Journal of Applied Analysis
Vol. 13, No. 1 (2007), pp. 83–96

A CRITERION FOR LOCAL RESOLVABILITY OF A SPACE AND THE ω-PROBLEM

S. P. PONOMAREV

Received October 26, 2005 and, in revised form, April 27, 2006

Abstract. The notion of the resolvability of a topological space was introduced by E. Hewitt [8]. Recently it was understood that this notion is also important in the study of ω-primitives, especially in the case of nonmetrizable spaces. In the present paper a criterion for the resolvability of a topological space at a point ("local resolvability") is given. This criterion, stated in terms of oscillation and quasicontinuity, permits to conclude, for instance, that on irresolvable spaces no positive continuous real-valued function has an ω-primitive. The result is strenghtened in the case of SI-spaces. It is also shown that every nonnegative upper semicontinuous function on a resolvable Baire space has an ω-primitive.

1. Basic definitions and preliminaries

Throughout the paper only dense-in-themselves topological spaces will be considered. Let $X = (X, \tau)$ be a topological space. To each function $F\colon X \to \mathbb{R}$ one associates the upper and lower Baire functions

$$M(F, \cdot)\colon X \to \overline{\mathbb{R}}, \quad m(F, \cdot)\colon X \to \overline{\mathbb{R}} \tag{1}$$

2000 *Mathematics Subject Classification.* Primary: 54C30, 54D80.
Key words and phrases. Resolvable spaces, quasicontinuity, oscillation, ω-primitive, SI-spaces, Baire spaces.

ISSN 1425-6908

defined by

$$M(F,x) = \inf_{U(x)\in\tau_x} \sup_{\xi\in U(x)} F(\xi), \tag{2}$$

$$m(F,x) = \sup_{U(x)\in\tau_x} \inf_{\xi\in U(x)} F(\xi) \tag{3}$$

where $U(x)$ ranges over a neighborhood base τ_x for the topology τ at $x \in X$. One can easily check that these values are independent of the choice of τ_x.

It is well known that $M(F,\cdot)$ is upper semicontinuous while $m(F,\cdot)$ is lower semicontinuous on X.

Recall that a function $\varphi\colon X \to \overline{\mathbb{R}}$ is said to be upper (lower) semicontinuous if for each $a \in \overline{\mathbb{R}}$ the set $\{x \in X\colon \varphi(x) < a\}$ (resp. $\{x \in X\colon \varphi(x) > a\}$) is open.

We will use the abbreviations USC and LSC for "upper semicontinuous" and "lower semicontinuous".

Definition 1. The value

$$\omega(F,x) = M(F,x) - m(F,x) \in [0,\infty]. \tag{4}$$

is called the oscillation of F at a point x.

Regarding definition (4), we adopt the convention: $+\infty - (-\infty) = +\infty$, and $+\infty - a = +\infty$, $a - (-\infty) = +\infty$ for $a \in \mathbb{R}$. Note that (4) can also be written in the form

$$\omega(F,x) = \inf_{U(x)\in\tau_x} \sup_{x',x''\in U(x)} (F(x') - F(x'')). \tag{5}$$

Definition 2. Let $X = (X,\tau)$ be a topological space and a USC function $f\colon X \to [0,\infty]$ be given. If there exists a function $F\colon X \to \mathbb{R}$ such that

$$\forall x \in X\colon \omega(F,x) = f(x)$$

then we call F an ω-primitive for f.

Note that f may take the value $+\infty$ while, according to our definition, an ω-primitive should be finite.

By the "ω-problem" (on a topological space X) we mean the problem of the existence of an ω-primitive for a given USC function $f\colon X \to [0,\infty]$. The ω-problem was completely solved in the case of metric spaces [5]. Namely, the following results were obtained.

