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STRONG AND WEAK CONVERGENCE OF IMPLICIT ITERATIVE PROCESS WITH ERRORS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. The purpose of this paper is to study the weak and strong convergence of an new implicit iteration process to a common fixed point for a finite family of asymptotically nonexpansive mappings in Banach spaces. The results presented in this paper extend and improve important known results in [1], [2], [4]–[9], [11]–[15] and others.

1. Introduction and preliminaries

Throughout this paper we assume that E is a real Banach space and $T: E \to E$ is a mapping. We denote by $F(T)$ and $D(T)$ the set of fixed points and the domain of T, respectively.

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Recall that E is said to satisfy *Opial condition*, if for each sequence $\{x_n\}$ in E, the condition that the sequence $x_n \to x$ weakly implies that

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$$

for all $y \in E$ with $y \neq x$.

Definition 1.1. Let D be a closed subset of E and $T\colon D \to D$ be a mapping.

1. T is said to be *demi-closed* at the origin, if for each sequence $\{x_n\}$ in D, the conditions $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly imply $Tx_0 = 0$.
2. T is said to be *semi-compact*, if for any bounded sequence $\{x_n\}$ in D such that $||x_n - Tx_n|| \to 0$ $(n \to \infty)$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to\ x^* \in D$.
3. T is said to be *asymptotically nonexpansive* [4], if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y|| \quad \forall x, y \in D,\ n \geq 1.$$

Proposition 1.1. *Let K be a nonempty subset of E, $\{T_i\}_{i=1}^N\colon K \to K$ be N asymptotically nonexpansive mappings. Then there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ $(n \to \infty)$ such that*

$$||T_i^n x - T_i^n y|| \leq k_n ||x - y||, \quad \forall n \geq 1, \tag{1.1}$$

for all $x, y \in K$, $i = 1, 2, \dots, N$.

Proof. (1) Since for each $i = 1, 2, \dots, N$, $T_i\colon K \to K$ is an asymptotically nonexpansive mapping, there exists a sequence $\{k_n^{(i)}\} \subset [1,\infty)$, with $k_n^{(i)} \to 1$ $(n \to \infty)$ such that

$$||T_i^n x - T_i^n y|| \leq k_n^{(i)} ||x - y||, \quad \forall x, y \in K,\ \ \forall n \geq 1,\ i = 1, 2, \dots, N.$$

Letting

$$k_n = \max\{k_n^{(1)}, k_n^{(2)}, \dots, k_n^{(N)}\},$$

then we have that $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ $(n \to \infty)$ and

$$||T_i^n x - T_i^n y|| \leq k_n^{(i)} ||x - y|| \leq k_n ||x - y||, \quad \forall n \geq 1,$$

for all $x, y \in K$, and for each $i = 1, 2, \dots, N$. $\square$

Definition 1.2. Let K be a nonempty closed convex subset of E with $K + K \subset K$, $x_0 \in K$ be any given point and $\{T_1, T_2, \dots, T_N\}\colon K \to K$ be N asymptotically nonexpansive mappings. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$, $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in K. Then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_n = (1-\alpha_n)x_{n-1} + \alpha_n T^n_{n(\mathrm{mod}\, N)} y_n + u_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n_{n(\mathrm{mod}\, N)} x_n + v_n, \qquad \forall n \geq 1 \end{cases} \tag{1.2}$$

is called the implicit iterative sequence with errors for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^N$.

Especially, if $\{T_1, T_2, \dots, T_N\}\colon K \to K$ be N asymptotically nonexpansive mappings, $\{\alpha_n\}$ be a sequence in [0, 1] and $x_0 \in K$ be a given point, then the sequence $\{x_n\}$ defined by

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T^n_{n(\mathrm{mod}\, N)} x_n, \quad \forall n \geq 1 \tag{1.3}$$

is called the implicit iterative sequence for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^N$.

Recently, concerning the convergence problems of an implicit (or non-implicit) iterative process to a common fixed point for a finite family of asymptotically nonexpansive mappings (or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been considered by several authors (see, for example, Bauschke [1], Chang and Cho [2], Goebel and Kirk [4], Górnicki [5], Halpern [6], Lions [7], Osilike [8], Reich [9], Schu [10], Sun [11], Tan and Xu [12], Wittmann [13], Xu and Ori [14] and Zhou and Chang [15]).

