \mathcal{M}_{\sigma\vee\tau}$ *if and only if* $q = p$.

Proof. Let $\theta \in \Theta$ and note that $p_t + q_t = p_t \vee q_t + p_t \wedge q_t$. Then

$$\begin{aligned}
\mathcal{M}_{(\sigma\wedge\tau)(\theta)} &= \sum_\theta \Delta\left(p_{t_{i+1}} \vee q_{t_{i+1}}\right)\mathcal{P}_{t_{i+1}} \\
&= \sum_\theta \Delta\left(p_{t_{i+1}} + q_{t_{i+1}} - p_{t_{i+1}} \wedge q_{t_{i+1}}\right)\mathcal{P}_{t_{i+1}} \\
&= \sum_\theta \Delta p_{t_{i+1}}\mathcal{P}_{t_{i+1}} + \sum_\theta \Delta q_{t_{i+1}}\mathcal{P}_{t_{i+1}} - \sum_\theta \Delta(p_{t_{i+1}} \wedge q_{t_{i+1}})\mathcal{P}_{t_{i+1}} \\
&= \mathcal{M}_{\sigma(\theta)} + \mathcal{M}_{\tau(\theta)} - \mathcal{M}_{(\sigma\vee\tau)(\theta)}.
\end{aligned}$$

Passing to the limit as θ refines yields that

$$\mathcal{M}_{\sigma\wedge\tau} = \mathcal{M}_\sigma + \mathcal{M}_\tau - \mathcal{M}_{\sigma\vee\tau}.$$

Now assume that $q = p$; using the Optional Stopping Theorem after multiplying $\mathcal{M}_\sigma$ or $\mathcal{M}_\tau$ on the left yields

$$\mathcal{M}_{\sigma\wedge\tau} = \mathcal{M}_\sigma . \mathcal{M}_\tau = \mathcal{M}_\tau . \mathcal{M}_\sigma = \mathcal{M}_\sigma \wedge \mathcal{M}_\tau,$$

and from this the other required relation follows easily. The converse follows immediately from the Optional Stopping Theorem. □

We see from the discussion above that the knowledge of the value of a time at infinity is necessary and sufficient to study the structure of time projections and their lattice properties. This leads to the following definition.

Definition 4.9. Two times σ and τ in a family $\mathbb{T}$ are said to be *equivalent* if $\sigma(+\infty) = \tau(+\infty)$. We write this fact as $\sigma \sim \tau$. Clearly, the relation $\sigma \sim \tau$ is an equivalence relation with partition $\{[q] : q \in \mathcal{A}_p\}$, where $\mathcal{A}_p$ is the projection lattice of $\mathcal{A}$. Accordingly, the family of random times is the class $[\mathbf{1}]$. The next result summarizes all above facts.

Corollary 4.10. *Let σ and τ be any two times in $[p]$ with $p \in \mathcal{A}_0$. Then, for each $\theta \in \Theta$, we have*

1. $\mathcal{M}_{\tau(\theta)} \bigwedge \mathcal{M}_{\sigma(\theta)} = \mathcal{M}_{(\sigma\wedge\tau)(\theta)}$;
2. $\mathcal{M}_{\tau(\theta)} \bigvee \mathcal{M}_{\sigma(\theta)} = \mathcal{M}_{(\sigma\vee\tau)(\theta)}$;
3. $\mathcal{M}_{\tau} \bigwedge \mathcal{M}_{\sigma} = \mathcal{M}_{\sigma\wedge\tau}$,
4. *If $\sigma \leq \tau$ then $\mathcal{M}_\sigma \leq \mathcal{M}_\tau$ (Optional Stopping Theorem).*

Note that the result for suprema does not need the condition $p \in \mathcal{A}_0$.

Definition 4.11. We define a family of von Neumann algebras $\{\mathcal{V}_p \colon p \text{ is a projection in } \mathcal{A}\}$ by: $\mathcal{V}_p$ is the von Neumann algebra generated by

$$\{\mathcal{M}_\tau : \tau \text{ is a time in } [p]\}.$$

Note that

$$\mathcal{V} = \left(\bigcup_p \mathcal{V}_p\right)''.$$

Lemma 4.12. *For $t \in [0, +\infty]$, $\mathcal{P}_t$ belongs to the commutant of $\mathcal{V}_p$ if and only if $p\mathcal{P}_t = \mathcal{P}_t p$.*