Theorem 1. ([5], Theorems 3, 4*) *Let $X = (X,d)$ be an arbitrary metric space and $f\colon X \to [0,\infty)$ (or $f\colon X \to [0,\infty]$) a USC function which vanishes*

*Correction: in the statement of Theorem 4 in [5] the inequality $-g < F \le f$ should be replaced by $-g < F < \infty$. The proof remains unchanged.

at each isolated point of X. Then for each LSC function $g\colon X \to (0,\infty)$ there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$ and $-g < F \le f$ (respectively: $-g < F < \infty$). Such a function F can always be found in at most Baire class 2.

It is worth noting that generally one cannot eliminate an LSC function $g > 0$ without failing Theorem 1 if the inequality $-g < F \le f$ we replace by $0 \le F \le f$. This will follow from Theorem 2. But first we recall some necessary definitions from [3].

A metric space X is called:

1) σ-discrete at a point $x \in X$ if x has a neighborhood which is a σ-discrete set;
2) locally σ-discrete if X is σ-discrete at each $x \in X$;
3) massive if X is not σ-discrete at any of its points.

It is clear that these definition are also valid for topological spaces. In [5] it was shown that

a) *Each locally σ-discrete metric space is σ-discrete.*
b) *Each metric space X can be written in the form of the disjoint union $X = \Sigma(X) \cup \mathcal{M}(X)$, where $\Sigma(X)$, $\mathcal{M}(X)$ are respectively σ-discrete and massive subspaces of X (one of them may be empty). Such a decomposition of X is unique.*

Observe that $\Sigma(X)$ is open while $\mathcal{M}(X)$ is closed and dense in itself (if nonempty).

Open problem. Do (or under what conditions) claims a), b) hold for non-metrizable spaces?

Now let us prove

Theorem 2. *Let $X = (X,d)$ be a dense-in-itself metric space. Assume that each USC function $f\colon X \to [0,\infty)$ has an ω-primitive F such that $0 \le F \le f$. Then X is a massive space.*

Proof. Assume that X is not massive. Then we have the decomposition $X = \Sigma(X) \cup \mathcal{M}(X)$ with $\Sigma(X) \neq \emptyset$. As $\Sigma(X)$ is σ-discrete, then by Lemma 2 of [3] it can be represented in the form

$$\Sigma(X) = \bigcup_{n=1}^{\infty} C_n$$

where $\Delta C_n := \inf\{d(x_1, x_2) \colon x_1, x_2 \in C_n,\ x_1 \neq x_2\} > 0$ for each $n \in \mathbb{N}$. Consider $f \colon X \to [0, \infty)$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathcal{M}(X); \\ \dfrac{n+1}{n} & \text{if } x \in C_n,\ n \in \mathbb{N}. \end{cases}$$

Since $\Delta C_n > 0$, $n \in \mathbb{N}$, we have that no point of $\Sigma(X)$ is an accumulation point of any C_n, $n \in \mathbb{N}$. It easily follows, in view of the construction of f, that

$$\forall x \in \Sigma(X) \colon f(x) - \limsup_{y \to x} f(y) > 0. \tag{6}$$

This obviously implies that f is USC on X. Note that since X is dense in itself, its open subset $\Sigma(X)$ is so too, and therefore the upper limit in (6) exists. By assumption, there is an ω-primitive F for f such that $0 \leq F \leq f$. It follows that (cf. (1))

$$\forall x \in X \colon m(F, x) = 0 \tag{7}$$

and

$$\forall x \in X \colon M(F, x) = f(x). \tag{8}$$

Note that $M(F, x) = \max\{F(x), \limsup_{y \to x} F(y)\}$. Since $F \leq f$, we get, in view of (6),

$$\forall x \in \Sigma(X) \colon \limsup_{y \to x} F(y) \leq \limsup_{y \to x} f(y) < f(x).$$

It follows by (8), that $F(x) = f(x)$ for each $x \in X$. This implies, taking into account the construction of f, that $F(x) > 1$ for each $x \in \Sigma(X)$, which contradicts (7). Our argument thus shows that there should be $\Sigma(X) = \emptyset$. Therefore $X = \mathcal{M}(X)$ what was to be shown. $\square$

Thus $0 \leq F \leq f$ cannot hold for X σ-discrete. It should also be noted that the metrizability of a space is only a sufficient condition for the existence of an ω-primitive. For instance, in [4] some special classes of nonmetrizable first countable T_1-spaces without isolated points were considered for which the question of the existence of ω-primitives is answered affirmatively.