The purpose of this paper is to study the weak and strong convergence of implicit iterative sequence $\{x_n\}$ defined by (1.2) and (1.3) to a common fixed point for a finite family of asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces. The results presented in this paper not only generalized and extend the corresponding results in [1], [2], [4]–[9], [11]–[15], but also give an affirmative to the open question suggested by Xu and Ori [14]. Moreover the results even in the case of $u_n = v_n = 0$ or $\beta_n = 0$, $v_n = 0$, $\forall n \geq 1$ are also new.

In order to prove the main results of this paper, we need the following Lemmas:

Lemma 1.1 ([3, 5, 12]). *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E and $T\colon K \to K$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is semi-closed at zero, i.e., for each sequence $\{x_n\}$ in K, if $\{x_n\}$ converges weakly to $q \in K$ and $\{(I-T)x_n\}$ converges strongly to 0, then $(I-T)q = 0$.*

Lemma 1.2 ([3, 12]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=0}^{\infty} c_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$. Then

(1) *the limit $\lim_{n\to\infty} a_n$ exists.*
(2) *In addition, if there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $a_n \to 0$ $(n \to \infty)$.*

Lemma 1.3 ([10]). *Let E be a uniformly convex Banach space, b, c be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ and $\{x_n\}$, $\{y_n\}$ are two sequence in E. Then the conditions:*

$$\begin{cases} \lim_{n\to\infty} ||(1-t_n)x_n + t_n y_n|| = d, \\ \limsup_{n\to\infty} ||x_n|| \leq d, \\ \limsup_{n\to\infty} ||y_n|| \leq d, \end{cases}$$

imply that $\lim_{n\to\infty} ||x_n - y_n|| = 0$, where $d \geq 0$ is some constant.

Lemma 1.4. *Let E be a real Banach space, K be a nonempty closed convex subset of E with $K+K \subset K$, $\{T_1, T_2, \dots, T_N\}: K \to K$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$, $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in K and $\{k_n\}$ be the sequence defined by (1.1) and $\sigma = \sup_{n\geq 1} k_n \geq 1$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} (k_n - 1)\alpha_n < \infty$;
(ii) $\tau = \sup\{\alpha_n : n \geq 1\} < \dfrac{1}{\sigma^2}$;
(iii) $\sum_{n=1}^{\infty} ||u_n|| < \infty, \quad \sum_{n=1}^{\infty} ||v_n|| < \infty$.

If $\{x_n\}$ is the implicit iterative sequence defined by (1.2), then for each $p \in F = \bigcap_{i=1}^{N} F(T_i)$ the limit $\lim_{n\to\infty} ||x_n - p||$ exists.

Proof. Since $F = \bigcap_{n=1}^{N} F(T_i) \neq \emptyset$, for any given $p \in F$, it follows from (1.2) and Proposition 1.1 that

$$\begin{aligned} ||x_n - p|| &= ||(1-\alpha_n)(x_{n-1} - p) + \alpha_n(T^n_{n(\text{mod } N)} y_n - p) + u_n|| \\ &\leq (1-\alpha_n)||x_{n-1} - p|| + \alpha_n ||T^n_{n(\text{mod } N)} y_n - p|| + ||u_n|| \\ &= (1-\alpha_n)||x_{n-1} - p|| + \alpha_n ||T^n_{n(\text{mod } N))} y_n - T^n_{n(\text{mod } N)} p|| + ||u_n|| \\ &\leq (1-\alpha_n)||x_{n-1} - p|| + \alpha_n k_n ||y_n - p|| + ||u_n||. \end{aligned} \tag{1.4}$$

Again it follows from (1.2) and Proposition 1.1 that

$$\begin{aligned}||y_n - p|| &= ||(1-\beta_n)(x_n - p) + \beta_n(T^n_{n(\bmod N)}x_n - p) + v_n|| \\ &\le (1-\beta_n)||x_n - p|| + \beta_n||T^n_{n(\bmod N)}x_n - p|| + ||v_n|| \\ &= (1-\beta_n)||x_n - p|| + \beta_n||T^n_{n(\bmod N)}x_n - T^n_{n(\bmod N)}p|| + ||v_n|| \\ &\le (1-\beta_n)||x_n - p|| + \beta_n k_n||x_n - p|| + ||v_n|| \\ &\le k_n||x_n - p|| + ||v_n||. \qquad (1.5)\end{aligned}$$