Proof. Let $t \in [0, +\infty]$, $\tau = (p_t)$ is a *p-time* and $\theta = \{0 = t_0, t_1, \ldots, t_n = +\infty\} \in \Theta$. Then

$$\exists k \in \{0, \cdots, n-1\} \text{ with } t_k \leq t < t_{k+1}.$$

So

$$\begin{aligned}
\mathcal{M}_{\tau(\theta)}\mathcal{P}_t &= \left[\sum_{i=0}^{n-1} \left(p_{t_{i+1}} - p_{t_i}\right)\mathcal{P}_{t_{i+1}}\right]\mathcal{P}_t \\
&= \sum_{i=0}^{k-1} \left(p_{t_{i+1}} - p_{t_i}\right)\mathcal{P}_{t_{i+1}}\mathcal{P}_t + \sum_{i=k}^{n-1} \left(p_{t_{i+1}} - p_{t_i}\right)\mathcal{P}_{t_{i+1}}\mathcal{P}_t \\
&\quad \text{(the first sum is equal to 0 if } k = 0) \\
&= \sum_{i=0}^{k-1} \left(p_{t_{i+1}} - p_{t_i}\right)\mathcal{P}_{t_{i+1}} + \sum_{i=k}^{n-1} \left(p_{t_{i+1}} - p_{t_i}\right)\mathcal{P}_t
\end{aligned}$$

$$
\begin{aligned}
&(\text{since } \mathcal{P}_t\text{'s are orthogonal projections that increase with } t)\\
&=\sum_{i=0}^{k-1}\left(p_{t_{i+1}}-p_{t_i}\right)\mathcal{P}_{t_{i+1}}+\left(p-p_{t_k}\right)\mathcal{P}_t.
\end{aligned}
$$

Now assume that $p\mathcal{P}_t=\mathcal{P}_t p$; since $\mathcal{P}_s$ lies in the commutant of $\mathcal{A}_s$, we get

$$
\begin{aligned}
\mathcal{M}_{\tau(\theta)}\mathcal{P}_t=&\sum_{i=0}^{k-1}\mathcal{P}_{t_{i+1}}\left(p_{t_{i+1}}-p_{t_i}\right)+\mathcal{P}_t\left(p-p_{t_k}\right)\\
=&\mathcal{P}_t\left[\sum_{i=0}^{k-1}\mathcal{P}_{t_{i+1}}\left(p_{t_{i+1}}-p_{t_i}\right)+\mathcal{P}_t\left(p-p_{t_k}\right)\right]\\
=&\mathcal{P}_t\left[\sum_{i=0}^{k-1}\left(p_{t_{i+1}}-p_{t_i}\right)\mathcal{P}_{t_{i+1}}+\mathcal{P}_t\sum_{i=k}^{n-1}\mathcal{P}_{t_{i+1}}\left(p_{t_{i+1}}-p_{t_i}\right)\right]\\
=&\mathcal{P}_t\mathcal{M}_{\tau(\theta)}.
\end{aligned}
$$

Now passing to the limit as θ refines yields that $\mathcal{M}_\tau\mathcal{P}_t=\mathcal{P}_t\mathcal{M}_\tau$ for each $t\in[0,+\infty]$. This means that $\mathcal{P}_t$ commutes with the generators of $\mathcal{V}_p$ and as a result $\mathcal{P}_t\in\mathcal{V}_p'$. Conversely, we see immediately that $\mathcal{M}_\tau\mathcal{P}_t=\mathcal{P}_t\mathcal{M}_\tau$ for each $\tau\in[p]$. But $p=\mathcal{M}_\tau$ for the *p-time* $\tau=(p_t)$ which is given by

$$
p_t=\begin{cases}0 & t\in[0,+\infty)\\ p & t=+\infty.\end{cases}
$$

So $p\mathcal{P}_t=\mathcal{P}_t p$, as required. □

As an immediate corollary, we have

Corollary 4.13. *For $t\in[0,+\infty]$, if $p\mathcal{P}_t=\mathcal{P}_t p$ then $p\mathcal{P}_t$ belongs to the centre of $\mathcal{V}_p$.*

Proof. Since $p\mathcal{P}_t=\mathcal{P}_t p$ we get that $\mathcal{P}_t\in\mathcal{V}_p'$ (Lemma 4.12). We have shown above that $\mathcal{M}_\tau\le p$ for each $\tau\in[p]$, so p commutes with the generators of $\mathcal{V}_p$ and as a result $p\in\mathcal{V}_p'$. This implies that $p\mathcal{P}_t\in\mathcal{V}_p'$. On the other side we know that $p\mathcal{P}_t$ is a time projection associated with p-deterministic time $t\in[0,+\infty)$, which is defined by

$$
p_s=\begin{cases}0 & s\le t\\ p & s>t,\end{cases}
$$

so $p\mathcal{P}_t\in\mathcal{V}_p$ and hence $p\mathcal{P}_t\in\mathcal{V}_p\cap\mathcal{V}_p'=$ centre of $\mathcal{V}_p$, as claimed. □

Remark 4.14. We would like to stress again at this point that when p is a projection in $\mathcal{A}_0$, then the time projections associated with simple times in $[p]$ form a lattice and satisfy all the results concerning the structure of time projections in [2, 3, 6, 10], within an arbitrary non-commutative filtration of von Neumann algebras as well as those employed in quantum stochastic theory of the canonical anticommutation relations (CAR) and the canonical commutation relations (CCR); the proofs for the case of CAR and CCR are quite routine (by using the martingale representation theorem for elements of $\mathcal{H}$, see [12]).