In the present paper it will be shown, in particular, that results analogous to Theorem 1 need not hold for nonmetrizable topological spaces. This is the case of the so-called irresolvable spaces. To this end we will use Theorem 3.

To simplify notations, we will sometimes write X instead of (X, τ), if no confusion could arise. The closure of a set $E \subset X$ will be denoted by $\overline{E}$ and by $\operatorname{Int} E$ we denote the interior of E.

Definition 3 ([8])**.** A topological space X is said to be resolvable if there exists a set $S \subset X$ which is dense and boundary in X, i.e. $\overline{S} = \overline{X \setminus S} = X$. Such a set (as well as its complementary $X \setminus S$) is called a CD-set. A space which is not resolvable is called irresolvable.

It is immediate from Definition 3 that a resolvable space is dense in itself and the following are equivalent:

(a) The space X is resolvable.
(b) There exist two disjoint sets $A,\ B \subset X$ such that each of them is dense in X.

E. Hewitt pointed out in [8] that "all commonly studied" dense-in-themselves topological spaces are resolvable. In particular, he proved that first countable T_0-spaces without isolated points are resolvable. It follows, by the way, that all nonmetrizable spaces considered in [4] are resolvable. Amazingly, it was shown in [8] that irresolvable spaces do exist. Since then, numerous research papers related to that topic have appeared (see, e.g., a survey article by W. W. Comfort [1]).

Definition 4 ([8])**.** A subset E of a topological space X is said to be resolvable if E is resolvable as a topological space equipped with the subspace topology.

We introduce the following

Definition 5. A topological space X is said to be resolvable at a point $x_0 \in X$ if each open neighborhood of x_0 contains a nonempty open subset which is resolvable.

Lemma 1 ([8], Theorem 20)**.** *A topological space X is resolvable if and only if every nonempty open subset of X contains a set which is resolvable.*

This proposition immediately implies, in view of Definition 5, the following corollary we will use later.

Corollary 1. *If a topological space X is resolvable at each point of a set E dense in X then X is resolvable.*

Observe that each nonempty open subset of a resolvable space is resolvable [8].

2. A criterion for local resolvability of a space

In this section we prove a necessary and sufficient condition for the resolvability of a space at a point, what may be called as a "local resolvability". In Section 3 some consequences will be deduced from that criterion. Recall the following

Definition 6 (see, e.g., [10])**.** Let $X = (X, \tau)$ be a topological space. A function $f : X \to \mathbb{R}$ is called quasicontinuous at $x_0 \in X$ if for each $\varepsilon > 0$ and each open neighborhood $W(x_0)$ of x_0 there is a nonempty open set $U \subset W(x_0)$ such that for each $x \in U$ we have $|f(x) - f(x_0)| < \varepsilon$.

It is obvious that each continuous function is quasicontinuous.

Theorem 3 (a criterion for local resolvability)**.** *Let $X = (X, \tau)$ be a topological space. In order that X be resolvable at a point x_0, it is necessary and sufficient that the following condition be satisfied.*

There exists an open neighborhood G of x_0 and a function $F : G \to \mathbb{R}$, such that $0 < \omega(F, x_0) < \infty$ and $\omega(F, \cdot)$ is quasicontinuous at x_0.