Substituting (1.5) into (1.4), we obtain that

$$\begin{aligned}||x_n - p|| \le& (1-\alpha_n)||x_{n-1} - p|| + \alpha_n k_n^2||x_n - p|| \\ &+ \alpha_n k_n||v_n|| + ||u_n||.\end{aligned}$$

which implies that

$$(1-\alpha_n k_n^2)||x_n - p|| \le (1-\alpha_n)||x_{n-1} - p|| + \mu_n, \qquad (1.6)$$

where

$$\mu_n = \alpha_n k_n||v_n|| + ||u_n||.$$

By the condition (iii) and the boundedness of the sequences $\{\alpha_n\}$ and $\{k_n\}$ we know that

$$\sum_{i=1}^{\infty} \mu_n < \infty.$$

From the condition (ii) we know that

$$\alpha_n k_n^2 \le \tau\sigma^2 < 1 \quad \text{and so} \quad 1 - \alpha_n k_n^2 \ge 1 - \tau\sigma^2 > 0,$$

hence from (1.6) we have

$$\begin{aligned}||x_n - p|| &\le \frac{1-\alpha_n}{1-\alpha_n k_n^2}||x_{n-1} - p|| + \frac{\mu_n}{1-\tau\sigma^2} \\ &= \left(1 + \frac{(k_n^2 - 1)\alpha_n}{1-\alpha_n k_n^2}\right)||x_{n-1} - p|| + \frac{\mu_n}{1-\tau\sigma^2} \\ &\le \left(1 + \frac{(k_n^2 - 1)\alpha_n}{1-\tau\sigma^2}\right)||x_{n-1} - p|| + \frac{\mu_n}{1-\tau\sigma^2} \\ &= (1+b_n)||x_{n-1} - p|| + c_n. \qquad (1.7)\end{aligned}$$

where

$$b_n = \frac{(k_n^2 - 1)\alpha_n}{1-\tau\sigma^2} \quad \text{and} \quad c_n = \frac{\mu_n}{1-\tau\sigma^2}.$$

By conditions (i) and (iii) we have that

$$\sum_{n=1}^{\infty} b_n = \frac{1}{1-\tau\sigma^2}\sum_{n=1}^{\infty}(k_n^2 - 1)\alpha_n$$

$$= \frac{1}{1-\tau\sigma^2}\sum_{n=1}^{\infty}(k_n+1)(k_n-1)\alpha_n$$
$$\leq \frac{1+\sigma}{1-\tau\sigma^2}\sum_{n=1}^{\infty}(k_n-1)\alpha_n < \infty$$

and

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty}\frac{\mu_n}{1-\tau\sigma^2} < \infty.$$

Taking $a_n = ||x_{n-1} - p||$ in inequality (1.7), we have

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq 1.$$

and the satisfy all conditions in Lemma 1.2. Therefore the limit $\lim_{n\to\infty} ||x_n - p||$ exists. Without loss of generality we may assume that

$$\lim_{n\to\infty} ||x_n - p|| = d, \quad p \in F. \tag{1.8}$$

where $d \geq 0$ is some constant. This completes the proof of Lemma 1.4. □

2. Main results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let E be a real Banach space, K be a nonempty closed convex subset of E with $K + K \subset K$, $\{T_1, T_2, \dots, T_N\}: K \to K$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1, T_2, \dots, T_N\}$). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$, $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in K, $\{k_n\}$ be the sequence defined by (1.1) and $\sigma = \sup_{n\geq 1} k_n \geq 1$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}(k_n - 1)\alpha_n < \infty$;

(ii) $\tau = \sup\{\alpha_n : n \geq 1\} < \dfrac{1}{\sigma^2}$;

(iii) $\sum_{n=1}^{\infty} ||u_n|| < \infty, \quad \sum_{n=1}^{\infty} ||v_n|| < \infty.$

Then the implicit iterative sequence $\{x_n\}$ defined by (1.2) converges strongly to a common fixed point $p \in F = \bigcap_{i=1}^{N} F(T_i)$ if and only if

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{2.1}$$

Proof. The necessity of condition (2.1) is obvious.