We finish this paper with a result on the range of the time projection associated with a *q-time*.

Theorem 4.15. *Let $\tau = (q_t)$ be a q-time. Then $\zeta \in \mathcal{H}$ is in the range of a time projection $\mathcal{M}_\tau$ if and only if $q_t\zeta \in \mathcal{H}_t$ for all $t \in [0, +\infty]$ and $q\zeta = \zeta$.*

Proof. If $\zeta \in \mathcal{M}_\tau(\mathcal{H})$, then $\zeta \in \mathcal{M}_{\tau(\theta)}(\mathcal{H})$ for each $\theta \in \Theta$. Suppose now that $t \in (0, +\infty)$ and take the partition $\theta_t = \{0 = t_0 < t_1 = t < t_2 = +\infty\}$. Then

$$\mathcal{M}_{\tau(\theta_t)} = (\mathbf{1} - q_t) + q_t\mathcal{P}_t,$$

and

$$\zeta = \mathcal{M}_{\tau(\theta_t)}(\zeta) = (\mathbf{1} - q_t)\,\zeta + q_t\mathcal{P}_t\zeta.$$

Applying q_t to both sides yields $q_t\zeta = q_t\mathcal{P}_t\zeta \in \mathcal{H}_t$. Concerning the second part, we see immediately that $q\zeta = \zeta$ (since $\mathcal{M}_\tau \leq q$). Conversely, if $q_t\zeta \in \mathcal{H}_t$ for all $t \in [0, +\infty]$ and $q\zeta = \zeta$, then

$$\begin{aligned}\mathcal{M}_{\tau(\theta)}\zeta &= \sum_\theta \Delta q_{t_i}\mathcal{P}_{t_i}\zeta = \sum_\theta \Delta q_{t_i} q_{t_i}\mathcal{P}_{t_i}\zeta \\ &= \sum_\theta \Delta q_{t_i}\mathcal{P}_{t_i}(q_{t_i}\zeta) = \sum_\theta \Delta q_{t_i} q_{t_i}\zeta \\ &= \sum_\theta \Delta q_{t_i}\zeta = q\zeta = \zeta.\end{aligned}$$

Taking the limit gives $\zeta = \mathcal{M}_\tau\zeta$. □

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References

[1] Barnett, C., Thakrar, B., *A non-commutative random stopping theorem*, J. Funct. Anal. **88** (1990), 342–350.

[2] Barnett, C., Wilde, I. F., *Random times and time projections*, Proc. Amer. Math. Soc. **110** (1990), 425–440.

[3] Barnett, C., Wilde, I. F., *Random times, predictable processes, and stochastic integration in finite von Neumann algebras*, Proc. London Math. Soc. (3) **67** (1993), 355–383.

[4] Barnett, C., Streater, R. F., Wilde, I. F., *Quasi-free quantum stochastic integrations for CAR and CCR*, J. Funct. Anal. **52** (1983), 19–47.

[5] Barnett, C, Wilde, I. F., *Quantum Doob-Meyer decompositions*, J. Operator Theory **20** (1988), 133–164.

[6] Barnett, C., Thakrar, B., *Time projections in a von Neumann algebra*, J. Operator Theory **18** (1987), 19–31.

[7] Barnett, C., Goldstein, S., Wilde, I. F., *Quantum stopping times and Doob-Meyer decompositions*, J. Operator Theory **35** (1996), 85–106.

[8] Kussmaul, A., *Stochastic Integration and Generalised Martingales*, Res. Notes Math. **11**, Pitman Publishing, London-San Francisco, Calif.-Melbourne, 1977.

[9] Lindsay, J. M., *Fermion martingales*, Probab. Theory Related Fields **71** (1986), 307–320.

[10] Mohammed, A. A. A., *On the order structure of time projection*, J. Appl. Anal. **8** (2002), 279–295.

[11] Mohammed, A. A. A., *Stopping and stochastic integrals as closable operators*, Probab. Math. Statist. **22** (2002), 167–180.

[12] Wilde, I. F., *Quantum martingales and stochastic integrals*, in "Quantum Probability and Applications, III" (Oberwolfach, 1987), Lecture Notes in Math. **1303**, Springer, Berlin, 1988.

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