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ON A JENSEN TYPE FUNCTIONAL EQUATION

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Abstract. We investigate the functional equation

$$\begin{aligned} &af \quad \frac{x+y+z}{3} \quad + f(x) + f(y) + f(z) \\ &= b\left[f \quad \frac{x+y}{2} \quad + f \quad \frac{y+z}{2} \quad + f \quad \frac{z+x}{2} \quad \right] \end{aligned} \tag{1}$$

for $f\colon M \to S$, where M is an Abelian 2- and 3-divisible group and S is an abstract convex cone. The motivation for studying this equation came from results due to Tiberiu Trief [8] and Young Whan Lee [3], where equation (1) was considered with constants $a = 3$, $b = 2$ and $a = 9$ and $b = 4$, respectively.

1. Let $(M, +)$ be an Abelian group in which the unique division by 2 and 3 is performable. Let $(S, +)$ be an Abelian semigroup. Suppose that S contains the identity element 0 and for each $\lambda \geq 0$ and $s \in S$, an element $\lambda \cdot s$ in S is defined. It is assumed that the multiplication $[0, \infty) \times S \ni (\lambda, s) \longmapsto \lambda \cdot s \in S$ satisfies the following axioms:

$$\begin{aligned} &1 \cdot s = s, && \lambda(\mu \cdot s) = (\lambda\mu) \cdot s, \\ &\lambda \cdot (s+t) = \lambda \cdot s + \lambda \cdot t, && (\lambda + \mu) \cdot s = \lambda \cdot s + \mu \cdot s \end{aligned}$$

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for all $s, t \in S$ and $\lambda, \mu \geq 0$. Then S is said to be an *abstract convex cone* (see e.g. [7]).

If $s, t, t' \in S$, $t + s = t' + s$ always implies that $t = t'$, then S is said to satisfy the *cancellation law*.

Suppose that an invariant with respect to translations and positively homogeneous metric ϱ is given in S, i.e.,

$$\varrho(t + s, t' + s) = \varrho(t, t')$$

and

$$\varrho(\lambda s, \lambda t) = \lambda \varrho(s, t)$$

for $\lambda > 0$ and $s, t, t' \in S$.

It is easy to see that the mappings $[0, \infty) \times S \ni (\lambda, s) \longmapsto \lambda \cdot s \in S$ and $S \times S \ni (s, t) \longmapsto s + t \in S$ are continuous in the metric topology.

We are going to examine functional equation (1) where a, b are non-negative constants and f is an unknown function defined in M with values in S. Equation (1) in the case $a = 3$, $b = 2$ was studied in the paper of Tiberiu Trif [8] in the class of functions $f\colon X \to Y$, where X and Y are real vector spaces. For the same a and b equation (1) was considered in [6] for functions $f\colon M \to S$. In paper [6] it has been shown that every solution of the equation

$$\begin{gathered} 3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) = \\ 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right] \end{gathered} \tag{2}$$

has to be of the form

$$f(x) = f(0) + a(x), \tag{3}$$

where $a\colon M \to S$ is an additive function. In the case $a = 9$ and $b = 4$ equation (1) was considered in paper [3] of Yong Whan Lee also in the class of functions $f\colon X \to Y$, where X, Y are real vector spaces.

One could believe that the natural domain of equation (2) is a convex set. The following example shows that there are solutions $f\colon M \to S$ of (2) which does not have to be of form (3). We take $f(x) = [0, 1-x]$ for $x \in [0, 1]$. This function has values in the convex cone $cc(\mathbb{R})$ of all non-empty convex compact subsets of $\mathbb{R}$ and is a solution of the Jensen functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}\left(f(x) + f(y)\right) \tag{4}$$

so it is a solution of (2) but it cannot be represented in the form $f(x) = f(0) + a(x)$, where a is an additive function (cf. [5]).

2. In the sequel we will assume that

(a) M is an Abelian group with zero in which the unique division by 2 and 3 is performable;
(b) S is an abstract cone satisfying the cancellation law;
(c) (S, ρ) is a complete metric space and ρ is translation invariant and positively homogeneous.