Next we prove the sufficiency of Theorem 2.1. For any given $p \in F$, it follows from (1.7) in Lemma 1.4 that

$$||x_n - p|| \leq (1 + b_n)||x_{n-1} - p|| + c_n \quad \forall n \geq 1. \tag{2.2}$$

where

$$b_n = \frac{(k_n^2 - 1)\alpha_n}{1 - \tau\sigma^2} \quad \text{and} \quad c_n = \frac{\mu_n}{1 - \tau\sigma^2}$$

with $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Hence, we have

$$d(x_n, F) \leq (1 + b_n)d(x_{n-1}, F) + c_n \quad \forall n \geq 1. \tag{2.3}$$

It follows from (2.3) and Lemma 1.2 that the limit $\lim_{n\to\infty} d(x_n, F)$ exists. By the condition (2.1), we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence in K. In fact, since $\sum_{n=1}^{\infty} b_n < \infty$, $1 + t \leq \exp\{t\}$ for all $t > 0$, and (2.2), therefore we have

$$||x_n - p|| \leq \exp\{b_n\}||x_{n-1} - p|| + c_n. \tag{2.4}$$

Hence, for any positive integers n, m, from (2.4) it follows that

$$\begin{aligned}
||x_{n+m} - p|| \leq& \exp\{b_{n+m}\}||x_{n+m-1} - p|| + c_{n+m} \\
\leq& \exp\{b_{n+m}\}[\exp\{b_{n+m-1}\}||x_{n+m-2} - p|| + c_{n+m-1}] + c_{n+m} \\
=& \exp\{b_{n+m} + b_{n+m-1}\}||x_{n+m-2} - p|| + \exp\{b_{n+m}\}c_{n+m-1} \\
&+ c_{n+m} \\
\leq& \dots \\
\leq& \exp\left\{\sum_{i=n+1}^{n+m} b_i\right\} ||x_n - p|| + \exp\left\{\sum_{i=n+2}^{n+m} b_i\right\} \sum_{i=n+1}^{n+m} c_i \\
\leq& W||x_n - p|| + W \sum_{i=n+1}^{\infty} c_i.
\end{aligned}$$

where $W = \exp\{\sum_{n=1}^{\infty} b_n\} < \infty$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} c_n < \infty$, for any given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\varepsilon}{4(W + 1)}, \quad \sum_{i=n+1}^{\infty} c_i < \frac{\varepsilon}{2W}, \quad \forall n \geq n_0.$$

Therefore there exists $p_1 \in F$ such that

$$d(x_n, p_1) < \frac{\varepsilon}{2(W + 1)}, \quad \forall n \geq n_0$$

Consequently, for any $n \geq n_0$ and for all $m \geq 1$ we have

$$\begin{aligned} ||x_{n+m} - x_n|| &\leq ||x_{n+m} - p_1|| + ||x_n - p_1|| \\ &\leq (1+W)||x_n - p_1|| + W \sum_{i=n+1}^{\infty} c_i \\ &< \frac{\varepsilon}{2(W+1)}(1+W) + W \cdot \frac{\varepsilon}{2W} \\ &= \varepsilon. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in K. By the completeness of K, we can assume that $x_n \to x^* \in K$. Since the set of fixed pints of a asymptotically nonexpansive mapping is closed, hence F is closed. This implies that $x^* \in F$, and so x^* is a common fixed point of $T_1, T_2, \ldots, T_N$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let E be a real uniformly convex Banach space satisfying Opial condition, K be a nonempty closed convex subset of E with $K + K \subset K$, $\{T_1, T_2, \ldots, T_N\}\colon K \to K$ be N asymptotically nonexpansive mappings with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$, $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in K, $\{k_n\}$ be the sequence defined by (1.1) and $\sigma = \sup_{n\geq 1} k_n \geq 1$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}(k_n - 1)\alpha_n < \infty$;

(ii) $0 < \tau_1 = \inf\{\alpha_n \colon n \geq 1\} \leq \sup\{\alpha_n \colon n \geq 1\} = \tau_2 < \dfrac{1}{\sigma^2}$;

(iii) $0 \leq \mu = \sup\{\beta_n \colon n \geq 1\} < \dfrac{1}{\sigma}$;

(iv) $\sum_{n=1}^{\infty} ||u_n|| < \infty, \quad \sum_{n=1}^{\infty} ||v_n|| < \infty$;

(v) *there exists constants $L > 0$ and $\alpha > 0$ such that, for any $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$,*

$$||T_i^n x - T_j^n y|| \leq L||x - y||^{\alpha}, \quad \forall n \geq 1,$$

for all $x, y \in K$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.2) converges weakly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

Proof. Since $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, for any given $p \in F$, it follows from (1.2) and (1.8) that

$$||x_n - p|| = ||(1-\alpha_n)[x_{n-1} - p + u_n] + \beta_n[T^n_{n(\text{mod } N)} y_n - p + u_n]|| \to d \quad (n \to \infty), \quad p \in F. \tag{2.5}$$

