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DIFFERENTIAL POLYNOMIALS GENERATED BY SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study fixed points of solutions of the differential equation

$$f'' + A_1(z) f' + A_0(z) f = 0,$$

where $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ are transcendental meromorphic functions with finite order. Instead of looking at the zeros of $f(z) - z$, we proceed to a slight generalization by considering zeros of $g(z) - \varphi(z)$, where g is a differential polynomial in f with polynomial coefficients, φ is a small meromorphic function relative to f, while the solution f is of infinite order.

1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory and with the basic Wiman Valiron theory as well (see

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[7], [8], [10], [13], [14]). In addition, we will use $\lambda(f)$ and $\lambda(1/f)$ to denote respectively the exponents of convergence of the zero-sequence and the pole-sequence of a meromorphic function f, $\rho(f)$ to denote the order of growth of f, $\bar{\lambda}(f)$ and $\bar{\lambda}(1/f)$ to denote respectively the exponents of convergence of the sequence of distinct zeros and distinct poles of f. A meromorphic function $\varphi(z)$ is called a small function of a meromorphic function $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$, where $T(r,f)$ is the Nevanlinna characteristic function of f. In order to express the rate of growth of meromorphic solutions of infinite order, we recall the following definition.

Definition 1.1 ([2], [12], [16])**.** Let f be a meromorphic function. Then the hyper order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \varlimsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}. \tag{1.1}$$

Definition 1.2 ([2], [9], [12])**.** Let f be a meromorphic function. Then the hyper exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\bar{\lambda}_2(f) = \varlimsup_{r\to+\infty} \frac{\log\log \bar{N}\left(r,\dfrac{1}{f}\right)}{\log r}, \tag{1.2}$$

where $\bar{N}(r,1/f)$ is the counting function of distinct zeros of $f(z)$ in $\{|z| < r\}$.

Consider the second order linear differential equation

$$f'' + A_1(z) f' + A_0(z) f = 0, \tag{1.3}$$

where $A_j(z)\ (\not\equiv 0)\ (j = 0,1)$ are transcendental meromorphic functions with finite order. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [17]). However, there are a few studies on the fixed points of solutions of differential equations. It was in year 2000 that Z. X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients (see [2]). In [15], Wang and Yi investigated fixed points and hyper order of differential polynomials generated by solutions of second order linear differential equations with meromorphic coefficients. In [11], Laine and Rieppo gave improvement of the results of [15] by considering fixed points and iterated order. In [3], Chen Zongxuan and Shon Kwang Ho have studied the differential equation

$$f'' + A_1(z) e^{az} f' + A_0(z) e^{bz} f = 0 \tag{1.4}$$

and have obtained the following results:

Theorem A ([3])**.** *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0,1)$ *be meromorphic functions with* $\rho(A_j) < 1$ $(j = 0,1)$, *a, b be complex numbers such that* $ab \neq 0$ *and* $\arg a \neq \arg b$ *or* $a = cb$ $(0 < c < 1)$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of the equation* (1.4) *has infinite order.*

In the same paper, Z. X. Chen and K. H. Shon have investigated the fixed points of solutions, their 1^{st}, 2^{nd} derivatives and differential polynomial and have obtained:

Theorem B ([3])**.** *Let* $A_j(z)$ $(j = 0,1)$, *a, b, c satisfy the additional hypotheses of Theorem A. Let* d_0, d_1, d_2 *be complex constants that are not all equal to zero. If* $f(z) \not\equiv 0$ *is any meromorphic solution of equation* (1.4), *then:*

(i) f, f', f'' *all have infinitely many fixed points and satisfy*

$$\bar{\lambda}(f - z) = \bar{\lambda}(f' - z) = \bar{\lambda}(f'' - z) = +\infty,$$

(ii) *the differential polynomial*

$$g(z) = d_2 f'' + d_1 f' + d_0 f$$

has infinitely many fixed points and satisfies $\bar{\lambda}(g - z) = +\infty$.