We start from the following lemma.

Lemma 1. *If $\lambda \cdot s = \mu \cdot s$ for some $\lambda, \mu \geq 0$ and $s \in S$, then $s = 0$ or $\lambda = \mu$.*

Proof. If $\lambda \neq \mu$, for example $\lambda > \mu$, then $\lambda \cdot s = ((\lambda - \mu) + \mu) \cdot s = (\lambda - \mu) \cdot s + \mu \cdot s$, whence $(\lambda - \mu) \cdot s = 0$. Thus $0 = \rho\left((\lambda - \mu) \cdot s, 0\right) = (\lambda - \mu)\rho(s, 0)$. Since $\lambda \neq \mu$, $\rho(s, 0) = 0$ and $s = 0$ follows. □

Of course, the zero function is a solution of (1).

Proposition. *If $f \colon M \to S$ is a non-zero solution of* (1)*, then*

$$a = 3(b - 1). \tag{5}$$

Proof. There exists an $x_0 \in M$ such that $f(x_0) \neq 0$. Setting $x = y = z = x_0$ in (1) we obtain

$$(a + 3)f(x_0) = 3bf(x_0).$$

Thus by Lemma 1 formula (5) follows. □ □

Since a, b are non-negative constants, $b \geq 1$.

Theorem 1. *Let $a = 3(b - 1)$. If $f \colon M \to S$ is a solution of* (4)*, then f satisfies* (1)*. Conversely, if $b \geq 1$, $b \neq 4$, and $f \colon M \to S$ is a solution of* (1)*, then f satisfies* (4)*.*

Proof. Suppose that $f \colon M \to S$ is a solution of (4). Since $f\left((1/3)(x + y + z)\right) = (1/3)\left(f(x) + f(y) + f(z)\right)$,

$$3(b - 1)f\left(\frac{x + y + z}{3}\right) + f(x) + f(y) + f(z) = b\left[f(x) + f(y) + f(z)\right].$$

The right hand side of (1), in virtue of (4), is also equal to $b\left[f(x) + f(y) + f(z)\right]$, so the first statement of the theorem follows.

Now, assume that $b = 1$ and that f is a solution of (1). Then $a = 0$ and f satisfies the equation

$$f(x) + f(y) + f(z) = f\left(\frac{x + y}{2}\right) + f\left(\frac{y + z}{2}\right) + f\left(\frac{z + x}{2}\right). \tag{6}$$

By letting $y = z = 0$ in (6), we infer

$$f(x) + f(0) = 2f\left(\frac{x}{2}\right).$$

Next putting $z = 0$ in (6) we obtain hence

$$\begin{aligned} f(x) + f(y) + f(0) &= f\left(\frac{x+y}{2}\right) + f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right) \\ &= f\left(\frac{x+y}{2}\right) + \frac{1}{2}f(x) + \frac{1}{2}f(0) + \frac{1}{2}f(y) + \frac{1}{2}f(0) \\ &= f\left(\frac{x+y}{2}\right) + \frac{1}{2}f(x) + \frac{1}{2}f(y) + f(0). \end{aligned}$$

Consequently f satisfies (4).

Now suppose that $b > 1$. Let us assume that f is a solution of (1) and write

$$g(x) := \frac{1}{2}\left(f(x) + f(-x)\right), \quad x \in M.$$

Of course g is a solution of (1), i.e.,

$$\begin{aligned} &3(b-1)g\left(\frac{x+y+z}{3}\right) + g(x) + g(y) + g(z) \\ &= b\left[g\left(\frac{x+y}{2}\right) + g\left(\frac{y+z}{2}\right) + g\left(\frac{z+x}{2}\right)\right], \end{aligned} \tag{7}$$

g is even and $g(0) = f(0)$.