From (1.8) and the condition (iv) we know that

$$\begin{aligned}&\limsup_{n\to\infty}||x_{n-1}-p+u_n||\\&\le\limsup_{n\to\infty}||x_{n-1}-p||+\limsup_{n\to\infty}||u_n||=d,\quad p\in F.\end{aligned}\tag{2.6}$$

It follows from (1.5) and the condition (iv) that

$$\begin{aligned}&\limsup_{n\to\infty}||T^n_{n(\mathrm{mod}\,N)}y_n-p+u_n||\\&\le\limsup_{n\to\infty}k_n||y_n-p||+\limsup_{n\to\infty}||u_n||\\&=\limsup_{n\to\infty}k_n||y_n-p||\\&\le\limsup_{n\to\infty}k_n\{k_n||x_n-p||+||v_n||\}\\&\le\limsup_{n\to\infty}k_n^2||x_n-p||+\limsup_{n\to\infty}k_n||v_n||\\&=d,\quad p\in F.\end{aligned}\tag{2.7}$$

Therefore, from the condition (ii), (2.5)–(2.7) and Lemma 1.3 we know that

$$\lim_{n\to\infty}||T^n_{n(\mathrm{mod}\,N)}y_n-x_{n-1}||=0.\tag{2.8}$$

Moreover, since

$$\begin{aligned}||x_n-x_{n-1}||&=||\alpha_n(T^n_{n(\mathrm{mod}\,N)}y_n-x_{n-1})+u_n||\\&\le\alpha_n||T^n_{n(\mathrm{mod}\,N)}y_n-x_{n-1}||+||u_n||,\end{aligned}\tag{2.9}$$

hence, from (2.8) and the condition (iv) we obtain

$$\lim_{n\to\infty}||x_n-x_{n-1}||=0\quad\forall j=1,2,\dots,N\tag{2.10}$$

and so

$$\lim_{n\to\infty}||x_n-x_{n+j}||=0\quad\forall j=1,2,\dots,N.\tag{2.11}$$

On the other hand, we have

$$\begin{aligned}||x_n-T^n_{n(\mathrm{mod}\,N)}x_n||\le&||x_n-x_{n-1}||+||x_{n-1}-T^n_{n(\mathrm{mod}\,N)}y_n||\\&+||T^n_{n(\mathrm{mod}\,N)}y_n-T^n_{n(\mathrm{mod}\,N)}x_n||.\end{aligned}\tag{2.12}$$

Now, we consider the third term on the right side of (2.12). From the Proposition 1.1, (1.2) and the condition (iii) we have

$$\begin{aligned}||T^n_{n(\mathrm{mod}\,N)}y_n-T^n_{n(\mathrm{mod}\,N)}x_n||&\le k_n||y_n-x_n||\qquad(2.13)\\&\le\sigma||\beta_n(T^n_{n(\mathrm{mod}\,N)}x_n-x_n)+v_n||\\&\le\sigma\beta_n||T^n_{n(\mathrm{mod}\,N)}x_n-x_n||+\sigma||v_n||\\&\le\sigma\mu||T^n_{n(\mathrm{mod}\,N)}x_n-x_n||+\sigma||v_n||.\end{aligned}$$

Substituting (2.13) into (2.12), we obtain that

$$\begin{aligned}||x_n - T^n_{n(\bmod N)}x_n|| \leq& ||x_n - x_{n-1}|| + ||x_{n-1} - T^n_{n(\bmod N)}y_n|| \\ &+ \sigma\mu||T^n_{n(\bmod N)}x_n - x_n|| + \sigma||v_n|| \end{aligned} \tag{2.14}$$

Hence, by virtue of the condition (iv), (2.8), (2.10) we have

$$\limsup_{n\to\infty} ||x_n - T^n_{n(\bmod N)}x_n|| \leq \sigma\mu \limsup_{n\to\infty} ||x_n - T^n_{n(\bmod N)}x_n|| \tag{2.15}$$

that is

$$(1-\sigma\mu)\cdot \limsup_{n\to\infty} ||x_n - T^n_{n(\bmod N)}x_n|| \leq 0 \tag{2.16}$$