Proof. (I) The condition is sufficient. Fix an ε, $0 < \varepsilon < \omega(F, x_0)/10$. Since $M(F, \cdot)$, $m(F, \cdot)$ are USC and LSC functions respectively, there exists an open neighborhood $W(x_0)$ of x_0, $W(x_0) \subset G$, in which these functions are finite and such that for each $x \in W(x_0)$ we have

$$M(F, x) < M(F, x_0) + \varepsilon \quad \text{and} \quad m(F, x) > m(F, x_0) - \varepsilon. \tag{9}$$

Since $\omega(F, \cdot)$ is quasicontinuous at x_0, there is a nonempty open set $U \subset W(x_0)$ such that for each $x \in U$ we have

$$|\omega(F, x) - \omega(F, x_0)| < \varepsilon. \tag{10}$$

Consider the following two sets:

$$A = \left\{ x \in U : F(x) > \frac{M(F, x) + m(F, x)}{2} \right\}, \tag{11}$$

$$B = \left\{ x \in U : F(x) < \frac{M(F, x) + m(F, x)}{2} \right\}. \tag{12}$$

Our aim is to prove that each of these disjoint sets is dense in U. First we will show that the inequalities

$$|m(F, x'') - m(F, x')| < 3\varepsilon, \tag{13}$$

$$|M(F, x'') - M(F, x')| < 3\varepsilon \tag{14}$$

hold for all x', $x'' \in U$.

1) Proof of (13). In view of (9), (10) we have that for each $x \in U$

$$\begin{aligned} m(F,x_0) - \varepsilon < m(F,x) &= m(F,x) - M(F,x) + M(F,x) \\ &< M(F,x_0) + \varepsilon - \omega(F,x) < M(F,x_0) - \omega(F,x_0) + 2\varepsilon. \end{aligned}$$

It follows that

$$\forall x \in U\colon m(F,x) \in (m(F,x_0) - \varepsilon, M(F,x_0) - \omega(F,x_0) + 2\varepsilon)$$

which, clearly, implies (13).

2) Proof of (14) is similar to the proof of (13). Indeed, using again (9), (10), we may write for each $x \in U$

$$\begin{aligned} M(F,x_0) + \varepsilon > M(F,x) &= \omega(F,x) + m(F,x) \\ &> \omega(F,x) + m(F,x_0) - \varepsilon > \omega(F,x_0) + m(F,x_0) - 2\varepsilon, \end{aligned}$$

whence it follows that

$$\forall x \in U\colon M(F,x) \in (\omega(F,x_0) + m(F,x_0) - 2\varepsilon, M(F,x_0) + \varepsilon)$$

which obviously implies (14).

Finally, let us prove that the sets A, B defined by (11), (12) are both dense in U.

3) Proof of "A is dense in U". Assume the contrary. Then there is a nonempty open set $D \subset U$ such that

$$\forall x \in D\colon m(F,x_0) - \varepsilon < m(F,x) \le F(x) \le \frac{M(F,x) + m(F,x)}{2}.$$

It follows by (9), (13) and (10) that

$$\begin{aligned} &\forall x', x'' \in D\colon F(x'') - F(x') \le \frac{M(F,x'') + m(F,x'')}{2} - m(F,x') \\ &= \frac{M(F,x'') - m(F,x'') + 2(m(F,x'') - m(F,x'))}{2} \\ &= \frac{\omega(F,x'') + 2(m(F,x'') - m(F,x'))}{2} \\ &< \frac{\omega(F,x'') + 6\varepsilon}{2} < \frac{\omega(F,x_0) + 7\varepsilon}{2}. \end{aligned}$$

Since D is open, this estimate yields (cf. (5)),

$$\forall x \in D\colon \omega(F,x) \le \frac{\omega(F,x_0) + 7\varepsilon}{2}, \tag{15}$$

and therefore, in view of (10), we get $\omega(F,x_0) \le 9\varepsilon$, a contradiction, since $\varepsilon < \omega(F,x_0)/10$. Thus A is dense in U.