In this paper, we study the relation between the small functions and solutions of equation (1.3) in the case when all meromorphic solutions are of infinite order and we obtain the following results:

Theorem 1.1. *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0,1)$ *be transcendental meromorphic functions with finite order such that all meromorphic solutions of equation* (1.3) *are of infinite order, let* d_j $(j = 0,1,2)$ *be polynomials that are not all equal to zero,* $\varphi(z)$ $(\not\equiv 0)$ *be a meromorphic function of finite order satisfying*

$$(d_1 - d_2 A_1)\varphi' - \left(d_2 A_1^2 - (d_2 A_1)' - d_2 A_0 - d_1 A_1 + d_0 + d_1'\right)\varphi \not\equiv 0. \quad (1.5)$$

If $f \not\equiv 0$ *is a meromorphic solution of equation* (1.3) *with* $\lambda(1/f) < +\infty$, *then the differential polynomial* $g(z) = d_2 f'' + d_1 f' + d_0 f$ *satisfies* $\bar{\lambda}(g - \varphi) = +\infty$.

Theorem 1.2. *Suppose that* $A_j(z)$ $(\not\equiv 0)$ $(j = 0,1)$, $\varphi(z) \not\equiv 0$ *satisfy the hypotheses of Theorem* 1.1. *If* $f(z) \not\equiv 0$ *is a meromorphic solution of* (1.3) *with* $\rho(f) = +\infty$ *and* $\rho_2(f) = \rho$, *then* f *satisfies*

$$\bar{\lambda}(f - \varphi) = \bar{\lambda}(f' - \varphi) = \bar{\lambda}(f'' - \varphi) = \rho(f) = +\infty, \quad (1.6)$$

$$\bar{\lambda}_2(f - \varphi) = \bar{\lambda}_2(f' - \varphi) = \bar{\lambda}_2(f'' - \varphi) = \rho_2(f) = \rho. \quad (1.7)$$

Corollary. *Let $A_0(z)$ be a transcendental entire function with $\rho(A_0) < 1$, let d_j $(j = 0, 1, 2)$ be polynomials that are not all equal to zero, $\varphi(z)$ $(\not\equiv 0)$ be an entire function of finite order. If f is a nontrivial solution of the equation*

$$f'' + e^{-z} f' + A_0(z) f = 0, \tag{1.8}$$

then the differential polynomial $g(z) = d_2 f'' + d_1 f' + d_0 f$ satisfies $\bar{\lambda}(g - \varphi) = +\infty$.

2. Auxiliary lemmas

We need the following lemmas in the proofs of our theorems.

Lemma 2.1 ([6])**.** *Let f be a transcendental meromorphic function of finite order ρ, let $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ for $i = 1, \dots, m$ and let $\varepsilon > 0$ be a given constant. Then the following two statements hold:*

(i) *There exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi) \backslash E_1$, then there is a constant $R_1 = R_1(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \geq R_1$ and for all $(k, j) \in \Gamma$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

(ii) *There exists a set $E_2 \subset (1, +\infty)$ that has finite logarithmic measure*

$$lm(E_2) = \int_1^{+\infty} \frac{\chi_{E_2}(t)}{t} dt,$$

where χ_{E_2} is the characteristic function of E_2, such that for all z satisfying $|z| \notin E_2 \cup [0, 1]$ and for all $(k, j) \in \Gamma$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.2}$$

Lemma 2.2. *Let A_j $(j = 0, 1)$ and A_2 $(\not\equiv 0)$ be meromorphic functions with $\rho(A_j) < 1$ $(j = 0, 1, 2)$. We denote $\psi_2(z) = A_0 + A_1 e^{-z}$. Let ψ_{21}, ψ_{20} have the form of ψ_2, and $\varphi \not\equiv 0$ be a meromorphic function of finite order. Then*

(i) $\psi_2(z) + A_2 e^{-2z} \not\equiv 0$;

(ii) $\psi_{21} \dfrac{\varphi'}{\varphi} + \psi_{20} + A_2 e^{-2z} \not\equiv 0$.