From the condition (iii), $0 \leq \sigma\mu < 1$, hence from (2.16) we have

$$\lim_{n\to\infty} ||x_n - T^n_{n(\bmod N)}x_n|| = 0. \tag{2.17}$$

By the condition (v), we have

$$||T^{n-1}_{n(\bmod N)}x_n - T^{n-1}_{(n-1)(\bmod N)}x_{n-1}|| \leq L||x_n - x_{n-1}||^\alpha. \tag{2.18}$$

From (2.10), (2.17), (2.18) and Proposition 1.1 that

$$\begin{aligned}||x_n - T_{n(\bmod N)}x_n|| \leq& ||x_n - T^n_{n(\bmod N)}x_n|| + ||T^n_{n(\bmod N)}x_n - T_{n(\bmod N)}x_n|| \\ \leq& ||x_n - T^n_{n(\bmod N)}x_n|| + k_1||T^{n-1}_{n(\bmod N)}x_n - x_n|| \\ \leq& ||x_n - T^n_{n(\bmod N)}x_n|| \\ &+ k_1\{||T^{n-1}_{n(\bmod N)}x_n - T^{n-1}_{(n-1)(\bmod N)}x_{n-1}|| \\ &+ ||T^{n-1}_{(n-1)(\bmod N)}x_{n-1} - x_{n-1}|| + ||x_{n-1} - x_n||\} \\ \leq& ||x_n - T^n_{n(\bmod N)}x_n|| + k_1L||x_n - x_{n-1}||^\alpha \\ &+ k_1||T^{n-1}_{(n-1)(\bmod N)}x_{n-1} - x_{n-1}|| \\ &+ k_1||x_{n-1} - x_n|| \to 0 \quad (n\to\infty),\end{aligned}$$

which implies that

$$||x_n - T_{n(\bmod N)}x_n|| \to 0 \quad (n\to\infty) \tag{2.19}$$

and so, from (2.10) and (2.19), it follows that, for any $j = 1, 2, \dots, N$,

$$\begin{aligned}||x_n - T_{n(\bmod N)+j}x_n|| \leq& ||x_n - x_{n+j}|| + ||x_{n+j} - T_{n(\bmod N)+j}x_{n+j}|| \\ &+ ||T_{n(\bmod N)+j}x_{n+j} - T_{n(\bmod N)+j}x_n|| \\ \leq& ||x_n - x_{n+j}|| + ||x_{n+j} - T_{n(\bmod N)+j}x_{n+j}|| \\ &+ k_1||x_{n+j} - x_n|| \to 0 \quad (n\to\infty).\end{aligned} \tag{2.20}$$

Since E is uniformly convex, every bounded subset of E is weakly compact. Again since $\{x_n\}$ is a bounded sequence in K, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q \in K$.

Without loss of generality, we can assume that $n_k = j(\text{mod}\, N)$, where j is some positive integer in $\{1, 2, \dots, N\}$. Otherwise, we can take a subsequence $\{x_{n_{k_i}}\} \subset \{x_{n_k}\}$ such that $n_{k_i} = j(\text{mod}\, N)$. For any $l \in \{1, 2, \dots, N\}$, there exists an integer $i \in \{1, 2, \dots, N\}$ such that $n_k + i = l(\text{mod}\, N)$. Hence, from (2.20) we have

$$\lim_{k\to\infty} ||x_{n_k} - T_l x_{n_k}|| = 0. \tag{2.21}$$

By Lemma 1.1, we know that $q \in F(T_l)$. By the arbitrariness of $l \in \{1, 2, \dots, N\}$, we know that $q \in F = \bigcap_{j=1}^{N} F(T_j)$.

Finally, we prove that the sequence $\{x_n\}$ converges weakly to q. In fact, suppose this not true. Then there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_1 \in K$ and $q_1 \neq q$. Then by the same method as given above, we can also prove that $q_1 \in F = \bigcap_{j=1}^{N} F(T_j)$.