We note that $x = y + z = 0$ in (7) gives

$$3(b-1)g(0) + g(0) + 2g(y) = 2bg\left(\frac{y}{2}\right) + bg(0).$$

Hence

$$2(b-1)g(0) + bg(0) + 2g(y) = 2bg\left(\frac{y}{2}\right) + bg(0)$$

and after cancelling $bg(0)$ we have

$$(b-1)g(0) + g(y) = bg\left(\frac{y}{2}\right),$$

whence

$$g(y) = \frac{1}{b}g(2y) + \left(1 - \frac{1}{b}\right)g(0). \tag{8}$$

Inserting $2y$ instead of y we get

$$g(2y) = \frac{1}{b}g(2^2y) + \left(1 - \frac{1}{b}\right)g(0).$$

We substitute the last equality into (8) to obtain

$$g(y) = \frac{1}{b^2} g(2^2 y) + \left(1 - \frac{1}{b^2}\right) g(0).$$

By induction

$$g(y) = \frac{1}{b^n} g(2^n y) + \left(1 - \frac{1}{b^n}\right) g(0) \tag{9}$$

for all $y \in M$ and $n \in \mathbb{N}$.

We will prove that the sequence $((1/b^n) g(2^n y))$ satisfies the Cauchy condition. For every positive integers m and n we have by (9)

$$\begin{aligned}
&\varrho\left(\frac{1}{b^{m+n}} g\left(2^{m+n} y\right), \frac{1}{b^n} g\left(2^n y\right)\right) \\
&= \varrho\left(\frac{1}{b^{m+n}} g\left(2^{m+n} y\right) + \left(1 - \frac{1}{b^{m+n}}\right) g(0), \frac{1}{b^n} g\left(2^n y\right) + \left(1 - \frac{1}{b^n}\right) g(0)\right. \\
&\left.+ \left(\frac{1}{b^n} - \frac{1}{b^{m+n}}\right) g(0)\right) = \varrho\left(g(y), g(y) + \left(\frac{1}{b^n} - \frac{1}{b^{m+n}}\right) g(0)\right) \\
&= \varrho\left(0, \left(\frac{1}{b^n} - \frac{1}{b^{m+n}}\right) g(0)\right) = \left(\frac{1}{b^n} - \frac{1}{b^{m+n}}\right) \varrho\left(0, g(0)\right).
\end{aligned}$$

Thus we may define the function $G\colon M \to S$ as follows

$$G(y) = \lim_{n\to\infty} \frac{1}{b^n} g\left(2^n y\right). \tag{10}$$

Of course $G(0) = 0, G$ is also even and by (9)

$$g(y) = g(0) + G(y) = f(0) + G(y), \quad y \in M. \tag{11}$$

Setting $2^n x, 2^n y, 2^n z$ instead of x, y, z, respectively, in (7) we obtain on letting $n \to \infty$,

$$\begin{aligned}
&3(b-1) G\left(\frac{x+y+z}{3}\right) + G(x) + G(y) + G(z) \\
&= b\left[G\left(\frac{x+y}{2}\right) + G\left(\frac{y+z}{2}\right) + G\left(\frac{z+x}{2}\right)\right],
\end{aligned} \tag{12}$$

i.e., G is a solution of (1). By (10)

$$G(2y) = \lim_{n\to\infty} \frac{1}{b^n} g\left(2^{n+1} y\right) = b \lim_{n\to\infty} \frac{1}{b^{n+1}} g\left(2^{n+1} y\right) = bG(y).$$

Thus (12) becomes

$$\begin{aligned}
&3(b-1) G\left(\frac{x+y+z}{3}\right) + G(x) + G(y) + G(z) \\
&= G(x+y) + G(y+z) + G(z+x).
\end{aligned} \tag{13}$$

If we put $y = z = 0$, then we obtain the condition

$$3(b-1)G\left(\frac{x}{3}\right) = G(x), \quad x \in M. \tag{14}$$

This equality and (13) lead to

$$G(x+y+z) + G(x) + G(y) + G(z) = G(x+y) + G(y+z) + G(z+x).$$

Taking $y + z = 0$ we obtain

$$2G(x) + 2G(y) = G(x+y) + G(x-y), \quad x, y \in M,$$

so G is a quadratic function. By (14)

$$3(b-1)G(x) = G(3x), \quad x \in M.$$

On the other hand $G(3x) = 9G(x)$ for each $x \in M$. Thus $3G(x) = (b-1)G(x)$, $x \in M$. Note that by the assumption $b \neq 4$. Consequently by Lemma 1, $G = 0$ in M. So by (11) we get $g(y) = f(0)$ for $y \in M$, whence $f(x) + f(-x) = 2f(0)$ for all $x \in M$.