Proof. Suppose that the claim fails. As for (i), if $A_2e^{-2z}+A_1e^{-z}+A_0\equiv 0$, then

$$\begin{aligned} 2T\left(r,e^{-z}\right) &\le T\left(r,A_2e^{-2z}\right)+T\left(r,\frac{1}{A_2}\right) \\ &= T\left(r,A_1e^{-z}+A_0\right)+T\left(r,\frac{1}{A_2}\right) \\ &\le T\left(r,e^{-z}\right)+\sum_{j=0}^{2}T\left(r,A_j\right)+O\left(1\right), \end{aligned} \tag{2.3}$$

hence $\rho\left(e^{-z}\right)<1$, a contradiction. As for (ii), the left hand side can be written as follows

$$\begin{aligned} &\psi_{21}\frac{\varphi'}{\varphi}+\psi_{20}+A_2e^{-2z} \\ &= A_{00}+A_{01}\frac{\varphi'}{\varphi}+\left(A_{10}+A_{11}\frac{\varphi'}{\varphi}\right)e^{-z}+A_2e^{-2z}, \end{aligned} \tag{2.4}$$

where A_{00}, A_{01}, A_{10}, A_{11} are meromorphic functions of order <1. Since φ is of finite order, then by the lemma of logarithmic derivative ([7])

$$m\left(r,\frac{\varphi'}{\varphi}\right)=O\left(\ln r\right). \tag{2.5}$$

Therefore, by a reasoning as to above, but using the proximity functions instead of the characteristic, a contradiction $\rho\left(e^{-z}\right)<1$ again follows. □

Lemma 2.3 ([4])**.** *If* $A_0\left(z\right)$ *is a transcendental entire function with* $\rho\left(A_0\right)<1$, *then every solution* $f\left(z\right)\not\equiv 0$ *of equation* (1.8) *has infinite order.*

Lemma 2.4 (Wiman-Valiron, [8], [14])**.** *Let* $f\left(z\right)$ *be a transcendental entire function, and let* z *be a point with* $|z|=r$ *at which* $|f\left(z\right)|=M\left(r,f\right)$. *Then the estimate*

$$\frac{f^{(k)}\left(z\right)}{f\left(z\right)}=\left(\frac{\nu\left(r,f\right)}{z}\right)^k\left(1+o\left(1\right)\right)\quad (k\ge 1 \text{ is an integer}), \tag{2.6}$$

holds for all $|z|$ *outside a set* E_3 *of* r *of finite logarithmic measure, where* $\nu\left(r,f\right)$ *denotes the central index of* f.

To avoid some problems caused by the exceptional set we recall the following lemma.

Lemma 2.5 ([5]). *Let* $g\colon [0,+\infty) \to \mathbb{R}$ *and* $h\colon [0,+\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $g(r) \le h(r)$ *for all* $r \notin E_4 \cup [0,1]$, *where* $E_4 \subset (1,+\infty)$ *is a set of finite logarithmic measure. Let* $\alpha > 1$ *be a given constant. Then there exists an* $r_0 = r_0(\alpha) > 0$ *such that* $g(r) \le h(\alpha r)$ *for all* $r \ge r_0$.

Lemma 2.6. *Let* $f(z)$ *be a meromorphic function with* $\rho(f) = +\infty$ *and the exponent of convergence of the poles of* $f(z)$, $\lambda(1/f) < +\infty$. *Let* $d_j(z)$ $(j = 0, 1, 2)$ *be polynomials that are not all equal to zero. Then*

$$g(z) = d_2(z) f'' + d_1(z) f' + d_0(z) f \tag{2.7}$$

satisfies $\rho(g) = +\infty$.