4) Proof of "B is dense in U". Just as in the previous argument, assume that the claim is not true. Then there exists a nonempty open set $D_1 \subset U$ such that

$$\forall x \in D_1 \colon M(F, x_0) + \varepsilon > M(F, x) \geq F(x) \geq \frac{M(F, x) + m(F, x)}{2}.$$

It follows, in view of (9), (14) and (10), that

$$\begin{aligned}
&\forall x', x'' \in D_1 \colon F(x'') - F(x') \leq M(F, x'') - \frac{M(F, x') + m(F, x')}{2} \\
&= \frac{2(M(F, x'') - M(F, x')) + \omega(F, x')}{2} < \frac{\omega(F, x_0) + 7\varepsilon}{2}.
\end{aligned}$$

Exactly like in the preceding proof, this estimation yields

$$\forall x \in D_1 \colon \omega(F, x) \leq \frac{\omega(F, x_0) + 7\varepsilon}{2},$$

which, as we have seen, leads to a contradiction $\omega(F, x_0) \leq 9\varepsilon$, thereby proving that the set B is dense in U.

Thus we have proved that each open neighborhood $W(x_0)$ of the point x_0 contains an open resolvable subset U (observe that, given any open set $\mathcal{O} \ni x_0$, we may obviously always choose $W(x_0) \subset G \cap \mathcal{O}$).

This means that the space X is resolvable at x_0 (cf. Definition 5), what was to be shown.

(II) The condition is necessary. To prove this, denote by $\mathcal{D}$ the union of all resolvable open subsets of X. Since X is resolvable at x_0, the set $\mathcal{D}$ is obviously nonempty. It is also clear that it can be written in the form

$$\mathcal{D} = \bigcup_{s \in S} U_s, \tag{16}$$

where every U_s is open and resolvable. Observe that from the definition of $\mathcal{D}$ it follows that this set is maximal in the sense that there are no open resolvable sets which are not contained in $\mathcal{D}$.

First let us show that $\mathcal{D}$ is resolvable. Let $\Omega \subset \mathcal{D}$ be a nonempty open set. It is clear from (16) that there is $s_0 \in S$ such that $U_{s_0} \cap \Omega \neq \emptyset$. Since U_{s_0} is resolvable, the set $U_{s_0} \cap \Omega$ is resolvable too. Since Ω was an arbitrary nonempty open subset of $\mathcal{D}$, it follows by Lemma 1 that $\mathcal{D}$ is resolvable.

Next let us prove that $x_0 \in \overline{\mathcal{D}}$. Assume this is not the case. Then $x_0 \in X \setminus \overline{\mathcal{D}}$, and $X \setminus \overline{\mathcal{D}}$ is obviously an open neighborhood of x_0. Since X is assumed to be resolvable at x_0, there is a nonempty open set $V \subset X \setminus \overline{\mathcal{D}}$, which is resolvable. Since $V \cap \mathcal{D} = \emptyset$, we have that $\mathcal{D}$ is a proper subset of a resolvable open set $\mathcal{D} \cup V$, which contradicts the maximality of $\mathcal{D}$. So, we conclude that $x_0 \in \overline{\mathcal{D}}$.

Now we will construct a function $F: G \to \mathbb{R}$ with the required properties. Since $\mathcal{D}$ is resolvable, there exists a CD-set $A \subset \mathcal{D}$ (in the subspace topology of $\mathcal{D}$). Put $G = X$ and define $F: G \to \mathbb{R}$ letting

$$F(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Since A and $X \setminus A$ are both dense in $\mathcal{D}$, it is easy to see that

$$\omega(F, x) = \begin{cases} 1 & \text{if } x \in \overline{\mathcal{D}} \\ 0 & \text{if } x \notin \overline{\mathcal{D}}. \end{cases} \tag{17}$$

Since $x_0 \in \overline{\mathcal{D}}$, we have, of course, $\omega(F, x_0) = 1$, and it is trivial that $\omega(F, \cdot)$ is quasicontinuous at x_0 because each open neighborhood of x_0 intersects $\mathcal{D}$ where the oscillation of F equals 1 in view of (17). Thus F is the required function. □

3. Irresolvable spaces and ω-primitives

In this short section we deduce some almost immediate consequences of Theorem 3. In particular, we will see that on every irresolvable space X there are USC functions $f: X \to [0, \infty)$ which have no ω-primitives. In this connection we also consider SI-spaces which form a special class of irresolvable spaces.