Putting $z = -y$ in (1) we obtain

$$3(b-1)f\left(\frac{x}{3}\right) + f(x) + f(y) + f(-y) = b\left[f\left(\frac{x+y}{2}\right) + f(0) + f\left(\frac{x-y}{2}\right)\right].$$

Hence

$$3(b-1)f\left(\frac{x}{3}\right) + f(x) + 2f(0) = b\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right)\right] + bf(0).$$

We observe that the left-hand side does not depend on y. So setting $y = 0$ in the above equality and comparing the right-hand sides we derive

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = 2f\left(\frac{x}{2}\right), \quad x, y \in M.$$

Consequently f is a solution of the Jensen functional equation (4). □

It is not difficult to check that every quadratic function $q\colon M \to S$ is a solution of the equation

$$9f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) = 4\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]. \tag{15}$$

Analysing the proof of Theorem 1 one may obtain the following result.

Theorem 2. *If $f\colon M \to S$ is a function, then f is an even solution of* (15) *if and only if there exists a quadratic function $q\colon M \to S$ such that*

$$f(x) = f(0) + q(x), \quad x \in M.$$

3. We do not know if every solution $f\colon M \to S$ of (15) has to be of the form $f(x) = f(0) + a(x) + q(x)$, $x \in M$, where $a\colon M \to S$ is an additive function and $q\colon M \to S$ is a quadratic one. To obtain more informations about solutions of equation (15) we will embed the abstract convex cone S into a real vector space. We use the idea of H. Rådström (cf. [4]).

The equivalence relation in $S \times S = S^2$ is defined as follows

$$(s,t) \sim (u,v) \Leftrightarrow s + v = t + u.$$

The equvalence class containing a pair (s,t) is denoted by $[s,t]$. The quotient space $S^2/\sim$ is denoted by X. We define the addition in X by the formula

$$[s,t] + [u,v] = [s+u, t+v]$$

and if $\lambda \geq 0$, then

$$\lambda[s,t] = [\lambda \cdot s, \lambda \cdot t]$$

while if $\lambda < 0$, then

$$\lambda[s,t] = [-\lambda \cdot t, -\lambda \cdot s].$$

With these operations the set X becomes a real vector space.

Suppose that $f\colon M \to S$ is a solution of (15). It is easy to check that the function $F\colon M \to X$

$$F(x) := [f(x), f(0)] \tag{16}$$

is a solution of the equation

$$\begin{aligned} &9F\left(\frac{x+y+z}{3}\right) + F(x) + F(y) + F(z) \\ &= 4\left[F\left(\frac{x+y}{2}\right) + F\left(\frac{y+z}{2}\right) + F\left(\frac{z+x}{2}\right)\right]. \end{aligned} \tag{17}$$

Conversely, if F given by (16) satisfies (17), then f is a solution of (15).

Theorem 3. *If $f\colon M \to S$ is a solution of* (15)*, then functions $a, q\colon M \to X$ defined as follows*

$$a(x) = \frac{1}{2}[f(x), f(-x)],$$
$$q(x) = \frac{1}{2}[f(x) + f(-x), 2f(0)]$$

satisfy equation (17) *and a is an additive function, q is a quadratic one. Moreover $F(x) = a(x) + q(x)$, $x \in M$, where F is given by formula* (16)*.*