Proof. We suppose that $\rho(g) < +\infty$ and then we obtain a contradiction. First, we suppose that $d_2(z) \not\equiv 0$. Set $f(z) = w(z)/h(z)$, where $h(z)$ is canonical product (or polynomial) formed with the non-zero poles of $f(z)$, $\lambda(h) = \rho(h) = \lambda(1/f) < +\infty$, $w(z)$ is an entire function with $\rho(w) = \rho(f) = +\infty$. We have

$$\begin{aligned} f'(z) &= \frac{w'}{h} - \frac{h'}{h^2} w, \\ f''(z) &= \frac{w''}{h} - 2\frac{h'}{h^2} w' + \left(2\frac{(h')^2}{h^3} - \frac{h''}{h^2}\right) w \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} f'''(z) &= \frac{w'''}{h} - 3\frac{h'}{h^2} w'' + \left(6\frac{(h')^2}{h^3} - 3\frac{h''}{h^2}\right) w' \\ &\quad + \left(6\frac{h'h''}{h^3} - 6\frac{(h')^3}{h^4} - \frac{h'''}{h^2}\right) w. \end{aligned} \tag{2.9}$$

Hence

$$\begin{aligned} \frac{f'''(z)}{f(z)} &= \frac{w'''}{w} - 3\frac{h'}{h}\frac{w''}{w} + \left(6\frac{(h')^2}{h^2} - 3\frac{h''}{h}\right)\frac{w'}{w} \\ &\quad + 6\frac{h'h''}{h^2} - 6\frac{(h')^3}{h^3} - \frac{h'''}{h}, \end{aligned} \tag{2.10}$$

$$\frac{f''(z)}{f(z)} = \frac{w''}{w} - 2\frac{h'}{h}\frac{w'}{w} + 2\frac{(h')^2}{h^2} - \frac{h''}{h}, \tag{2.11}$$

$$\frac{f'(z)}{f(z)} = \frac{w'}{w} - \frac{h'}{h}. \tag{2.12}$$

Differentiating both sides of (2.7), we obtain

$$g' = d_2 f''' + \left(d_2' + d_1\right) f'' + \left(d_1' + d_0\right) f' + d_0' f. \tag{2.13}$$

Writing $g' = (g'/g)g$ and substituting $g = d_2 f'' + d_1 f' + d_0 f$ into (2.13), we get

$$d_2 f''' + \left(d_2' - \frac{g'}{g} d_2 + d_1\right) f'' + \left(d_1' - \frac{g'}{g} d_1 + d_0\right) f' + \left(d_0' - \frac{g'}{g} d_0\right) f = 0. \tag{2.14}$$

This leads to

$$d_2 \frac{f'''}{f} + \left(d_2' - \frac{g'}{g} d_2 + d_1\right) \frac{f''}{f} + \left(d_1' - \frac{g'}{g} d_1 + d_0\right) \frac{f'}{f} + \left(d_0' - \frac{g'}{g} d_0\right) = 0. \tag{2.15}$$

Substituting (2.10)–(2.12) into (2.15), we obtain

$$\begin{aligned} & d_2 \left(\frac{w'''}{w} - 3\frac{h'}{h}\frac{w''}{w} + \left(6\frac{(h')^2}{h^2} - 3\frac{h''}{h}\right)\frac{w'}{w} + \left(6\frac{h'h''}{h^2} - 6\frac{(h')^3}{h^3} - \frac{h'''}{h}\right)\right) \\ & + \left(d_2' - \frac{g'}{g} d_2 + d_1\right)\left(\frac{w''}{w} - 2\frac{h'}{h}\frac{w'}{w} + 2\frac{(h')^2}{h^2} - \frac{h''}{h}\right) \\ & + \left(d_1' - \frac{g'}{g} d_1 + d_0\right)\left(\frac{w'}{w} - \frac{h'}{h}\right) + \left(d_0' - \frac{g'}{g} d_0\right) = 0. \end{aligned} \tag{2.16}$$