Taking $p = q$ and $p = q_1$ and using the same method given in the proof of (1.8), we can prove that the following two limits exist and

$$\lim_{n\to\infty} ||x_n - q|| = d_1, \quad \lim_{n\to\infty} ||x_n - q_1|| = d_2$$

where d_1 and d_2 are two nonnegative numbers. By virtue of the Opial condition of E, we have

$$\begin{aligned} d_1 &= \limsup_{n_k\to\infty} ||x_{n_k} - q|| < \limsup_{n_k\to\infty} ||x_{n_k} - q_1|| = d_2 \\ &= \limsup_{n_j\to\infty} ||x_{n_j} - q_1|| < \limsup_{n_j\to\infty} ||x_{n_j} - q|| = d_1. \end{aligned}$$

This is a contradiction. Hence $q_1 = q$. This implies that $\{x_n\}$ converges weakly to q. This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let E be a real uniformly convex Banach space satisfying Opial condition, K be a nonempty closed convex subset of E, $\{T_1, T_2, \dots, T_N\}: K \to K$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$, $\{k_n\}$ be the sequence defined by (1.1) and $\sigma = \sup_{n\geq 1} k_n \geq 1$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} (k_n - 1)\alpha_n < \infty$;

(ii) $0 < \tau_1 = \inf\{\alpha_n : n \geq 1\} \leq \sup\{\alpha_n : n \geq 1\} = \tau_2 < \dfrac{1}{\sigma^2}$;

(iii) *there exists constants $L > 0$ and $\alpha > 0$ such that, for any $i, j \in \{1, 2, \dots, N\}$ with $i \neq j$,*

$$||T_i^n x - T_j^n y|| \leq L||x - y||^{\alpha}, \quad \forall n \geq 1,$$

for all $x, y \in K$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.3) converges weakly to a common fixed point of $\{T_1, T_2, \dots, T_N\}$.

Proof. Taking $\beta_n = 0$, $u_n = v_n = 0$, $\forall n \geq 1$ in Theorem 2.2, then the conclusion of Theorem 2.3 can be obtained from Theorem 2.2 immediately. This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E with $K + K \subset K$, $\{T_1, T_2, \ldots, T_N\}: K \to K$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and there exist an T_j, $1 \leq j \leq N$, which is semi-compact (without loss of generality, we can assume that T_1 is semi-compact). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$, $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in K, $\{k_n\}$ be the sequence defined by (1.1) and $\sigma = \sup_{n\geq 1} k_n \geq 1$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}(k_n - 1)\alpha_n < \infty$;

(ii) $0 < \tau_1 = \inf\{\alpha_n : n \geq 1\} \leq \sup\{\alpha_n : n \geq 1\} = \tau_2 < \dfrac{1}{\sigma^2}$;

(iii) $0 \leq \mu = \sup\{\beta_n : n \geq 1\} < \dfrac{1}{\sigma}$;

(iv) $\sum_{n=1}^{\infty} ||u_n|| < \infty, \quad \sum_{n=1}^{\infty} ||v_n|| < \infty$;

(v) *there exists constants $L > 0$ and $\alpha > 0$ such that, for any $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$,*
$$||T_i^n x - T_j^n y|| \leq L||x - y||^{\alpha}, \quad \forall n \geq 1,$$
for all $x, y \in K$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.2) converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$ in K.

Proof. For any given $p \in F = \bigcap_{i=1}^N F(T_i)$, by the same method as given in proving (1.8) and (2.21), we can prove that
$$\lim_{n\to\infty} ||x_n - p|| = d, \tag{2.22}$$
where $d \geq 0$ is some nonnegative number, and
$$\lim_{k\to\infty} ||x_{n_k} - T_l x_{n_k}|| = 0, \quad \forall l = 1, 2, \ldots, N. \tag{2.23}$$
Especially, we have
$$\lim_{k\to\infty} ||x_{n_k} - T_1 x_{n_k}|| = 0. \tag{2.24}$$
By the assumption, T_1 is semi-compact, therefore it follows from (2.24) that there exists a subsequence $\{x_{n_{k_i}}\} \subset \{x_{n_k}\}$ such that $x_{n_{k_i}} \to x^* \in K$. Hence from (2.23) we have that
$$||x^* - T_l x^*|| = \lim_{k_i\to\infty} ||x_{n_{k_i}} - T_l x_{n_{k_i}}|| = 0, \quad \forall l = 1, 2, \ldots, N,$$

which implies that

$$x^* \in F = \bigcap_{i=1}^{N} F(T_i).$$

Take $p = x_*$ in (1.8), similarly we can prove that

$$\lim_{n\to\infty} ||x_n - x^*|| = d_1,$$

where $d_1 \geq 0$ is some nonnegative number. From $x_{n_{k_i}} \to x^*$ we know that $d_1 = 0$, i.e., $x_n \to x^*$. This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E, $\{T_1, T_2, \dots, T_N\}\colon K \to K$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exist an T_j, $1 \leq j \leq N$, which is semi-compact (without loss of generality, we can assume that T_1 is semi-compact). Let $\{\alpha_n\}$ be a sequence in $[0,1]$, $\{k_n\}$ be the sequence defined by (1.1) and $\sigma = \sup_{n\geq 1} k_n \geq 1$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}(k_n - 1)\alpha_n < \infty$;