Theorem 4. *Let $X = (X, \tau)$ be a dense-in-itself irresolvable space. Assume that there exists a USC function $f: X \to [0, \infty)$ and a dense set $E \subset X$ such that the following conditions are satisfied:*

(a) *f is quasicontinuous at each $x \in E$;*
(b) *$f(x) > 0$ for each $x \in E$.*

Then f has no ω-primitive.

In particular, no continuous function $f: X \to (0, \infty)$ has an ω-primitive.

Proof. Suppose that f has an ω-primitive $F: X \to \mathbb{R}$. Then conditions (a), (b) imply that all assumptions of Theorem 3 are satisfied for F and each $x \in E$ (with $G = X$). Therefore X is resolvable at each point $x \in E$. Since E is dense in X, we conclude by Corollary 1 that X is resolvable. This contradiction completes the proof. □

Corollary 2. *Let X be a dense-in-itself topological space. If there exists a continuous function $f: X \to (0, \infty)$ having an ω-primitive, then X is resolvable.*

Definition 7. Let X be a topological space. We say that X is ω-regular at $x_0 \in X$ if there exists an open neighborhood U of x_0 such that each USC function $f\colon U \to [0,\infty)$ has an ω-primitive $F\colon U \to \mathbb{R}$.

Observe that Definition 7 simply describes the localization of the ω-problem.

Theorem 5. *Assume that a topological space $X = (X,\tau)$ is ω-regular at each point of a set $E \subset X$ dense in X. Then X is resolvable.*

Proof. First note that X is dense in itself what is immediate from assumptions in view of Definition 7. For each $x \in E$ there is a neighborhood $U_x \in \tau_x$ in which the ω-problem is solvable. Now for each $x \in E$ we let $f\colon U_x \to [0,\infty)$ be equal to 1. Clearly, f is USC on the subspace U_x. Since X is ω-regular at x, f has an ω-primitive $F_x\colon U_x \to \mathbb{R}$. Now it is obvious that all assumptions of Theorem 3 are satisfied, hence X is resolvable at each $x \in E$. Since E is dense in X, it remains to apply Corollary 1. $\square$

Open problem. Assume that a topological space X is ω-regular at each of its points. Is the ω-problem solvable on X?

But of course the main open problem is whether the ω-problem is solvable on each resolvable space.

Definition 8 ([8])**.** A dense-in-itself topological space X is called an SI-space (or, simply, SI) if X has no resolvable subsets.

In [8] it was shown that SI-spaces exist and each SI-space is irresolvable.

Theorem 6. *Let X be an SI-space. If a USC function $f\colon X \to [0,\infty)$ is quasicontinuous and positive at some point $x_0 \in X$ then f has no ω-primitives.*

Proof. Indeed, if f had an ω-primitive then by Theorem 3, X would be resolvable at x_0 which is impossible because X is an SI-space. $\square$

4. ω-problem for resolvable Baire spaces and quasicontinuity

First we will show that each boundary subset of a resolvable space can be extended to a CD-set.

Theorem 7. *Let X be a resolvable space. Then for each boundary set $E \subset X$ there exists a CD-set $A \subset X$ such that $E \subset A$.*

Proof. If $\overline{E} = X$ then obviously it suffices to put $A = E$. So, assume that $\mathrm{Int}\,(X \setminus E) \neq \emptyset$. Since X is resolvable, $\mathrm{Int}\,(X \setminus E)$ is resolvable too. Therefore there exists a CD-set $S \subset \mathrm{Int}\,(X \setminus E)$ (in the subspace topology of $\mathrm{Int}\,(X \setminus E)$). It remains to check that $A = E \cup S$ is the required CD-set.