By Lemma 2.1 (ii), there exists a set $E_1 \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$, we have

$$\left|\frac{g'(z)}{g(z)}\right| = O\left(r^{\alpha}\right), \quad \left|\frac{h^{(j)}(z)}{h(z)}\right| = O\left(r^{\alpha}\right) \quad (j = 1, 2, 3), \tag{2.17}$$

where $0 < \alpha < +\infty$ is some constant. By Lemma 2.4, there exists a set $E_2 \subset (1, +\infty)$ with logarithmic measure $lm\,(E_2) < +\infty$ and we can choose z satisfying $|z| = r \notin E_2 \cup [0, 1]$ and $|w\,(z)| = M\,(r, w)$, such that

$$\frac{w^{(j)}\,(z)}{w\,(z)} = \left(\frac{\nu\,(r, w)}{z}\right)^j (1 + o\,(1)) \quad (j = 1, 2, 3). \tag{2.18}$$

Now we take point z satisfying $|z| = r \notin E_1 \cup E_2 \cup [0, 1]$ and $w\,(z) = M\,(r, w)$, by substituting (2.17) and (2.18) into (2.16), we get

$$\nu\,(r, w) = O\left(r^{\beta}\right), \tag{2.19}$$

where $0 < \beta < +\infty$ is some constant. By Lemma 2.5, we conclude that (2.19) holds for a sufficiently large r. This is a contradiction by $\rho\,(w) = +\infty$. Hence $\rho\,(g) = +\infty$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$. Using a similar reasoning as above, we get a contradiction. Hence $\rho(g) = +\infty$.

Finally, if $d_2 \equiv 0$, $d_1 \equiv 0$, $d_0 \not\equiv 0$, then we have $g(z) = d_0(z) f(z)$ and by d_0 is a polynomial, then we get $\rho(g) = +\infty$. □

Lemma 2.7 ([1]). *Let $A_0, A_1, \dots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\rho(f) = +\infty$ of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.20}$$

then $\bar{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.

Lemma 2.8. *Let $A_0, A_1, \dots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of the equation (2.20) with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies $\bar{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho$.*

Proof. By equation (2.20), we can write

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \tag{2.21}$$

If f has a zero at z_0 of order α $(> k)$ and if $A_0, A_1, \dots, A_{k-1}$ are all analytic at z_0, then F has a zero at z_0 of order at least $\alpha - k$. Hence,

$$n\left(r, \frac{1}{f}\right) \le k\bar{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} n(r, A_j) \tag{2.22}$$

and

$$N\left(r, \frac{1}{f}\right) \le k\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} N(r, A_j). \tag{2.23}$$

By (2.21), we have

$$m\left(r, \frac{1}{f}\right) \le \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=0}^{k-1} m(r, A_j) + m\left(r, \frac{1}{F}\right) + O(1). \tag{2.24}$$

Applying the lemma of the logarithmic derivative (see [7]), we have

$$m\left(r, \frac{f^{(j)}}{f}\right) = O(\log T(r, f) + \log r) \quad (j = 1, \dots, k), \tag{2.25}$$

holds for all r outside a set $E \subset (0, +\infty)$ with a finite linear measure $m(E) < +\infty$. By (2.23), (2.24) and (2.25), we get

$$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$$

$$\leq k\bar{N}\left(r,\frac{1}{f}\right)+\sum_{j=0}^{k-1}T\left(r,A_j\right)+T\left(r,F\right)+O\left(\log\left(rT\left(r,f\right)\right)\right)$$
$$\left(|z|=r\notin E\right). \tag{2.26}$$

Since $\rho(f)=+\infty$, then there exists $\{r'_n\}$ $(r'_n\to+\infty)$ such that

$$\lim_{r'_n\to+\infty}\frac{\log T\left(r'_n,f\right)}{\log r'_n}=+\infty. \tag{2.27}$$