Proof. We observe that $a(0) = 0$ and $q(0) = 0$, where the second zero in the last equalities is zero of the vector space X. Further,

$$a(-x) = \frac{1}{2}[f(-x), f(x)] = -\frac{1}{2}[f(x), f(-x)] = -a(x)$$

and

$$q(-x) = \frac{1}{2}[f(x) + f(-x), 2f(0)] = q(x).$$

Now we are going to show that a is a solution of (17). By the definition of operations "+" and "·" we have

$$\begin{aligned}
&18a\left(\frac{x+y+z}{3}\right) + 2a(x) + 2a(y) + 2a(z) \\
&= 9\left[f\left(\frac{x+y+z}{3}\right), f\left(-\frac{x+y+z}{3}\right)\right] \\
&+ [f(x), f(-x)] + [f(y), f(-y)] + [f(z), f(-z)] \\
&= \left[9f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z),\right. \\
&\left. 9f\left(-\frac{x+y+z}{3}\right) + f(-x) + f(-y) + f(-z)\right] \\
&= 4\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right),\right. \\
&\left. f\left(-\frac{x+y}{2}\right) + f\left(-\frac{y+z}{2}\right) + f\left(-\frac{z+x}{2}\right)\right] \\
&= 4\left[f\left(\frac{x+y}{2}\right), f\left(-\frac{x+y}{2}\right)\right] \\
&+ 4\left[f\left(\frac{y+z}{2}\right), f\left(-\frac{y+z}{2}\right)\right] + 4\left[f\left(\frac{z+x}{2}\right), f\left(-\frac{z+x}{2}\right)\right] \\
&= 8a\left(\frac{x+y}{2}\right) + 8a\left(\frac{y+z}{2}\right) + 8a\left(\frac{z+x}{2}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&9a\left(\frac{x+y+z}{3}\right) + a(x) + a(y) + a(z) \\
&= 4a\left(\frac{x+y}{2}\right) + 4a\left(\frac{y+z}{2}\right) + 4a\left(\frac{z+x}{2}\right).
\end{aligned} \tag{18}$$

In order to prove that a is additive at first we put $y+z=0$ in (18). Then

$$9a\left(\frac{x}{3}\right) + a(x) = 4a\left(\frac{x+y}{2}\right) + 4a\left(\frac{x-y}{2}\right). \tag{19}$$

Next, putting $y = x$ in (19) we have

$$9a\left(\frac{x}{3}\right) = 3a(x).$$

Thus

$$a(3x) = 3a(x), \quad x \in M. \tag{20}$$

Further letting $x = 3y$ in (19) we get

$$9a(y) + a(3y) = 4a(2y) + 4a(y).$$

This equality and (20) imply

$$a(2y) = 2a(y), \quad y \in M. \tag{21}$$

From (18) taking into account (20) and (21) we obtain

$$3a(x+y+z) + a(x) + a(y) + a(z) = 2a(x+y) + 2a(y+z) + 2a(z+x).$$

For $z = -x - y$ we have hence

$$a(x) + a(y) - a(x+y) = 2a(x+y) - 2a(x) - 2a(y),$$

whence the additivity of a follows.

Now we will show that q is also a solution of (17). To see this note that

$$\begin{aligned}
&18q\left(\frac{x+y+z}{3}\right) + 2q(x) + 2q(y) + 2q(z)\\
&= 9\left[f\left(\frac{x+y+z}{3}\right) + f\left(-\frac{x+y+z}{3}\right), 2f(0)\right]\\
&+ [f(x) + f(-x), 2f(0)] + [f(y) + f(-y), 2f(0)] + [f(z) + f(-z), 2f(0)]\\
&= \left[9f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) + 9f\left(-\frac{x+y+z}{3}\right)\right.\\
&\left. + f(-x) + f(-y) + f(-z), 24f(0)\right]\\
&= \left[4f\left(\frac{x+y}{2}\right) + 4f\left(\frac{y+z}{2}\right) + 4f\left(\frac{z+x}{2}\right)\right.\\
&\left. + 4f\left(-\frac{x+y}{2}\right) + 4f\left(-\frac{y+z}{2}\right) + 4f\left(-\frac{z+x}{2}\right), 24f(0)\right]\\
&= 4\left[f\left(\frac{x+y}{2}\right) + f\left(-\frac{x+y}{2}\right), 2f(0)\right]\\
&+ 4\left[f\left(\frac{y+z}{2}\right) + f\left(-\frac{y+z}{2}\right), 2f(0)\right]\\
&+ 4\left[f\left(\frac{z+x}{2}\right) + f\left(-\frac{z+x}{2}\right), 2f(0)\right]\\
&= 8q\left(\frac{x+y}{2}\right) + 8q\left(\frac{y+z}{2}\right) + 8q\left(\frac{z+x}{2}\right).
\end{aligned}$$