1) Proof of $\overline{A} = X$. Suppose this is not the case. Then there is a nonempty open set $D \subset X$ such that $D \cap A = \emptyset$ (we may take $D = X \setminus \overline{A}$). Then obviously $D \cap E = \emptyset$, whence we get

$$\emptyset \neq D \subset X \setminus \overline{E} \subset \mathrm{Int}\,(X \setminus E) \Longrightarrow D \cap S \neq \emptyset$$

(because S is dense in $\mathrm{Int}\,(X \setminus E)$). But this contradicts the equality $D \cap A = \emptyset$. Consequently, $\overline{A} = X$.

2) Proof of $\overline{X \setminus A} = X$, or, what amounts to the same, $\mathrm{Int}\ A = \emptyset$. Suppose on the contrary that $\mathrm{Int}\ A \neq \emptyset$. Then we will prove that

$$S \cap \mathrm{Int}\ A = \emptyset. \tag{18}$$

Assume that this equality does not hold. Then, since $S \subset \mathrm{Int}\,(X \setminus E)$, it follows that

$$W = \mathrm{Int}\ A \cap \mathrm{Int}\,(X \setminus E) \neq \emptyset,$$

(observe that $W \subset A = E \cup S$). We then have

$$\emptyset \neq W \subset \mathrm{Int}\ A \cap (X \setminus E) = (\mathrm{Int}\ A) \setminus E \subset A \setminus E = S,$$

a contradiction, because $\mathrm{Int}\ S = \emptyset$. Thus (18) holds.

Recall that we have supposed that $\mathrm{Int}\ A \neq \emptyset$. Then by (18) we get

$$\emptyset \neq \mathrm{Int}\ A = A \cap \mathrm{Int}\ A = (E \cup S) \cap \mathrm{Int}\ A = E \cap \mathrm{Int}\ A \subset E.$$

In other words, we have $\emptyset \neq \mathrm{Int}\ A \subset E$, whence $\mathrm{Int}\ E \neq \emptyset$, a contradiction, since E is a boundary set. Thus the equality $\overline{X \setminus A} = X$ is true too, and we conclude that A is the required CD-set. □

Theorem 8. *Let $X = (X, \tau)$ be a resolvable Baire space. Then every USC function $f : X \to [0, \infty)$ has an ω-primitive.*

Proof. Let $D(f)$ denote the set of all points at which f is not continuous. The set $D(f)$ is of first category (cf. [7]), and since X is Baire, the set $D(f)$ is boundary. By Theorem 7, there exists a CD-set $A \subset X$ which contains $D(f)$. We claim that the function $F\colon X \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \in X \setminus A \end{cases} \tag{19}$$

is an ω-primitive for f.

First of all observe that since $F \le f$, the function f is USC, and as $X \setminus A$ is dense in X, we obviously have

$$\forall x \in X\colon M(F,x) \le M(f,x) = f(x) \tag{20}$$

and

$$\forall x \in X\colon m(F,x) = 0. \tag{21}$$

1) Let $x \in A$. Since $F(x) = f(x)$, we have $M(F,x) \ge f(x)$. It follows, in view of (20), (21), that

$$\forall x \in A\colon \omega(F,x) = f(x). \tag{22}$$

2) Let $x \in X \setminus A$. Then f is continuous at x. Fix an $\varepsilon > 0$. There exists an open neighborhood U_ε of x such that

$$\forall \xi \in U_\varepsilon\colon f(\xi) > f(x) - \varepsilon. \tag{23}$$

Since A is dense in each neighborhood of x, particularly in U_ε, it is immediate from (23) that $M(F,x) \ge f(x) - \varepsilon$. Then in view of (20), we may write

$$\forall \varepsilon > 0\colon f(x) - \varepsilon \le M(F,x) \le f(x)$$

which clearly implies $M(F,x) = f(x)$. Therefore, in view of (21), we obtain $\omega(F,x) = f(x)$. We conclude that

$$\forall x \in X \setminus A\colon \omega(F,x) = f(x)$$

which, combined with (22), completes the proof. $\square$

Remark 1. The above theorem generalizes essentially Theorem 3 of [2] where it was additionally assumed that X is T_1 and each $x \in X$ has a neighborhood base τ_x well ordered by inclusion relation.