Set the linear measure of E, $m(E)=\delta<+\infty$, then there exists a point $r_n\in[r'_n,r'_n+\delta+1]-E$. From

$$\frac{\log T\left(r_n,f\right)}{\log r_n}\geq\frac{\log T\left(r'_n,f\right)}{\log\left(r'_n+\delta+1\right)}=\frac{\log T\left(r'_n,f\right)}{\log r'_n+\log\left(1+\frac{\delta+1}{r'_n}\right)}, \tag{2.28}$$

it follows that

$$\lim_{r_n\to+\infty}\frac{\log T\left(r_n,f\right)}{\log r_n}=+\infty. \tag{2.29}$$

Set $\sigma=\max\{\rho(A_j)\ (j=0,\dots,k-1),\rho(F)\}$. Then for a given arbitrary large $\beta>\sigma$,

$$T\left(r_n,f\right)\geq r_n^{\beta} \tag{2.30}$$

holds for sufficiently large r_n. On the other hand, for any given ε with $0<2\varepsilon<\beta-\sigma$, we have

$$T\left(r_n,A_j\right)\leq r_n^{\sigma+\varepsilon}\ \left(j=0,\dots,k-1\right),\quad T\left(r_n,F\right)\leq r_n^{\sigma+\varepsilon} \tag{2.31}$$

for sufficiently large r_n. Hence, we have

$$\max\left\{\frac{T\left(r_n,F\right)}{T\left(r_n,f\right)},\frac{T\left(r_n,A_j\right)}{T\left(r_n,f\right)}\ \left(j=0,\dots,k-1\right)\right\}\leq\frac{r_n^{\sigma+\varepsilon}}{r_n^{\beta}}\to0,\quad r_n\to+\infty. \tag{2.32}$$

Therefore,

$$T\left(r_n,F\right)\leq\frac{1}{k+3}T\left(r_n,f\right),$$
$$T\left(r_n,A_j\right)\leq\frac{1}{k+3}T\left(r_n,f\right)\quad\left(j=0,\dots,k-1\right) \tag{2.33}$$

holds for sufficiently large r_n. From

$$O\left(\log r_n+\log T\left(r_n,f\right)\right)=o\left(T\left(r_n,f\right)\right), \tag{2.34}$$

we obtain that

$$O\left(\log r_n + \log T\left(r_n, f\right)\right) \leq \frac{1}{k+3} T\left(r_n, f\right) \tag{2.35}$$

also holds for sufficiently large r_n. Thus, by (2.26), (2.33), (2.35), we have

$$T\left(r_n, f\right) \leq k\left(k+3\right) \bar{N}\left(r_n, \frac{1}{f}\right). \tag{2.36}$$

It yields $\bar{\lambda}_2\left(f\right) = \lambda_2\left(f\right) = \rho_2\left(f\right) = \rho$. □

3. Proof of Theorem 1.1

First, we suppose $d_2 \not\equiv 0$. Suppose that f $(\not\equiv 0)$ is a meromorphic solution of equation (1.3) with $\rho\left(f\right) = +\infty$ and $\lambda\left(1/f\right) < +\infty$. Set $w = g - \varphi = d_2 f'' + d_1 f' + d_0 f - \varphi$, then by Lemma 2.6 we have $\rho\left(w\right) = \rho\left(g\right) = \rho\left(f\right) = +\infty$. In order to the prove $\bar{\lambda}\left(g - \varphi\right) = +\infty$, we need to prove only $\bar{\lambda}\left(w\right) = +\infty$.