Thus q satisfies the functional equation

$$\begin{aligned}
&9q\left(\frac{x+y+z}{3}\right) + q(x) + q(y) + q(z)\\
&= 4q\left(\frac{x+y}{2}\right) + 4q\left(\frac{y+z}{2}\right) + 4q\left(\frac{z+x}{2}\right).
\end{aligned} \tag{22}$$

Now we proceed to show that q is quadratic. We substitute $x+z=0$ in (22) to obtain

$$9q\left(\frac{y}{3}\right)+2q(x)+q(y)=4q\left(\frac{x+y}{2}\right)+4q\left(\frac{y-x}{2}\right). \tag{23}$$

Taking $y=0$ in (23) one has

$$q(x)=4q\left(\frac{x}{2}\right),$$

whence the relation

$$q(2x)=4q(x) \tag{24}$$

follows. Further putting $x=0$ in (23) leads to the relation

$$9q\left(\frac{y}{3}\right)+q(y)=8q\left(\frac{y}{2}\right).$$

whence by (24)

$$q(3y)=9q(y). \tag{25}$$

Finally, with (24) and (25), equality (23) becomes

$$2q(x)+2q(y)=q(x+y)+q(x-y), \tag{26}$$

i.e., q is a quadratic function.

At the end notice that

$$\begin{aligned}&a(x)+q(x)=\frac{1}{2}\left[f(x),f(-x)\right]+\frac{1}{2}\left[f(x)+f(-x),2f(0)\right]\\&=\frac{1}{2}\left[2f(x)+f(-x),f(-x)+2f(0)\right]=\frac{1}{2}[2f(x),2f(0)]\\&=[f(x),f(0)]=F(x).\end{aligned}$$

□

Theorem 4. *A function $f\colon M\to S$ is a solution of* (15) *if and only if f satisfies the system of the functional equations*

$$f(x+y)+f(-x)+f(-y)=f(-x-y)+f(x)+f(y) \tag{27}$$

$$2f(0)+f(x+y)+f(x-y)=2f(x)+f(y)+f(-y). \tag{28}$$

Proof. Assume that f is a solution of (15). Theorem 3 says that

$$a(x)=\frac{1}{2}[f(x),f(-x)]$$

is an additive function what means $a(x+y)=a(x)+a(y)$ for $x,y\in M$, or

$$[f(x+y),f(-x-y)]=[f(x),f(-x)]+[f(y),f(-y)]. \tag{29}$$

(27) is an immediate consequence of (29). Similarly

$$q(x) = \frac{1}{2}[f(x) + f(-x), 2f(0)]$$

is quadratic. Thus q satisfies equation (26). Consequently

$$\begin{aligned}&[f(x) + f(-x), 2f(0)] + [f(y) + f(-y), 2f(0)]\\ &= \frac{1}{2}[f(x+y) + f(-x-y), 2f(0)] + \frac{1}{2}[f(x-y) + f(-x+y), 2f(0)].\end{aligned}$$

The last equality may be rewritten as

$$\begin{aligned}&f(x+y) + f(-x-y) + f(x-y) + f(-x+y) + 4f(0)\\ &= 2f(x) + 2f(-x) + 2f(y) + 2f(-y)\end{aligned}$$

or

$$\begin{aligned}&f(x+y) + f(x-y) + f(-x-y) + f(x) + f(y) + f(-x+y)\\ &+ f(x) + f(-y) + 4f(0) = 4f(x) + 2f(-x) + 3f(y) + 3f(-y).\end{aligned}$$

From (27) the left-hand side of the above relation may be rewritten as

$$f(x+y)+f(x-y)+f(x+y)+f(-x)+f(-y)+f(x-y)+f(-x)+f(y)+4f(0).$$

Cancelling $2f(-x) + f(y) + f(-y)$ we obtain

$$2f(x+y) + 2f(x-y) + 4f(0) = 4f(x) + 2f(y) + 2f(-y),$$

whence (28) follows.