Consider the following example. Assume the ZFC theory, and let $X = (\mathbb{R}^n, \tau_d)$ where τ_d is the usual density topology. It is well known that X is Baire, Hausdorff, but not first countable [9]. Moreover, X is resolvable. Indeed, it is clear that every Bernstein set $\mathcal{B} \subset \mathbb{R}^n$ (which exists in the ZFC theory) is a CD-subset of X. Therefore by virtue of Theorem 8, we have

Corollary 3. *Every USC function* $f\colon (\mathbb{R}^n, \tau_d) \to [0, \infty)$ *has an* ω-*primitive.*

Remark 2. In [6], Theorem 2, one can find a different and direct proof of the statement of Corollary 3.

Next result shows that we can give up on the Baireness of the space in Theorem 8, but instead we will assume the quasicontinuity of f.

Theorem 9. *Let* X *be a resolvable space. Then each quasicontinuous USC function* $f\colon X \to [0, \infty)$ *has an* ω-*primitive.*

Proof. Proof is more simple than that of Theorem 8 because we need not make use of Theorem 7. Fix a CD-set $A \subset X$ and put $F = fh$ where h is the characteristic function of A. It is clear that $\omega(F, x) = f(x)$ at each $x \in A$ (the argument is the same as in 1) of Theorem 8). So, let $x \in X \setminus A$. Fix an $\varepsilon > 0$. Since f is quasicontinuous, and A is dense in X, it is obvious that every neighborhood U of x contains a point $y \in A$ for which $f(y) > f(x) - \varepsilon$. This yields $M(F, x) \geq f(x) - \varepsilon$ and since ε was arbitrary, we have $M(F, x) \geq f(x)$. Since the reverse inequality $M(F, x) \leq f(x)$ is trivial, we conclude that $M(F, x) = f(x)$. But the equality $m(F, x) = 0$ holds for each $x \in X$ because $X \setminus A$ is dense in X. Therefore $\omega(F, x) = f(x)$ for $x \in X \setminus A$. We conclude that $F = fh$ is an ω-primitive for f. □

Remark 3. Theorem 9 generalizes, for instance, Theorem 1 of [2] which was proved for continuous f.

References

[1] Comfort, W. W., *Edwin Hewitt as topologist: An appreciation*, Topological Commentatory **6**(1) (2001), available at Topology Atlas `http://at.yorku.ca/topology/`.

[2] Di Bari, C., Vetro, C., *Primitive rispetto all'oscillazione*, Rend. Circ. Mat. Palermo (2) **51** (2002), 175–178.

[3] Duszyński, Z., Grande, Z., Ponomarev, S. P., *On the* ω*-primitive*, Math. Slovaca **51** (2001), 469–476.

[4] Ewert, J., Ponomarev, S. P., *Oscillation and* ω*-primitives*, Real Anal. Exchange **26**(2) (2001/2002), 687–702.

[5] Ewert, J., Ponomarev, S. P., *On the existence of* ω*-primitives on arbitrary metric spaces*, Math. Slovaca **53**(1) (2003), 51–57.

[6] Ewert, J., Ponomarev, S. P., *On the convergence of* ω*-primitives*, Math. Slovaca **53**(1) (2003), 59–66.

[7] Fort, M. K., Jr., *Category theorems*, Fund. Math. **42** (1955), 276–288.

[8] Hewitt, E., *A problem of set-theoretic topology*, Duke Math. J. **10** (1943), 309–333.

[9] Lukeš, J., Malý, J., Zajíček, L., *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Math. **1189**, Springer-Verlag, Berlin, 1986.

[10] Neubrunn, T., *Quasi-continuity*, Real Anal. Exchange **14** (1988/1989), 259–306.

S. P. Ponomarev
Institute of Mathematics
Pedagogical University
Arciszewskiego 22b
76-200 Słupsk, Poland
e-mail: stapon@pap.edu.pl