(ii) $0 < \tau_1 = \inf\{\alpha_n \colon n \geq 1\} \leq \sup\{\alpha_n \colon n \geq 1\} = \tau_2 < \dfrac{1}{\sigma^2}$;

(iii) *there exists constants $L > 0$ and $\alpha > 0$ such that, for any $i, j \in \{1, 2, \dots, N\}$ with $i \neq j$,*

$$||T_i^n x - T_j^n y|| \leq L||x - y||^{\alpha}, \quad \forall n \geq 1,$$

for all $x, y \in K$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.3) converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_N\}$ in K.

Proof. Taking $\beta_n = 0$, $u_n = v_n = 0$, $\forall n \geq 1$ in Theorem 2.4, then the conclusion of Theorem 2.5 can be obtained from Theorem 2.4 immediately. This completes the proof of Theorem 2.5. □

Remark 2.1. Since $0 \leq (k_n - 1)\alpha_n \leq k_n - 1$, therefore, it is easy to see that if condition (ii) is replaced by (ii'):

(ii') $\sum_{n=1}^{\infty}(k_n - 1) < \infty$,

then the conclusion of Theorem 2.1–2.5 all are holds.

Remark 2.2. It is pointed out Xu and Ori [14] that is unclear what assumptions on the mappings $\{T_1, T_2, \dots, T_N\}$ and / or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$. However, Theorem 2.4 and Theorem 2.5 answered this open question to some extent.

Remark 2.3. Theorem 2.2 improves and extends Theorem 3.1 of Chang and Cho [2] in its two aspects:

(1) The key condition "$\sum_{n=1}^{\infty}(k_n - 1) < \infty$" is replaced by more weak condition "$\sum_{n=1}^{\infty}(k_n - 1)\alpha_n < \infty$".
(2) The implicit iterative process $\{x_n\}$ in [2] is replaced by the more general and new implicit iterative process $\{x_n\}$ with errors defined by (1.2).

Remark 2.4. Theorem 2.2 improves and extends Theorem 1 of Zhou and Chang [15] in the following ways:

(1) The key condition "$\sum_{n=1}^{\infty}(k_n - 1) < \infty$" is replaced by more weak condition "$\sum_{n=1}^{\infty}(k_n - 1)\alpha_n < \infty$".
(2) The condition (v) in [15, Theorem 1]: there exists a constant $L > 0$ such that for any $i, j \in \{1, 2, \dots, N\}$, $i \neq j$

$$||T_i^n x - T_j^n y|| \leq L||x - y||, \quad \forall n \geq 1, \quad \forall x, y \in K$$

is replaced by the more general condition (v) in Theorem 2.2.
(3) The implicit iterative process $\{x_n\}$ in [15] is replaced by the more general and new implicit iterative process $\{x_n\}$ with errors defined by (1.2).

Remark 2.5. Theorem 2.4 improves and extends Theorem 3 of Zhou and Chang [15] in its three aspects:

(1) The mappings $\{T_1, T_2, \dots, T_N\}$: $K \to K$ be N semi-compact in [15, Theorem 3] is extended to requiring only one member T in the family $\{T_1, T_2, \dots, T_N\}$ to be semi-compact.
(2) The class of nonexpansive mappings is extended to more general asymptotically nonexpansive mappings.
(3) The implicit iterative process $\{x_n\}$ in [15] is replaced by the more general and new implicit iterative process $\{x_n\}$ with errors defined by (1.2).

Remark 2.6. Theorem 2.1–2.5 generalize and improve the main results of Bauschke [1], Halpern [6], Lions [7], Reich [9], Wittmann [13], Xu and Ori [14] in the following aspects:

(1) The class of Hilbert spaces is extended to that of Banach spaces satisfying Opial's or semi-compactness condition.
(2) The class of nonexpansive mappings is extended to that of asymptotically nonexpansive mappings.
(3) The implicit iterative process $\{x_n\}$ is replaced by the more general and new implicit iterative process $\{x_n\}$ with errors defined by (1.2).

(4) The methods of the proofs used in this paper are different from those of used in the papers of [1], [6], [7], [9], [13] and [14].

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