Substituting $f'' = -A_1 f' - A_0 f$ into w, we get

$$w = \left(d_1 - d_2 A_1\right) f' + \left(d_0 - d_2 A_0\right) f - \varphi. \tag{3.1}$$

Differentiating both a sides of equation (3.1), we obtain

$$\begin{aligned} w' = & \left(d_2 A_1^2 - \left(d_2 A_1\right)' - d_2 A_0 - d_1 A_1 + d_0 + d_1'\right) f' \\ & + \left(d_2 A_1 A_0 - d_1 A_0 - \left(d_2 A_0\right)' + d_0'\right) f - \varphi'. \end{aligned} \tag{3.2}$$

Set

$$\begin{aligned} \alpha_1 &= d_1 - d_2 A_1, \\ \alpha_0 &= d_0 - d_2 A_0, \\ \beta_1 &= d_2 A_1^2 - \left(d_2 A_1\right)' - d_2 A_0 - d_1 A_1 + d_0 + d_1', \\ \beta_0 &= d_2 A_1 A_0 - d_1 A_0 - \left(d_2 A_0\right)' + d_0'. \end{aligned}$$

Then we have

$$\alpha_1 f' + \alpha_0 f = w + \varphi, \tag{3.3}$$

$$\beta_1 f' + \beta_0 f = w' + \varphi'. \tag{3.4}$$

Set

$$h = \alpha_1 \beta_0 - \beta_1 \alpha_0. \tag{3.5}$$

We divide it into two cases to prove.

Case 1. If $h \equiv 0$, then by (3.3)–(3.5), we get

$$\alpha_1 w' - \beta_1 w = -(\alpha_1 \varphi' - \beta_1 \varphi) = F. \tag{3.6}$$

By the hypotheses of Theorem 1.1, we have $\alpha_1\varphi' - \beta_1\varphi \not\equiv 0$ and then $F \not\equiv 0$. By $\alpha_1 \not\equiv 0$, $F \not\equiv 0$, and Lemma 2.7, we obtain $\bar{\lambda}(w) = \lambda(w) = \rho(w) = +\infty$, i.e., $\bar{\lambda}(g - \varphi) = +\infty$.

Case 2. If $h \not\equiv 0$, then by (3.3)–(3.5), we get

$$f = \frac{\alpha_1(w' + \varphi') - \beta_1(w + \varphi)}{h}. \tag{3.7}$$

Substituting (3.7) into equation (1.3) we obtain

$$\begin{aligned} &\frac{\alpha_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w \\ &= -\left(\left(\frac{\alpha_1\varphi' - \beta_1\varphi}{h}\right)'' + A_1\left(\frac{\alpha_1\varphi' - \beta_1\varphi}{h}\right)' + A_0\left(\frac{\alpha_1\varphi' - \beta_1\varphi}{h}\right)\right) \\ &= F, \end{aligned} \tag{3.8}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho(\phi_j) < +\infty$ $(j = 0, 1, 2)$. By the hypotheses of Theorem 1.1, we have $\alpha_1\varphi' - \beta_1\varphi \not\equiv 0$ and by

$$\rho\left(\frac{\alpha_1\varphi' - \beta_1\varphi}{h}\right) < +\infty,$$

it follows that $F \not\equiv 0$. By Lemma 2.7, we obtain $\bar{\lambda}(w) = \lambda(w) = \rho(w) = +\infty$, i.e., $\bar{\lambda}(g - \varphi) = +\infty$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$ or $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$. Using a similar reasoning to that above we get $\bar{\lambda}(w) = \lambda(w) = \rho(w) = +\infty$, i.e., $\bar{\lambda}(g - \varphi) = +\infty$. □

4. Proof of Theorem 1.2

Suppose that $f \not\equiv 0$ is a meromorphic solution of equation (1.3) with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$. Set $w_j = f^{(j)} - \varphi$ $(j = 0, 1, 2)$. Since $\rho(\varphi) < +\infty$, then we have $\rho(w_j) = \rho(f) = +\infty$, $\rho_2(w_j) = \rho_2(f) = \rho$ $(j = 0, 1, 2)$. By using a similar reasoning to that in the proof of Theorem 1.1, we obtain that

$$\bar{\lambda}(f - \varphi) = \bar{\lambda}(f' - \varphi) = \bar{\lambda}(f'' - \varphi) = \rho(f) = +\infty \tag{4.1}$$

and by Lemma 2.8, we get

$$\bar{\lambda}_2(f - \varphi) = \bar{\lambda}_2(f' - \varphi) = \bar{\lambda}_2(f'' - \varphi) = \rho_2(f) = \rho. \tag{4.2}$$