Conversely, if $f\colon M \to S$ is a solution of system (27)–(28), then a given by formula

$$a(x) = \frac{1}{2}[f(x), f(-x)]$$

is an additive function by (27) and

$$q(x) = \frac{1}{2}[f(x) + f(-x), 2f(0)]$$

is a quadratic one by (28) and (27). Thus both of them are solutions of equation (17). Consequently their sum

$$a(x) + q(x) = F(x) = [f(x), f(0)]$$

is also a solution of this equation, whence it follows that f satisfies (15). □

Remark 1. If f is a solution of (15), then $g(y) = (1/2)\,(f(y) + f(-y))$ is an even solution of this equation. From Theorem 2, $g(y) = g(0) + q(y)$, where q is a quadratic function. Thus (28) may be rewritten as follows

$$f(x+y) + f(x-y) = 2f(x) + 2q(y).$$

Remark 2. We are not able to solve the system of equations (27)–(28) in the class of functions $f\colon M \to S$. It is known that the only solutions of equation (15) in the class of functions $f\colon M \to Y$, where Y is a real vector space, are of the form

$$f(x) = b + a(x) + q(x), \tag{30}$$

where $b \in Y$, a is additive and q is quadratic (cf. [3]). Thus with respect to Theorem 4 all solutions of the system of (27)–(28) are of form (30).

4. Let X be a real Banach space and let $\mathrm{clb}(X)$ denote the hyperspace of all non-empty convex closed bounded subset of X. It is clear that $\mathrm{clb}(X)$ is an abstract convex cone with the addition given by

$$A \overset{*}{+} B = \mathrm{cl}(A+B),$$

where $\mathrm{cl}\,A$ denotes the closure of the set A, and with the multiplication λA by non-negative numbers λ. The identity element of $\mathrm{clb}(X)$ is the singleton $\{0\}$. In this convex cone the cancellation law holds true. This is a consequence of a generalization (cf. [7], also [1, Theorem II-17, p. 48]) of the Rådström's lemma (cf. [4]). The convex cone $\mathrm{clb}(X)$ may be endowed with the Hausdorff metric

$$h(A,B) = \inf\{t > 0\colon A \subset B + tK, B + tK\},$$

where K is the closed unit ball in X. The metric h is translation invariant (cf. [2]), positively homogeneous and complete (cf. [1]) in $\mathrm{clb}(X)$). Consequently all the obtained results may be transfered to the set-valued case. For example we can derive the following result from Theorem 1.

Theorem 5. *Assume that $b \geq 1$. If $F\colon M \to \mathrm{clb}(X)$ is a solution of the functional equation*

$$F\left(\frac{x+y}{2}\right) = \frac{1}{2}\left(F(x) \overset{*}{+} F(y)\right), \quad x, y \in M, \tag{31}$$

then F is a solution of the functional equation

$$\begin{aligned} &3(b-1)F\left(\frac{x+y+z}{3}\right) \overset{*}{+} F(x) \overset{*}{+} F(y) \overset{*}{+} F(z) \\ &= b\left[F\left(\frac{x+y}{2}\right) \overset{*}{+} F\left(\frac{y+z}{2}\right) \overset{*}{+} F\left(\frac{z+x}{2}\right)\right]. \end{aligned} \tag{32}$$

Conversely, if $b \geq 1$, $b \neq 4$ and $F\colon M \to \mathrm{clb}(X)$ is a solution of (32)*, then F satisfies the Jensen functional equation* (31)*.*

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