□

5. Proof of Corollary

Suppose that $f \not\equiv 0$ is a solution of equation (1.8). Then by Lemma 2.3, f is of infinite order. Let $\varphi \not\equiv 0$ be a finite order entire function. Then by using Lemma 2.2, we have

$$\begin{aligned}&\left(d_1 - d_2 e^{-z}\right) \frac{\varphi'}{\varphi}\\ &- \left(d_2 e^{-2z} - \left(d_1 + d_2' - d_2\right) e^{-z} - d_2 A_0 + d_0 + d_1'\right) \not\equiv 0. \end{aligned} \tag{5.1}$$

Hence,

$$\begin{aligned}&\left(d_1 - d_2 e^{-z}\right) \varphi'\\ &- \left(d_2 e^{-2z} - \left(d_1 + d_2' - d_2\right) e^{-z} - d_2 A_0 + d_0 + d_1'\right) \varphi \not\equiv 0. \end{aligned} \tag{5.2}$$

By Theorem 1.1, the differential polynomial $g(z) = d_2 f'' + d_1 f' + d_0 f$ satisfies $\bar{\lambda}(g - \varphi) = +\infty$. □

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References

[1] Chen, Z. X., *Zeros of meromorphic solutions of higher order linear differential equations*, Analysis (Oxford) **14** (1994), 425–438.

[2] Chen, Z. X., *The fixed points and hyper order of solutions of second order complex differential equations* (in Chinese), Acta Math. Sci. Ser. A Chin. Ed. **20**(3) (2000), 425–432.

[3] Chen, Z. X., Shon, K. H., *On the growth and fixed points of solutions of second order differential equations with meromorphic coefficients*, Acta Math. Sin. (Engl. Ser.) **21**(4) (2005), 753–764.

[4] Gundersen, G. G., *On the question of whether $f'' + e^{-z} f' + Q(z) f = 0$ can admit a solution $f \not\equiv 0$ of finite order*, Proc. Roy. Soc. Edinburg Sect. A **102** (1986), 9–17.

[5] Gundersen, G. G., *Finite order solutions of second order linear differential equations*, Trans. Amer. Math. Soc. **305** (1988), 415–429.

[6] Gundersen, G. G., *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc. (2) **37** (1988), 88–104.

[7] Hayman, W. K., *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[8] Hayman, W. K., *The local growth of power series: a survey of the Wiman-Valiron method*, Canad. Math. Bull. **17** (1974), 317–358.

[9] Kinnunen, L., *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math. **22**(4) (1998), 385–405.

[10] Laine, I., *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin-New York, 1993.

[11] Laine, I., Rieppo, J., *Differential polynomials generated by linear differential equations*, Complex Var. Theory Appl. **49**(12) (2004), 897–911

[12] Liu, M. S., Zhang, X. M., *Fixed points of meromorphic solutions of higher order Linear differential equations*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 191–211.
[13] Nevanlinna, R., *Eindeutige Analytische Funktionen* (in German), Zweite Auflage. Reprint. Grundlehren Math. Wiss. **46**, Springer-Verlag, Berlin-New York, 1974.
[14] Valiron, G., *Lectures on the General Theory of Integral Functions*, Chelsea, New York, 1949.
[15] Wang, J. and Yi, H. X., *Fixed points and hyper order of differential polynomials generated by solutions of differential equation*, Complex Var. Theory Appl. **48**(1) (2003), 83–94.
[16] Yang, C. C., Yi, H. X., *Uniqueness theory of meromorphic functions*, Math. Appl. **557**, Kluwer Acad. Publ. Group, Dordrecht, 2003.
[17] Zhang, Q. T., Yang, C. C., *The Fixed Points and Resolution Theory of Meromorphic Functions* (in Chinese), Beijing Univ. Press, Beijing, 1988.

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